Homework II

1. Let $a > 0$. Using the Riesz map

$$f(s) = \frac{1}{2\pi j}\int_{-\infty}^{\infty} \frac{f(j\omega)}{s - j\omega}\, d\omega, \qquad s \in \Pi$$

show that the Laguerre functions

$$B_n(s) = \frac{\sqrt{2a}}{s+a}\left(\frac{s-a}{s+a}\right)^{n-1}, \qquad n = 1, 2, \cdots$$

are complete in $\mathcal{H}_2(\Pi)$.

2. Consider the following continuous-time interpolation problem:

Given: noise-free samples of $f(s)$ and its derivatives at L distinct points s_k in the open right-half plane

$$f^{(j)}(s_k) = w_{kj}, \qquad j = 0, \cdots, N_k; \quad l = 1, \cdots, L.$$

Find: a quadruplet $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ that is a minimizal realization of $f(s)$ assuming that f is analytic on the closed right-half plane and has McMillan degree n.

Explain how a continuous-time subspace-based interpolation algorithm can be developed by utilizing the bilinear map

$$s = \lambda\frac{z-1}{z+1} \qquad (\lambda > 0)$$

and the discrete-time subspace-based interpolation algorithm.

3. Suppose (C, A) is observable and A is stable. Let $L(z) = (C(zI_n - A)^{-1} \;\; I_m)$. Consider the following optimization problem:

$$\min_{Q,S,R} \sum_{k=0}^{2N-1} \|S_k - L(e^{j2\pi k/2N}) \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} L^H(e^{j2\pi k/2N})\|_F^2$$

constrained to

$$\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \geq 0.$$

Let

$$P = APA^T + Q, \qquad G = APC^T + S, \qquad \Lambda_0 = CPC^T + R.$$

Then, show that a spectral factor can be computed from A, G, C, and Λ_0.

4. Let $\mathcal{H}$ be a Hilbert space whose elements are real or complex valued functions defined on a set S. We shall call $\mathcal{H}$ a functional Hilbert space if for every $x \in S$, the point evaluation functional $f \to f(x)$ on $\mathcal{H}$ is bounded. This means that there is a constant M_x such that for all $f \in \mathcal{H}$, we have $|f(x)| \leq M_x\|f\|$. By the Riesz representation

theorem, every bounded linear functional on $\mathcal{H}$ arises from an inner product, and so if $x \in S$ there is an element $k_x \in \mathcal{H}$ such that

$$f(x) = (f, K_x), \qquad \text{for every } f.$$

The function K on $S \times S$ defined by

$$K(x, y) = (K_y, K_x) = K_y(x)$$

is called the kernel function or the reproducing kernel of $\mathcal{H}$.

(a) Let S be the set of the natural numbers and $\mathcal{H} = \ell_2$. Let $\{e_n\}$ denote the natural basis for ℓ_2. Show that K is given by

$$K(m, n) = (e_m, e_n) = \delta_{mn}.$$

(b) Let $\mathcal{H}^2$ be the space of complex functions which are analytic inside the unit circle and have square-integrable boundary values. Show that the reproducing kernel for $\mathcal{H}^2$ is given by

$$K(z, w) = \sum_{n=0}^{\infty} z^n w^{-n} = \frac{1}{1 - z\bar{w}}.$$

K is called the Szegö kernel. (Hint: for $|\beta| < 1$, note that $g(z) = \sum_{n=0}^{\infty} \overline{\beta}^n z^n \in \mathcal{H}^2$ and $f(\beta) = (f, g)$ for every $f \in \mathcal{H}^2$.)