Homework I

1. Suppose that $Z = [T \;\; U] \in \mathbf{R}^{n\times n}$ is non-singular where $T \in \mathbf{R}^{n\times r}$, $U \in \mathbf{R}^{n\times(n-r)}$. Let the inverse matrix of Z be given by

$$Z^{-1} = \begin{bmatrix} L \\ V \end{bmatrix}, \qquad L \in \mathbf{R}^{r\times n},\ V \in \mathbf{R}^{(n-r)\times n}.$$

Then, it follows that $TL + UV = I_n$ and

$$\begin{bmatrix} L \\ V \end{bmatrix} [T \;\; U] = \begin{bmatrix} LT & LU \\ VT & VU \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & I_{n-r} \end{bmatrix}.$$

Show that $P = TL$ is the oblique projection onto $Im(T)$ along $Ker(L)$ and that $Q = UV$ is the oblique projection onto $Ker(L)[= Im(U)]$ along $Im(T)[= Ker(V)]$.

2. In the above problem, define

$$T = \begin{bmatrix} I_r \\ 0 \end{bmatrix}, \qquad U = \begin{bmatrix} -X \\ I_{n-r} \end{bmatrix}.$$

Compute the projection $P = TL$ by means of L and V. Show that $P^2 = P$ is satisfied if P has the following representation

$$P = \begin{bmatrix} I_r & X \\ 0 & 0 \end{bmatrix}, \qquad X \in \mathbf{R}^{r\times n}.$$

3. Suppose that $A \in \mathbf{R}^{p\times N}$, $B \in \mathbf{R}^{q\times N}$ where $p, q < N$. Let the orthogonal projection of the row vectors of A onto the space spanned by the row vectors of B be defined by $\hat{E}[A|B]$. Prove the following

$$\hat{E}[A|B] = AB^T(BB^T)^{\dagger}B.$$

If B has full row rank, the pseudo-inverse is replaced by the inverse.

4. Let A and B be defined as in Problem 3. Consider the LQ decomposition:

$$\begin{bmatrix} B \\ A \end{bmatrix} = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix}.$$

Suppose that B has full row rank. Show that the orthogonal projection is given by

$$E[A|B] = L_{21}Q_1^T = L_{21}L_{11}^{-1}B = A(Q_1Q_1^T).$$

Let $B^{\perp}$ be the orthogonal complement of the space spanned by the row vectors of B. Then, the orthogonal projection of the row vectors of A onto $B^{\perp}$ is expressed as

$$E[A|B^{\perp}] = L_{22}Q_2^T = A(Q_2Q_2^T).$$