0. Background Material:

Linearization and Hartman - Groβman Theorem

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In this section an important technique for analyzing nonlinear systems is discussed. First the notion of an equilibrium is defined. Then it is shown how in the neighborhood of a static equilibrium point a system may be linearized. In many situations (e.g., trajectory control for spacecraft) one may want to build systems that automatically steers a perturbed trajectory back to the nominal one. If the perturbations are sufficiently small, analysis and design problems benefit from linearization about the nominal trajectory. Generically such situations lead to *time varying* linear models, even if the original nonlinear model is time invariant.

1 Equilibria for Nonlinear Systems

Consider a nonlinear system given by the state equations

$$\dot{x} = f(x,u) \tag{1}$$
$$y = h(x), \tag{2}$$

where $x \in \mathbb{R}$, $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$. Let the input u be identically zero, then if there exists a vector x such that $f(x,0) = 0$, it follows that $\dot{x} = 0$, so that the system will remain in the state x. One says that the system is in equilibrium and such a state is called an equilibrium state.

Example: The scalar system $\dot{x} = x(x^2 - 1)$ has three equilibrium states: -1, 0 and $+1$.

Note that a linear system, $\dot{x} = Ax + bu$, always has the zero state as equilibrium state. It is possible that there are infinitely many equilibria, but then the A-matrix must be degenerate (singular). Any state in the nullspace $\mathcal{N}(A)$ is then an equilibrium. Note however

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that a linear system cannot have *isolated* equilibria.
The definition of an equilibrium can be generalized. Suppose that instead of $u(t) \equiv 0$ one allows also constant inputs u_0.

Definition: *The system (2) has the equilibrium solution* (u_0, x_0) *if* u_0 *is constant and*

$$f(x_0, u_0) = 0.$$

It is possible that no equilibria exist. For example: $\dot{x} = 1 + u^2$ has no equilibrium solution. There is however a certain "steadyness" present in this system: for $u \equiv 0$, the second derivate of x is identically zero, so that one could speak of a *uniform motion.*
The system $\dot{x} = x^2 + u^2$ has only one equilibrium solution: (0,0), whereas $\dot{x} = Ax + bu$ with nonsingular A has infinitely many equilibrium solutions $(-A^{-1}bu_0, u_0)$, one for each value of u_0.

2 Linearization about an Equilibrium

Consider again the system (2), and assume that it has the equilibrium solution (x_0, u_0). Let the state x be in a neighborhood of x_0, and set

$$x(t) = x_0 + \tilde{x}(t).$$

Likewise, let

$$u(t) = u_0 + \tilde{u}(t).$$

Substituting in (2) one gets, since x_0 is constant,

$$\begin{aligned} \tilde{x} &= f(x_0 + \tilde{x}, u_0 + \tilde{u}) && (3) \\ &= f(x_0, u_0) + \frac{\partial f(x_0, u_0)}{\partial x}\tilde{x} + \frac{\partial f(x_0, u_0)}{\partial u}\tilde{u} + h.o.t. && (4) \end{aligned}$$

where *h.o.t.* stands for 'higher order terms'. If $\tilde{x}$ and $\tilde{u}$ are sufficiently small, then the above is well approximated by the *linear system*

$$\dot{z} = \frac{\partial f(x_0, u_0)}{\partial x} z + \frac{\partial f(x_0, u_0)}{\partial u}\tilde{u}$$

This system is called the *linearized* system about the equilibrium (x_0, u_0). Note that in general, the realization parameters A and b

$$A = \frac{\partial f(x_0, u_0)}{\partial x}, \quad b = \frac{\partial f(x_0, u_0)}{\partial u}$$

depend on the particular equilibrium. The matrix $\frac{\partial f(x_0,u_0)}{\partial x}$ is referred to as the *Jacobian* of f w.r.t. x evaluated at (x_0, u_0).

Example: Consider the pendulum of mass m, length l, and friction k. The tangential forces are: gravity component: $-mg\sin\theta$, inertia: $ml\ddot{\theta}$ and the friction: $-kl\dot{\theta}$. Balancing these tangential forces gives

$$ml\ddot{\theta} = -kl\dot{\theta} - mg\sin\theta.$$

Letting $r = k/m$, one gets

$$\ddot{\theta} + r\dot{\theta} + \frac{g}{l}\sin\theta = 0.$$

A state space model is

$$\dot{\theta} = \omega \quad (5)$$

$$\dot{\omega} = -r\omega + \frac{g}{l}\sin\theta. \quad (6)$$

The equilibria follow from $\sin\theta = 0$, i.e., for $\theta = 0$, we get the stable equilibrium, while for $\theta = \pi$, the equilibrium is unstable. A small perturbation away from the equilibrium will not tend to restore the equilibrium.

Consider now the linearization about the stable point $\theta_0 = 0, \omega_0 = 0$. The Taylor expansion of $\sin\theta$ about 0 yields $\sin\theta \sim \theta$. The linearized model is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \quad (7)$$

If we consider the pendulum without friction, the linearized second order equation in θ is

$$\ddot{\theta} + \frac{g}{l}\theta = 0.$$

The solution is oscillatory with period $T_p = 2\pi\sqrt{\frac{l}{g}}$, which is independent of the amplitude. Careful: the period of the *linearized* system does not depend on the amplitude of the oscillation. One cannot deduce from this that this still holds for the exact but nonlinear system. In fact, detailed analysis shows such an amplitude dependence!.

One can also linearize the equations about the other (unstable) equilibrium. Letting now $\omega_0 = 0, \theta_0 = \pi$ one finds, since now

$$\sin(\theta) = \sin(\pi + \tilde{\theta}) = -\sin\tilde{\theta} \approx -\tilde{\theta}$$

the linearized state model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \quad (8)$$

or equivalently the second order equation

$$\ddot{\tilde{\theta}} + r\dot{\tilde{\theta}} - \frac{g}{l}\tilde{\theta} = 0.$$

Exercise: We have set up state equations for the free pendulum with mass m, length l and friction k, from the dynamical equation

$$\ddot{\theta} + r\dot{\theta} + \frac{g}{l}\sin\theta = 0, \quad r = \frac{k}{m}$$

by linearization about the equilibrium position $\theta_0 = 0$. This is a model for the 'grandfather clock' (pendulum clock). It is well known that if the clock is placed in a constant draft, it goes faster. Show that this effect is theoretically predictable, by computing the factor by which the period of the free *frictionless* ($k = 0$) pendulum

$$T = 2\pi\sqrt{\frac{l}{g}}$$

is decreased, assuming that the pendulum is suspended in a constant horizontal wind, imparting a *constant* force w on the bob. (Show first that the corresponding force balance leads to

$$\ddot{\theta} + r\dot{\theta} + \frac{g}{l}\sin\theta - \frac{w}{m}\cos\theta = 0,$$

and then linearize about the equilibrium, letting $k = 0$).

3 Linearization about a Nominal Trajectory

Here we consider the system (2) for which a nominal trajectory $x_{\rm n}(t)$ corresponding to a nominal input $u_{\rm n}(t)$ is known. Again we define the deviations away from the nominal as

$$\begin{aligned} \tilde{u}(t) &= u(t) - u_{\rm n}(t) && (9)\\ \tilde{x}(t) &= x(t) - x_{\rm n}(t) && (10) \end{aligned}$$

and substitute in the nonlinear state equation:

$$\begin{aligned} \dot{\tilde{x}}(t) + \dot{x}_{\rm n}(t) &= f(\tilde{x}(t) + x_{\rm n}(t), \tilde{u}(t) + u_{\rm n}(t)) && (11)\\ &= f(x_{\rm n}(t), u_{\rm n}(t)) + \frac{\partial f(x_{\rm n}(t), u_{\rm n}(t))}{\partial x}\tilde{x}(t) + \frac{\partial f(x_{\rm n}(t), u_{\rm n}(t))}{\partial u}\tilde{u}(t) + h.o.t. && (12)\\ & && (13) \end{aligned}$$

Since the nominal variables satisfy

$$\dot{x}_{\rm n}(t) = f(x_{\rm n}(t), u_{\rm n}(t)), \tag{14}$$

we obtain the linearized equations

$$\dot{z}(t) = A(t)z(t) + b(t)\tilde{u}(t), \tag{15}$$

such that $\tilde{x}(t) \approx z(t)$ for sufficiently small $\tilde{x}(t)$. Note that now the matrices A and b are time varying, even though the original nonlinear system is not.

Example: The system $\dot{x} = x^2 + u^2$ has no equilibrium solution, except for (0,0). A linearization about this equilibrium yields

$$\begin{aligned}\dot{z} &= \left.\frac{\partial(x^2+u^2)}{\partial x}\right|_{(0,0)} z + \left.\frac{\partial(x^2+u^2)}{\partial u}\right|_{(0,0)} \tilde{u} \\ &= 0 \cdot z + 0 \cdot \tilde{u} = 0.\end{aligned}$$

This is a bad situation, since this gives constant solutions. Let us now consider a nominal trajectory for $u_n(t) = 0$. Since $\dot{x} = x^2$ has the solution $x_n(t) = x(0)/(1 - tx(0))$, we find for the linearized system ($\tilde{u} = u$)

$$\begin{aligned}\dot{z} &= \left.\frac{\partial(x^2+u^2)}{\partial x}\right|_{(x_n(t),0)} z + \left.\frac{\partial(x^2+u^2)}{\partial u}\right|_{(x_n(t),0)} u \\ &= \frac{2x(0)}{1 - tx(0)} z.\end{aligned}$$

There still is no linear term in u. If however we consider the nominal trajectory for $u \equiv 1$, then

$$x_n(t) = \frac{x(0) + \tan t}{1 - x(0)\tan t}$$

and the linearized system is

$$\begin{aligned}\dot{z} &= \left.\frac{\partial(x^2+u^2)}{\partial x}\right|_{(x_n(t),0)} z + \left.\frac{\partial(x^2+u^2)}{\partial u}\right|_{(x_n(t),0)} \tilde{u} \\ &= \frac{2(x(0) + \tan t)}{1 - x(0)\tan t} z + 2\tilde{u}.\end{aligned}$$

A more justifiable procedure would be not to break off the Taylor expansion after the linear term, but after the first nonzero term of smallest order. Obviously, a linear system does not result in that case.

4 Stability Properties of the Equilibrium

In this section the asymptotic stability of an equilibrium solution (x_e, u_e) is investigated. One wants to analyze if for $u(t) = u_e$ the solution $x(t)$ will converge towards x_e, or more generally, what is the behavior of $x(t)$ for sufficiently small deviations $x(0) - x_e$? Let $A = \frac{\partial f}{\partial x}$ evaluated at (x_e, u_e). The linearized system is $\dot{\zeta} = A\zeta$. What does asymptotic stability of A imply about the stability of the equilibrium for the nonlinear system? Note, that if we can say anything at all about this, it surely will have to be a *local* result in view of the fact that the deviation $x(t) - x_e$ needs to be kept small to warrant the linearization.

The following holds:
If $\mathrm{Re}\,\lambda(A) < 0$, then (x_e, u_e) is locally asymptotically stable.
If there is an eigenvalue of A with positive real part, then the equilibrium is locally unstable. However, in the latter case, it may be possible that $x(t)$ in the long run does not diverge towards infinity. This because of the *local* nature of the instability property.
If an eigenvalue of A is purely imaginary, then no conclusion can be drawn from the linearized model about the stability properties of the equilibrium.
We summarize:

Definition: The equilibrium (x_e, u_e) of $\dot{x} = f(x, u)$ is said to be *hyperbolic* if $\mathrm{Re}\,\lambda(A) \neq 0$, where $A = \frac{\partial f}{\partial x}$ evaluated at (x_e, u_e).

Theorem: *If* (x_e, u_e) *is a* hyperbolic *equilibrium, then global stability of the linearized model implies the local stability of the equilibrium for the nonlinear system.*

A stronger result exists: The orbits (trajectories) of the original nonlinear system with fixed input $u(t) = u_e$, in a neighborhood of the equilibrium (x_e, u_e) are qualitatively similar to those of the linear system near the origin. More precisely:

Hartman-Groβman Theorem: *For a hyperbolic equilibrium, the orbits about* (x_e, u_e) *can be continuously deformed in some neighborhood of* (x_e, u_e) *into the corresponding orbits of the linearized system near the origin.*