Examination

1. Let $A \in \mathbf{R}^{m\times n}$ with $m \geq n$. By using the SVD, show that there exists an orthogonal matrix $Q \in \mathbf{R}^{m\times n}$ and a non-negative matrix $\Pi \in \mathbf{R}^{n\times n}$ such that $A = Q\Pi$.

2. Suppose that $A \in \mathbf{R}^{p\times N}, B \in \mathbf{R}^{q\times N}, C \in \mathbf{R}^{r\times N}$ where $p, q, r < N$. Suppose that B and C have full rank and $span\{B\} \cap span\{C\} = \{0\}$. Then, $E_{||C}[A|B]$, the oblique projection of the row vectors of A onto B along C is expressed as

$$E_{||C}[A|B] = A[B^T \;\; C^T] \begin{bmatrix} BB^T & BC^T \\ CB^T & CC^T \end{bmatrix}^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix}.$$

3. By applying the Müntz-Szász theory and using the completeness of the Laguerre functions in $\mathcal{H}_2(\Pi)$, show that the following functions

$$B_a(s) = \frac{1}{s+a}, \qquad 1 \leq a \leq 2$$

span $H_2(\Pi)$.

4. Consider the SISO system:

$$\begin{aligned} x(t+1) &= Ax(t) + b\nu(t), \\ y(t) &= 2c^T x(t) + d\nu(t) \end{aligned} \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the state, $\nu(t) \in \mathbf{R}$ and $y(t) \in \mathbf{R}$ are respectively the input and the output of the system. Let $f(z)$ denote the transfer function of (1), which is assumed to be stable and minimal. Let $u(\theta)$ denote the real part of $f(z)$ at $z = e^{j\theta}$, $\theta \in [0, \pi]$. Assume that N noise-corrupted samples of $u(\theta)$ are given as

$$\hat{u}_k = u(\theta_k) + \eta_k, \qquad k = 1, \cdots, N. \tag{2}$$

   (a) Show that the subspace-based spectral estimation algorithm can be used to consistently estimate $f(z)$ from the samples (2).

   (b) Assume $\eta_k = 0$ for all k in (2). What is the smallest value of N such that u can be retrived exactly if $\theta_k \neq 0, \pi$ for all k. Explain your answer.