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Optimal Control for Maximal Accuracy with an Arbitrary Control Space Metric

Erik I. Verriest*

Georgia Tech Lorraine
Technôpole de Metz 2000
F-57070 Metz, France
erik.verriest@ee.gatech.edu.

W. Steven Gray

Department of Electrical and Computer Engineering
Drexel University
Philadelphia, Pennsylvania 19104-2884
steven_gray@ece.drexel.edu

Abstract

The finite dimensional theory of minimal sensitivity design is extended to infinite dimensions. The high accuracy control of the state vector of a system is a practical application of this problem. First the discrete time high accuracy control problem is solved for a single input system with fixed bound on the relative error of the control. The optimal steering is characterized as one that is zero for the longest possible time. The continuous time problem is solved via the maximum principle and the example of the rocket car with relative control error is solved in detail. The maximum accuracy, and the accuracy/time problem have a solution of bang-zero-bang type. The accuracy/energy problem also exhibits a coasting period.

1. Introduction

The sensitivity of a particular design, given some design objectives, can typically be minimized by proper choice of the design. A geometrically pleasing solution has been given in [1, 2, 3], for the case of a *finite* number of degrees of freedom (the dimension of the parameter space). In many cases the parameter space is not finite dimensional. Whereas a great deal of effort has been spent on *analyzing* the sensitivity in the case of a Banach space valued parameter, (see e.g. [5] and references therein), it seems that the literature is less abundant about the problem of *minimizing* the sensitivity of a design with respect to perturbations in the parameter space. Unlike the theory of robust control, the effect of perturbations is minimized by tuning these (perturbation) parameters. In this paper, the finite dimensional theory of extremal sensitivity [2, 3], summarized below, is *extended* to the infinite case.
In section 2, the infinite dimensional extension is introduced as a problem of high accuracy steering. Section 3 bridges the gap between the finite dimensional and infinite dimensional case for *discrete* systems. The tools of optimal control theory are used in section 4 to solve the equivalent continuous time problem, and in section 5, the example of the rocket car is worked out in detail for different performance indices, combined with the high accuracy requirement.

Extremal sensitivity theorem (EST)
Let Θ be the parameter space, which is endowed with a Riemannian structure, g, allowing to speak of *direction* and *magnitude* of perturbations. An *observable* f is a smooth real valued function on (Θ, g), which lacks critical points. The inverse image $f^{-1}(0)$ contains equivalent *realizations* of f, and is a smooth submanifold of Θ. The design objective is to find in $f^{-1}(0)$ the parameter which exhibits the least sensitivity with respect to perturbations in Θ. The EST states that at points $\theta^* \in \Theta$ of extremal sensitivity, the gradient of the observable f must be in the eigenspace of the Hessian operator of f. (Both gradient and Hessian are defined with respect to the metric g.) This purely geometric criterion gives a viable alternative to the usual stochastic methods, since it handles the *worst case* performance whereas the stochastic theory optimizes the *average* performance.

2. Extremal Accuracy Control: The Problem

Consider the problem of steering the state of

$$\dot{x} = f(x, u, t) \tag{1}$$

from a fixed initial state to a terminal manifold $\Psi(x) = 0$ in a fixed time T. Acknowledging that in general several different steering policies perform the same task if a reachability condition is satisfied, one usually tries to find an *optimal* steering with respect to some performance index, trading off control effort and state excursions. In this paper we consider the optimal control policy if it is known a priori that, in addition, any desired policy can only *inaccurately* be generated, as is the case in a finite precision environment. This could be due to fast but inaccurate computer control (finite wordlength effects), or due to crudely imprecise actuators. In stone throwing, the angle of departure and initial velocity can only be inaccurately set. This example was used in [3] to illustrate the minimum sensitivity (of target position) design problem. The present note can also be seen as a generalization of the minimum sensitivity design problem [2] to the dynamic case. As expected this problem becomes an infinite dimensional one, and the conscise characterization of the extremal sensitivity as a gradient of a Hessian is more elusive. The problem will be approached by the calculus of variations and/or the Pontryagin maximum principle.

Thus, consider the system (1) where x is n-dimensional and the control u assumes its values in a space $\mathcal{U}$ which we shall endow with a Riemannian metric g: At each point u the tangent space $T_u\mathcal{U}$ is endowed with an inner product $g_u(\cdot,\cdot) : T_u\mathcal{U} \times T_u\mathcal{U} \rightarrow$

*also : School of Electrical and Computer Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0250.

IR. Note that this Riemannian space $(\mathcal{U}, g)$ is still finite dimensional . This may not be entirely realistic, in practice there usually is some temporal coherence in the control perturbations, as in A-D conversion, but it simplifies the analysis. Moreover by letting the worst case perturbation be defined at *each* instant, more conservative results are obtained than if coherences are taken into account.

The stage of the problem being set, let the desired nominal problem be the minimization of $\Phi(x(T)) + \int_0^T L(x,u,t)\,dt$ with the exact terminal constraint $\Psi(x(T)) = 0$. Assume this problem has a solution, which we shall denote by $u_o(t)$, and let $x_o(t)$ be the corresponding optimal trajectory.

3. High Accuracy Steering: Discrete Case

In order to bridge the gap from minimum sensitivity design with respect to a finite number of parameters, and the minimum sensitivity design with respect to a function, to be discussed in the next section, we shall consider the high accuracy control of a discrete n-th order m-input time invariant linear system $x_{k+1} = Ax_k + Bu_k$. Let the initial state be x_0. It is desired to reach the state x_f at the N-th step. It is well known that this can be achieved for *arbitrary* initial and final state if and only if the (A, B) is reachable and $N \geq n$, where n is the system order. For $N > n$, redundancy can be exploited to minimize the performance index. Iterating the state equations gives, $x_f - A^n x_0 = [B, AB, \ldots, A^{N-1}B]\mathcal{U}_N$ where $\mathcal{U}_N' = [u_{n-1}', u_{n-2}', \ldots, u_0']'$. This plays the role of a vector observable (of dimension n), while the space of the mN input components plays the role of the space Θ of design parameters. The maximal accuracy problem is in this framework exactly the minimal sensitivity problem discussed in [1, 2, 3].

Let us for simplicity consider the case of a first order system with scalar input, i.e., A and $B \neq 0$ are scalar. As is usually the case in practice, we shall assume that the inputs can only be inaccurately generated, with a given bound on the *relative error*. This means that the metric in the parameter space $\Theta = \mathbb{R}^N$ is given by the weighting matrix $G = \mathrm{diag}(\theta_i^{-2})$. In [2] we derived also an easier to use, but equivalent matrix criterion for extremal sensitivity, in terms of the *pseudo-gradient* vector and the *pseudo-Hessian* matrix, respectively df and d^2f. The criterion states that df must be in a left (note that df is a row vector) eigenspace of the matrix d^2f. Application to our problem gives, with $\theta_{N-k} = u_k$ and $f = \mathcal{R}_N\theta - \Delta_N$, where $\mathcal{R}_N$ is the reachability matrix for N steps, and $\Delta_N = x_f - A^N x_0$,

$$df = \mathcal{R}_N \,\mathrm{diag}(|\theta|) = [B|\theta_1|, \ldots, A^{N-1}B|\theta_N|] \quad (2)$$

$$d^2 f = \mathrm{diag}(B|\theta_1|, \ldots, A^{N-1}B|\theta_N|). \quad (3)$$

The resulting minimum sensitivity necessary conditions are with $\zeta = [\zeta_1, \ldots, \zeta_N]'$, of the form:

$$\mathrm{diag}(\zeta)\zeta = \lambda\zeta, \quad \zeta_i = A^{i-1}B|\theta_i|, \quad (4)$$

which has many solutions: Indeed, if $\mathcal{I} \subset \{1, 2, \ldots, N\}$ is an arbirary index set, then $\zeta_i = \lambda$ for $i \in \mathcal{I}$ and $\zeta_j = 0$ for $j \notin \mathcal{I}$ is a solution. Hence if $\mathcal{I}_+$ and $\mathcal{I}_-$ are arbitrary disjoint subsets of $\{1, 2, \ldots, N\}$, then it is readily seen that a solution is

$$A^{i-1}B\theta_i = \lambda, \quad \forall i \in \mathcal{I}_+ \quad (5)$$

$$A^{i-1}B\theta_i = -\lambda, \quad \forall i \in \mathcal{I}_- \quad (6)$$

$$A^{j-1}B\theta_j = 0, \quad \forall j \notin \mathcal{I}_+ \cup \mathcal{I}_-. \quad (7)$$

The λ is determined by the state constraint $\sum_i A^{i-1}\theta_i = \Delta_N$, i.e., we have $\lambda = \frac{\Delta_N}{\#\mathcal{I}_+ - \#\mathcal{I}_-}$. Are all candidate solutions really maximizing the accuracy? Lemma 2.13 in [2] can provide an answer. In this scalar case, it can be easily checked that, with the relative metric, and a nominal control of the above class: $u_k = \theta^*_{N-k}$, the worst perturbation of the final state is proportional to

$$\sum_i |A^{i-1}Bu_{N-i}| = |\Delta_N| \left| \frac{\#\mathcal{I}_+ + \#\mathcal{I}_-}{\#\mathcal{I}_+ - \#\mathcal{I}_-} \right|. \quad (8)$$

There is clearly a benefit in taking either $\mathcal{I}_+$ or $\mathcal{I}_-$ empty. Therefore all (equivalent) high accuracy controls are of the form (if $A = 0$, we limit N to 1)

$$\theta_i^* = \frac{\Delta_N}{(\#\mathcal{I})A^{i-1}B}, \quad i \in \mathcal{I} \quad (9)$$

$$\theta_i^* = 0 \quad , \quad i \notin \mathcal{I} \quad (10)$$

for an arbitrary index set $\mathcal{I}$.

How does this generalize to the maximal accuracy steering of higher order single input systems (still with a relative error criterion)? In the framework of our minimal sensitivity design problem, the *observable*, f, is now multi-dimensional. A method was introduced in [1, 2, 3] to deal with this. Restating this in the current context, one considers for all unit vectors ν of dimension n the *scalar* observable $\nu' f$. Solving the problem in this fashion for fixed ν gives again a necessary condition in the form

$$\mathrm{diag}(\zeta)\zeta = \lambda\zeta, \quad (11)$$

where now instead $\zeta_k = \nu' A^{N-k}B|\theta_k| = \nu' A^{N-k}B|u_{N-k}|$. Hence the solution from the previous paragraph can be adapted to find

$$u_k^* \in \{0, \lambda(\nu' A^{N-k-1}B)^{-1}\}, \quad (12)$$

with

$$\lambda = \frac{\nu'\Delta_N}{\#\mathcal{I}} \quad \text{or} \quad -\frac{\nu'\Delta_N}{\#\mathcal{I}}. \quad (13)$$

Clearly, this solution *depends* on the choice of the unit vector ν, unlike in the application of the extremal sensitivity theorem to optimum system implementation, where the optimum solution was ν-independent. Moreover, only the constraint $\nu'\mathcal{R}_N\mathcal{U}_N = \nu'(x_f - A^N x_0)$ is satisfied. Consequently one cannot conclude that all solutions to the above minimize the accuracy while satisfying the final state *vector* observable. One could argue that the introduction of an alternate scalar constraint $\phi(\mathcal{U}_N) = \frac{1}{2}(\mathcal{R}_N\mathcal{U}_N - \Delta_N)'(\mathcal{R}_N\mathcal{U}_N - \Delta_N)$ might solve the problem, since after all $\phi(\theta) = 0$ iff $\mathcal{R}_N\mathcal{U}_N = \Delta_N$. There is however a catch: this observable has a critical point, and in order for our theory to be applicable, we had to exclude this (we called this a 'technical condition' in [2]). This application shows that the extremal sensitivity condition does not carry any information at a critical point. The full multi-dimensional observable minimum sensitivity theory developed in [6] is required. There is however also a way around this. If with the above analysis, one can find a vector ν for which the *whole* state constraint is satisfied, then a candidate solution with maximal accuracy is found.

For the ν-dependent solution, $u_{N-i}^{(\nu)} = \theta_i = \frac{\nu'\Delta_N}{\#I}\frac{1}{\nu'A^{i-1}B}$ for $i \in \mathcal{I}$ and zero else, one gets at the final step the state (recall that $\Delta_N = x_f - A^N x_0$),

$$x_N = A^N x_0 + \sum_{i\in\mathcal{I}} \frac{\nu'\Delta_N}{\#I}\frac{A^{i-1}B}{\nu'A^{i-1}B}. \qquad (14)$$

Therefore, one needs to find a ν such that

$$\sum_{i\in\mathcal{I}} \frac{\nu'\Delta_N}{\#I}\frac{A^{i-1}B}{\nu'A^{i-1}B} = \Delta_N. \qquad (15)$$

This is in itself a nice linear algebra problem. It can be cast in the form:

Problem ν:
Given a set of $k \le n$ linearly independent vectors $\{\beta_1,\ldots,\beta_k\}$, and a vector x, all in $\mathbb{R}^n$. Find $\nu \in \mathbb{R}^n$ of unit norm such that

$$\left(\frac{1}{k}\sum_{i=1}^{k}\frac{\beta_i\nu'}{\nu'\beta_i} - I\right)x = 0. \qquad (16)$$

Note that in our application, an arbitrary set of vectors $\{A^{i-1}B, i \in \mathcal{I}\}$ is not necessarily a linearly independent set, but since the system is reachable, there will always exist a set of at most n such vectors, constituting a basis for $\mathbb{R}^n$. Below we shall give a constructive solution to problem ν.

Solution
First, it is obvious from (16) that for a solution to exist, x cannot have components outside the span of $\{\beta_1,\ldots,\beta_k\}$. We shall therefore make the additional assumption

$$x \in \text{span}\{\beta_1,\ldots,\beta_k\}. \qquad (17)$$

The set of $k \le n$ vectors, $\{\beta_1,\ldots,\beta_k\}$, can be augmented with $\{\beta_{k+1},\ldots,\beta_n\}$ to obtain a full basis. Let $\{\alpha_1,\ldots,\alpha_n\}$ be the dual basis, i.e., $\alpha_i'\beta_j = \delta_{ij}$. Express the unknown vector ν in terms of the dual basis: $\nu = \sum_{j=1}^n \mu_j\alpha_j$, and substitute in (16) to get

$$\left(\frac{1}{k}\sum_{i=1}^{k}\frac{\sum_{j=1}^n \mu_j\beta_i\alpha_j'}{\mu_i} - I\right)x = 0, \qquad (18)$$

taking the properties of the reciprocal basis into account. We shall temporarily assume that the $\mu_i, i = 1\ldots,k$ are nonzero, and later verify consistency. Express now the given vector x in terms of the (primal) basis taking (17) into account

$$x = \sum_{\ell=0}^{k}\chi_\ell\beta_\ell. \qquad (19)$$

If in addition x has *nonzero components* along all $\beta_i, i = 1,\ldots,k$, then all $\chi_i, i = 1\ldots,k$ are nonzero. Substituting this expansion in the equation (18) yields

$$\frac{1}{k}\sum_{i=1}^{k}\sum_{j=1}^{n}\frac{\mu_j}{\mu_i}\chi_j\beta_i = \sum_{i=1}^{k}\chi_i\beta_i, \qquad (20)$$

from which it follows that for all $i = 1,\ldots,k$: $\frac{1}{k}\sum_{j=1}^n \frac{\mu_j}{\mu_i}\chi_j = \chi_i$. A trivial solution is given by $\mu_i = \begin{cases} \frac{c}{\chi_i}, & i = 1\ldots,k \\ 0, & i = k+1,\ldots,n, \end{cases}$ This is consistent with our temporary assumption, and thus

$$\nu = c\sum_{i=1}^{k}\frac{1}{\chi_i}\alpha_i. \qquad (21)$$

c is a normalization constant, $||\nu|| = 1$, and $\chi_i = \alpha_i'x$ is exactly the component of x along β_i, with respect to the basis $\{\beta_1,\ldots,\beta_n\}$. □

The quantity $\nu'x$ is required below. We get from (19) and (21) $\nu'x = ck$.
Let us now apply this result to the high accuracy reachability problem: Given the initial and final states, respectively x_0 and x_f, a *maximum accuracy* transition from x_0 to x_f in N steps is given by the following (preliminary) algorithm:

i) Compute $\Delta_N = x_f - A^N x_0$, and express it with a *minimal* number of Krylov vectors $B, AB, A^2B,\ldots$, i.e., find a *minimal* index set $\mathcal{I}$ so that the corresponding Krylov vectors are *linearly independent*, and have Δ_N in their span. It follows that then none of these Krylov vectors can be orthogonal to Δ_N.

ii) Compute the vector ν as outlined in the solution of Problem ν. Note that all conditions for a solution to exists are satisfied because of step i).

iii) Generate the open loop control

$$u_k = \begin{cases} \frac{\nu'\Delta_N}{\#I}\frac{1}{\nu'A^{N-k-1}B}, & k \in \mathcal{I} \\ 0, & k \notin \mathcal{I} \end{cases} \qquad (22)$$

iv) As only *necessary* conditions are analyzed, the above steps generate *candidate* solutions. For each index set, the maximal perturbation associated with this index set $\mathcal{I}$ needs to be checked in order to close in on the optimal $\mathcal{I}^*$ and its associated control.

Steps ii) and iii) can be simplified: For this selection of ν from ii), the corresponding control is

$$u_{N-\ell_i}^{(\nu)} = \frac{c}{\nu'A^{\ell_i-1}B} = \frac{c}{c\sum_{\mathcal{I}}\frac{1}{\chi_i}\alpha_i'A^{\ell_i-1}B} = \chi_i. \quad (23)$$

The optimal control is the component of Δ_N *along the vector* $\beta_i = A^{\ell_i-1}B$, *w.r.t. the basis indicated by* $\mathcal{I}$. If there is no such component, the control is consistently set to zero. It thus follows that the maximum accuracy control is a control using the *least number of nonzero controls.* For the selection $\mathcal{I}$, the performance, i.e., the worst perturbation (in the relative metric) is proportional to

$$\begin{aligned} J_{\text{accuracy}} &= \max_{\epsilon_i=\pm1}||\sum_{\mathcal{I}}A^{\ell_i-1}Bu_{N-\ell_i}\epsilon_i|| \\ &= \max_{\epsilon_i=\pm1}\sum_{\mathcal{I}}||\beta_i\alpha_i'\Delta_N\epsilon_i|| \\ &= \max_{E}||\underline{\beta}\,E\,\underline{\beta}^{-1}\Delta_N|| \end{aligned} \qquad (24)$$

where in the last equality the maximization is over all *signature* matrices E. Obviously, a large combinatorial problem remains.

4. Extremal Accuracy Control: The Solution

Returning to the continuous time problem, we solve the problem in two stages. First we derive the *worst case perturbation* in the metric g, about some nominal trajectory, and compute the induced effect on the terminal objective. Next this induced perturbation effect is combined with the given performance index, and the optimal steering is computed.

Part 1: Perturbation Control
Let the effect of an infinitesimal perturbation $\delta u(t)$ for $0 \leq t \leq T$ be a change in the final state $\delta x(T)$, which may cause a change from $\Psi(x_0(T))$ to $\Psi(x_0(T)+\delta x(T))$. To first order:

$$\dot{\delta x} = f_x(x_o, u_o, t)\delta x + f_u(x_0, u_o, t)\delta u. \tag{25}$$

Since the perturbation system is affine (f_x and f_u are functions of u_o, but not δu) it is a time varying linear system. *Maximize* for this perturbation system the quantity $||\delta\Psi||_M$ with the constraint $||\delta u(t)||_U \leq 1$. Note that we also can endow the space of the terminal state with a Riemannian structure, M, in order to weigh the cost of the inaccuracy. However we shall assume that this is already incorporated in the definition of Ψ, so that the uniform metric (i.e. the usual Euclidean metric) is understood.

The above perturbation problem is a standard problem in linear optimal control, for which the Hamiltonian is formulated as

$$H = \lambda_\delta'(f_x\delta x + f_u\delta u), \tag{26}$$

and with terminal cost

$$\Phi(\delta x(T)) = ||\delta\Psi(x(T))|| = ||\Psi_x(x(T))\delta x(T)|| \tag{27}$$

The worst case perturbation is the maximizer of $\lambda_\delta' f_u \delta_u$ over the set $\mathcal{U} = \{g_{u_o}(\delta u, \delta u) \leq 1\}$. Since a linear function is maximized, the solution δ_u must occur at the boundary. It is known that a singular solution cannot exist [4, p.257]. For a single input system, this implies that the worst perturbation will be of bang-bang type, a situation that may be important in practical situations of relay control. The extremal perturbation for multi-input systems is derived from the locally (with respect to time) defined Lagrangian optimization problem

$$\lambda_\delta' f_u \delta_u + \mu g_{u_o}(\delta u, \delta u), \tag{28}$$

where μ is a Lagrange multiplier. Two solutions exist, one minimizing, the other maximizing. Here $\mu = -\frac{1}{2}||G^{-1}f_u'\lambda_\delta||_G$. so that

$$\delta u^* = \frac{G^{-1}f_u'\lambda_\delta}{||G^{-1}f_u'\lambda_\delta||_G}. \tag{29}$$

The costate equation is

$$\dot{\lambda_\delta} = -H_{\delta x}' = -f_x'\lambda_\delta. \tag{30}$$

The boundary conditions on state and costates are

$$\delta x(0) = 0 \quad ; \quad \lambda_\delta(T) = \Psi_x'(x(T))\Psi_x(x(T))\delta x(T), \tag{31}$$

since minimizing (27) is equivalent to minimizing $\frac{1}{2}||\Psi_x(x(T))\delta x(T)||^2$.

Part 2: Nominal Control
Now we solve for the minimizing solution $u^*(t)$ to the *combined* performance index

$$J_{\text{acc}} = \rho||\Psi_x(x(T))\delta x(T)|| + \Phi(x(T)) + \int_0^T L(x,u,t)\,dt. \tag{32}$$

A parameter, ρ, is introduced in order to weigh the relative importance of high accuracy. This minimization is now subject to *three* sets of dynamical constraints with their respective boundary conditions from part 1.

$$\begin{aligned} \dot{x} &= f(x,u) & (33)\\ \dot{\delta x} &= f_x\delta x + ||\lambda_\delta||^{-1}_{f_u G^{-1} f_u'} f_u G^{-1} f_u' \lambda_\delta & (34)\\ \dot{\lambda_\delta} &= -f_x'\lambda_\delta & (35)\end{aligned}$$

This problem is solved by adjoining these dynamic constraints to (32). If a *uniform* metric is used in the control value space $\mathcal{U}$, the inclusion of the maximal accuracy requirement has no effect on the nominal control. In the case of a more general metric, the nominal solution for high accuracy will be different from the nominal solution without the accuracy inclusion.

5. Double Integrator Example (Rocket Car)

In order to illustrate the above ideas, we have worked out the case of the rocket car (double integrator) for the high accuracy steering for several different nominal problems: but all in the case for the relative error metric on the control.

Let x denote position and v velocity. The system equations are, assuming that the car has unit mass,

$$\begin{aligned} \dot{x} &= v & (36)\\ \dot{v} &= u. & (37)\end{aligned}$$

The objective is to transfer from $x_0 = 0$ and $v_0 = 0$ to position $x_f > 0$ at time T with $v(T) = 0$.
For the relative metric, $G^{-1} = u^2$. The accuracy will be obtained by adjoining a term $\rho\frac{1}{2}(\delta x(T)^2 + \delta v(T)^2)$ to the nominal performance index. One finds, since $||\lambda_\delta||_G = |u||\lambda_{\delta v}|$, the worst perturbation dynamics

$$\begin{aligned} \dot{\delta x} &= \delta v & (38)\\ \dot{\delta v} &= |u|\text{sgn}(\lambda_{\delta v}) & (39)\end{aligned}$$

with $\delta x(0) = 0$ and $\delta v(0) = 0$. The perturbation costate equations are easily solved: one finds

$$\begin{aligned} \lambda_{\delta x}(t) &= \delta x(T), \qquad \forall t & (40)\\ \lambda_{\delta v}(t) &= \delta v(T) + (T-t)\delta x(T) & (41)\end{aligned}$$

and for the perturbations:

$$\begin{aligned} \delta v(t) &= \int_0^t |u(\tau)|\text{sgn}(\lambda_{\delta v}(\tau))\,d\tau & (42)\\ \delta x(t) &= \int_0^t (t-\tau)|u(\tau)|\text{sgn}(\lambda_{\delta v}(\tau))\,d\tau. & (43)\end{aligned}$$

We used Laplace's integral to express the double integral. Note also that $\lambda_{\delta v}$ changes sign at most once. We therefore consider separately the cases where $\lambda_{\delta v}$ does and does not change sign. It is found that there are two maximizing perturbations (for $\epsilon = -1$ and $\epsilon = 1$),

$$\begin{aligned} \delta v(T) &= \epsilon\int_0^T |u(t)|\,dt & (44)\\ \delta x(T) &= \epsilon\int_0^T (T-t)|u(t)|\,dt & (45)\end{aligned}$$

with $\delta u^*(t) = \epsilon u(t)$ Since both are equivalent, we shall work with $\epsilon = 1$ in what follows. One can either state the accuracy for this problem in terms of the above integrals over the control, or work with the dynamics of the (worst) perturbation states:

$$\dot{\delta v} = |u| \quad (46)$$

$$\dot{\delta x} = \delta v \quad (47)$$

with terminal objective function $\frac{1}{2}(\delta v^2(T)+\delta x^2(T))$. Now we are ready to consider the different optimal policies corresponding to the various performance indices.

Maximum Accuracy Control

Here the Hamiltonian for the combined problem is

$$\mathcal{H} = \Lambda_x v + \Lambda_v u + \lambda_{\delta x}\delta v + \lambda_{\delta v}|u|. \quad (48)$$

For each t the Hamiltonian is minimal if $\Lambda_v u + \lambda_{\delta v}|u|$ is minimal. So, if u is not constrained, the optimal input u must be impulsive. From first principles, it is clear that such a control will involve at least two impulses: $u(t) = u_0\delta(t) + u_T\delta(t-T)$. We get

$$\begin{array}{lll} v(0+) = u_0 & v(T-) = u_0 & v(T+) = u_0 + u_T \\ x(0+) = 0 & x(T-) = u_0 T & x(T+) = u_0 T, \end{array} \quad (49)$$

from which $u_0 = -u_T = \frac{x_f}{T}$. This gives for the accuracy measure with (34) and (45)

$$\frac{1}{2}(\delta v^2 + \delta x^2) = \frac{1}{2}(1+\frac{4}{T^2})x_f^2. \quad (50)$$

If the control is constrained, say $|u| \le u_0$, then it follows from (48) that the control only assumes the values $-u_0$, 0 and u_0. For $u = 0$, the u-dependent term is zero. For $u = -u_0$ and $u = u_0$ respectively, this term is *linear* in t. Thus the control policy is one of the sequences from the set:

$$\{(-u_0,0,u_0),(u_0,0,-u_0),(-u_0,u_0),(u_0,-u_0),(0,-u_0),$$
$$(-u_0,0),(u_0,0),(0,u_0),(u_0),(-u_0),(0)\}.$$

In order to match the given boundary conditions on x and v, one must start with $u = u_0$ and end with $u = -u_0$. So let's assume we have the policy:

$$u = \begin{cases} u_0 & t \in [0, t_0] \\ 0 & t \in [t_0, T-t_0] \\ -u_0 & t \in [T-t_0, T] \end{cases}. \quad (51)$$

The symmetry of the nominal control is rather obvious and is here used as a shortcut. We get

$$t_0^* = \frac{T \pm \sqrt{T^2 - 4\frac{x_f}{u_0}}}{2}. \quad (52)$$

There are two solutions, but checking the accuracy term gives with (45)

$$\delta v(T) = 2u_0 t_0 \quad (53)$$

$$\delta x(t) = u_0 t_0 T, \quad (54)$$

which shows that the perturbation term is proportional to t_0^2, hence the solution with the minus sign,

$$t_0^* = \frac{T - \sqrt{T^2 - 4\frac{x_f}{u_0}}}{2} \quad (55)$$

must be chosen to maximize accuracy. The accuracy is in this case $\frac{1}{8}u_0^2T^2(4+T^2)\left[1-\sqrt{1-\frac{4x_f}{u_0T^2}}\right]^2$. In the limit for $u_0 \to \infty$, we retrieve the impulsive input solution with accuracy $\frac{1}{2}\left(\frac{4}{T^2}+1\right)x_f^2$. For $T \to \infty$, the accuracy gets bigger and bigger with the above policy, but $t_0^* \to 0$ as $\frac{1}{T}$. We conclude:

With $x_0 = 0, v_0 = 0$ and goal state $x_f > 0, v_f = 0$, then as long as the reachability condition $4\frac{x_f}{u_0} < T^2$ holds, a solution exists. The optimal control consists of a maximal acceleration for a time $t_1^* = \frac{1}{2}[T - \sqrt{T^2 - 4\frac{x_f}{u_0}}]$, followed by coasting until time $t_2^* = T - t_1^*$, and deceleration for the remainder time.

Maximum Accuracy - Minimum Time Control

The Hamiltonian for the combined problem is simply

$$\mathcal{H} = 1 + \Lambda_x v + \Lambda_v u + \lambda_{\delta x}\delta v + \lambda_{\delta v}|u|. \quad (56)$$

The solution is virtually the same as in the maximal accuracy case. Since the final time T is free, the transversality condition $\mathcal{H} = 0$ needs to be added. Alternatively, one may take advantage of the solution obtained in the previous maximal accuracy problem. To this effect, consider the performance we obtained for *fixed* T and with *input constraint* $|u(t)| \le u_0$. This was $\frac{1}{2}u_0^2t_0^2(4+T^2)$ where $\frac{x_f}{u_0} = Tt_0 - t_0^2$. Thus the accuracy-time trade off gives a performance index

$$J_{\text{acc}} = \frac{\rho}{2}u_0^2t_0^2(4+T^2) + T. \quad (57)$$

Substituting

$$T = \frac{t_0^2 + \frac{x_f}{u_0}}{t_0} \quad (58)$$

yields

$$J_{\text{acc}} = \frac{\rho}{2}u_0^2t_0^2\left[4 + \left(t_0 + \frac{X_f}{u_0t_0}\right)^2\right] + t_0 + \frac{X_f}{u_0t_0}. \quad (59)$$

Minimizing over t_0 and introducing the new parameters

$$R = 2\rho u_0^2 \quad (60)$$

$$X = \frac{x_f}{u_0} \quad (61)$$

the optimality condition yields

$$Rt_0^5 + R(2+X)t_0^3 + t_0^2 - X = 0, \quad (62)$$

which can be solved for t_0, and finally the optimum time follows from

$$T^* = t_0 + \frac{X}{t_0}. \quad (63)$$

Graphically, the solution can be obtained by plotting for given X, the parametrized curve $(R^*(t_0), T^*(t_0))$

$$R^* = \frac{X - t_0^2}{t_0^3[2+X+t_0^2]}. \quad (64)$$

Figure 1 gives the optimal time T^* for this problem as function of R for the case $X = 100$. For $R = 0$, the accuracy is not important, and the solution is the well known bang-bang solution [4], giving a minimal time $T^* = 20$. The rapid increase of the optimal time with R (requirement of higher accuracy) is striking, but is to be expected from the sensitivity of the bang-bang solution.

Maximum Accuracy - Minimum Fuel Control

Here the Hamiltonian for the combined problem is

$$\mathcal{H} = |u| + \Lambda_x v + \Lambda_v u + \lambda_{\delta x}\delta v + \lambda_{\delta v}|u|. \quad (65)$$

The solution will be of the same form as in the previous two problems, i.e. a bang-zero-bang solution. This problem is rephrased as the optimization of

$$J_{\text{acc}} = \frac{\rho}{2}u_0^2t_0^2(4+T^2) + 2u_0t_0. \quad (66)$$

The second term in the above expression gives the fuel consumed during the (symmetrical) acceleration and deceleration time. In order to reach the required final state, we need to impose the constraint (55). Substituting, one finds, using parameters R and X:

$$J_{\text{acc}} = \frac{1}{4}R[(t_0^2 + X)^2 + 4t_0^2] + 2u_0 t_0. \qquad (67)$$

This function is increasing with t_0, and has no minimum in the open set $t_0 > 0$. The smaller t_0 the less fuel is used. For an arbitrarily small acceleration time, a nonzero but infinesimal velocity is reached, allowing an extremely long coasting period, after which an infinitesimal amount of fuel needs to be spent in the deceleration period. These near-extreme trajectories have the property that the smaller the boost period is, the higher the accuracy will be. The minimum fuel, free terminal time problem turns out to be *not well posed*, since accuracy and fuel requirements are not antagonistic. In case the terminal time is fixed, the bang-zero-bang policy is the optimal one. The final constraint imposes $t_0^* = \frac{1}{2}\left[T - \sqrt{T^2 - 4X}\right]$ and there is no freedom left to optimize (67). The solution for the control $u^*(t)$ is R-independent.

Maximum Accuracy - Minimum Energy

In this case the Hamiltonian for the combined problem is

$$\mathcal{H} = \frac{1}{2}u^2 + \Lambda_x v + \Lambda_v u + \lambda_{\delta x}\delta v + \lambda_{\delta v}|u|. \qquad (68)$$

For comparison, let again $|u(t)| \leq u_0$. The u-dependent part of the Hamiltonian consists now of two segments of convex parabolas, joined at $u = 0$. Consider first the segment for $u < 0$. The minimal value can either occur at one of the the boundaries $u = -u_0$, or $u = 0$; or else it occurs at the coordinate of the top of this parabola, (if the value of $\frac{1}{2}u^2 + \Lambda_v u + \lambda_{\delta v}|u|$ is negative there). A similar statement holds for the the branch $u > 0$.
The costate equations for Λ_x and Λ_v are

$$\begin{aligned} \dot{\Lambda}_x &= 0 & (69) \\ \dot{\Lambda}_v &= -\Lambda_x & (70) \end{aligned}$$

which gives a solution $\Lambda_v(t) = \Lambda_v(T) + (T-t)\Lambda_x(T)$ which is similar to the perturbation costate (41). Hence both $\Lambda_v + \lambda_{\delta v}$ and $\Lambda_v - \lambda_{\delta v}$ are *linear* in t, and can therefore change sign *at most once*.

In order to meet the final and initial conditions, one must have for $t = 0$, the control $u(0) > 0$, while at the final time T, $u(T) < 0$, the type of solution is a sequence of the form

$$(\text{sat}_{|u_0|}[-2(\Lambda_v + \lambda_{\delta v})], 0, \text{sat}_{|u_0|}[-2(\Lambda_v - \lambda_{\delta v})]). \quad (71)$$

where $\text{sat}_{|u_0|}(\cdot)$ is the *saturation* function:

$$\text{sat}_{|u_0|}(x) = \begin{cases} -u_0 & x < -u_0 \\ x & |x| \leq u_0 \\ u_0 & x > u_0 \end{cases} \qquad (72)$$

We shall solve the problem by matching the parameters on a symmetric control

$$u(t) = \begin{cases} u_0 & t \in [0, t_1] \\ \alpha t + \beta & t \in [t_1, t_2] \\ 0 & t \in [t_2, T - t_2] \\ -\alpha(T-t) - \beta & t \in [T - t_2, T - t_1] \\ -u_0 & t \in [T - t_1, T] \end{cases} \qquad (73)$$

and integrating. One constraint expresses that the terminal position needs to be matched, another that the control is continuous at the point where saturation sets in. The energy spent is

$$U = u_0^2 t_1 + \frac{1}{3}\alpha^2(t_2^3 - t_1^3) + \alpha\beta(t_2^2 - t_1^2) + \beta^2(t_2 - t_1). \quad (74)$$

Finally the perturbation term follows from (45). A constrained parameter optimization problem is then solved to find α, β, t_1, and t_2 for the fixed terminal time problem. If the terminal time is free, optimization with respect to T is required in addition.

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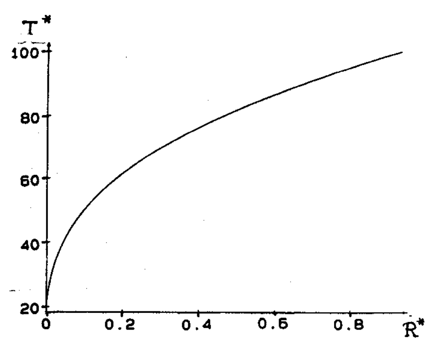

Figure 1. Minimum Time (T^*) versus Accuracy Weight (R^*) (normalized) for the Rocket Car

$$R^* = 2\rho u_0^2 \qquad X = \frac{x_f}{u_0} = 100$$