Examination: Solutions

1. Let $A = U\Sigma V^T$, where $\Sigma = diag(\sigma_1, \cdots, \sigma_n)$, and $U \in \mathbf{R}^{m\times n}$, $V \in \mathbf{R}^{n\times n}$. Then, we have $Q = UV^T$ and $\Pi = V\Sigma V^T$. Note that $VV^T = V^TV = I_n$.

2. Let $D = \begin{bmatrix} B \\ C \end{bmatrix}$. Then, D has full row rank. Thus from HW I.3,

$$\hat{E}\{A|D\} = A[B^T \ \ C^T] \begin{bmatrix} BB^T & BC^T \\ CB^T & CC^T \end{bmatrix}^{-1} \begin{bmatrix} B \\ C \end{bmatrix}.$$

We see that the above equations are expressed as

$$\hat{E}\{A|D\} = A[B^T \ \ C^T] \begin{bmatrix} BB^T & BC^T \\ CB^T & CC^T \end{bmatrix}^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} + A[B^T \ \ C^T] \begin{bmatrix} BB^T & BC^T \\ CB^T & CC^T \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ C \end{bmatrix}.$$

Since $span\{B\} \cap span\{C\} = \{0\}$, the first term of the right-hand side of the above equation is the oblique projection of the row vectors of A onto the space spanned by the row vectors of B along the row vectors of C. Thus, we have

$$\hat{E}_{||C}\{A|B\} = A[B^T \ \ C^T] \begin{bmatrix} BB^T & BC^T \\ CB^T & CC^T \end{bmatrix}^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix}.$$

3. Assume that there exists a non-trivial $f \in \mathcal{H}(\Pi)$ orthogonal to all $B_a(s)$:

$$\int_{-\infty}^{\infty} \frac{\overline{f(j\omega)}}{j\omega + a}\, d\omega = 0, \qquad 1 \le a \le 2.$$

Then, evaluating the Riesz map

$$g(s) = \frac{1}{2\pi j} \int_{-\infty}^{\infty} \frac{f(j\omega)}{s - j\omega}\, d\omega$$

on the interval $[1, 2]$ and taking the complex conjugates, we get $\overline{g(s)} = 0$ or $g(s) = 0$ for all $s \in [1, 2]$. Since g is analytic on Π, this implies g vanishes on Π. Thus, evaluating the derivatives of $g(s)$ at $s = a$ we obtain

$$0 = \frac{jg^{(k)}(a)}{(-1)^k k!} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{f(j\omega)}{(a - j\omega)^{k+1}}\, d\omega, \qquad k = 0, 1, \cdots$$

which imply that the Laguerre functions are not complete in $\mathcal{H}_2(\Pi)$. We reach a contradiction.

4. (a) The transfer function of the system is computed as $f(z) = d + 2c^T(zI - A)^{-1}b$. Using the fact that $f(z)$ is a scalar function, $u(\theta)$ can be written as

$$u(\theta) = \frac{1}{2}[f(z) + \overline{f(z)}] = h(z) + h(z^{-1})$$

with $h(z) = \frac{d}{2} + c^T(zI - A)^{-1}b$ and $z = e^{j\theta}$ as in the splitting of a power spectrum in terms of its spectral summands. Thus, our problem can be viewed as a first step in the spectral estimation problem of determining f from the samples of $|f|^2$; but, easier since there is no need to enforce the positivity requirement on S.

(b) Observe that the interpolation properties of the spectral estimation algorithm is determined by checking the minimality of the spectral summands, which is inherited from the minimality of the spectral factors. Thus, from the proof of Lemma 2 in Akçay and T¨urkay (2004, Automatica), the interpolation condition is $2N > p + 2n - 1$. Pick $p = 2n + 1$. Then, it suffices to let $N = 2n + 1$. Since $p > 2n$, it is the smallest possible number.