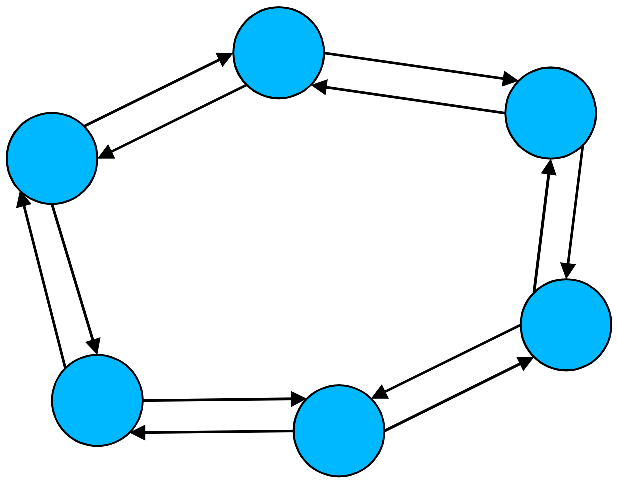
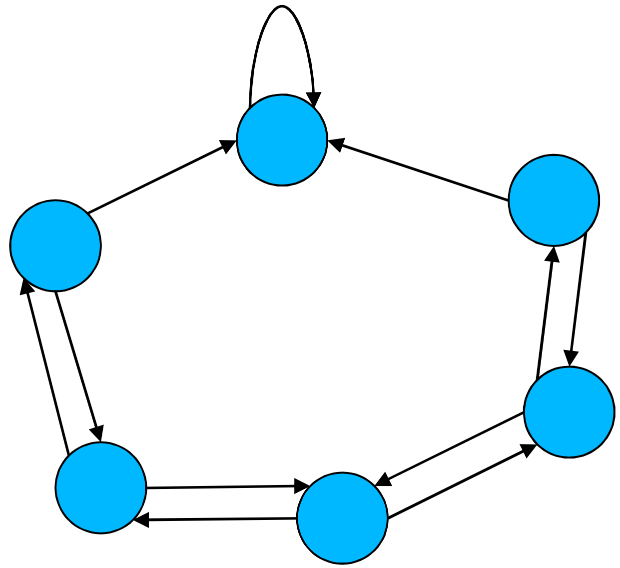
Introduction: An average consensus problem

- Agents move in discrete time
- They average their neighbours' positions
- They converge to average consensus



A rogue agent takes power

- One agent changes its connections
- End position: far from average
- Conditions for perturbation to preserve (close to) average?
- Enough to stay connected?



Opinion dynamics and dictators

- Political opinion: number in [-1,1]
- People talk with friends and change their opinion
- Same dynamics as consensus
- How can someone impose his opinion to everybody else ?
- How to preserve democracy?

Consensus=opinion dynamics=Markov chains

Consensus: $x_{t+1} = Ax_t$, A stochastic

Opinion dynamics: $x_{t+1} = Ax_t$, A stochastic

Typically: A sparse = small outdegree

Markov chains: $\pi_{t+1} = \pi_t A$

Entries of π = influence of agents in final decision

Democracy in Markov chains and its preservation under local perturbations

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This is not classical perturbation theory

- Matrix A is
  - stochastic
  - sparse (few entries per row)
  - large
- Left dominant eigenvector π
- Classical perturbation theory:

  If we change <u>all entries</u> by a <u>small ε</u>, what happens to π ?

This is ‘combinatorial perturbation’ theory

- Matrix A is
  - stochastic
  - sparse (few entries per row)
  - large
- Left dominant eigenvector π
- Combinatorial perturbation theory:

  If we change a few entries by any amount, what happens to π ?

Or rather: ‘asymptotic’ combinatorial perturbation theory

- We wish we had a bound on $\|\pi - \pi'\|$ depending on number of entries changed, dimension of A, etc.
- Hard!
- Simpler:
  - Large -> Larger and larger
  - Few -> bounded
  - What happens to $\|\pi\|, \|\pi'\|, \|\pi - \pi'\|$?

Sequences of chains and democracy

- Set of nodes $V_1 \subset V_2 \subset V_3 \ldots \subset V_\infty$
- Mixing Markov chain G_n on V_n
- G_n stabilises when n ! 1

  For all nodes x,y: $G_n(x,y)$ eventually constant
- π_n = stationary distribution of G_n
- $\|\pi_n\|_\infty$= largest entry of π_n

Democracy in Markov chains

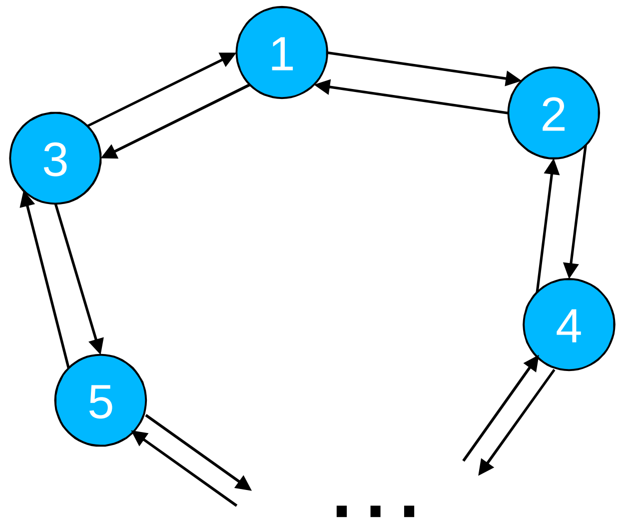
- Democratic iff $\|\pi_n\|_\infty \to 0$

- Democracy for opinion/consensus
  = no dictator
  = no agent has a dominant influence on the final opinion/consensus

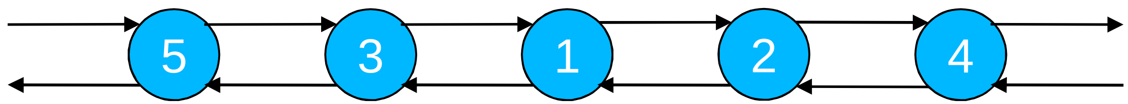
- Similar to ‘wisdom of crowds’ (M.O. Jackson)

The ring is democratic

G_n:



G_1:



Reversible random walks

- G_n: random walk on a connected undirected graph
- π_n: normalised degrees
- If bounded degree, then democratic

- The infinite graph can be weighted
- If weights bounded from above and below, and bounded degree, then democratic

- All those examples are reversible
- In fact, general for reversible chains

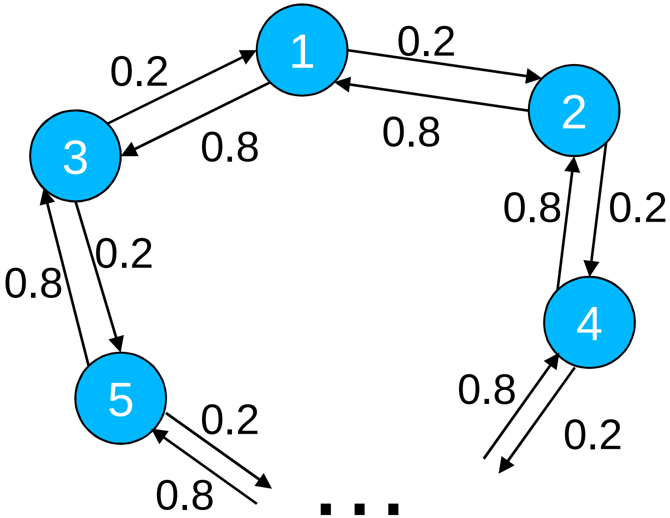
Finite perturbation

- Finite set of nodes S
- In all G_n, rewire all edges from S arbitrarily (independently of n)
- We assume the rewired G'_n is still mixing

- Is democracy preserved? Not always.

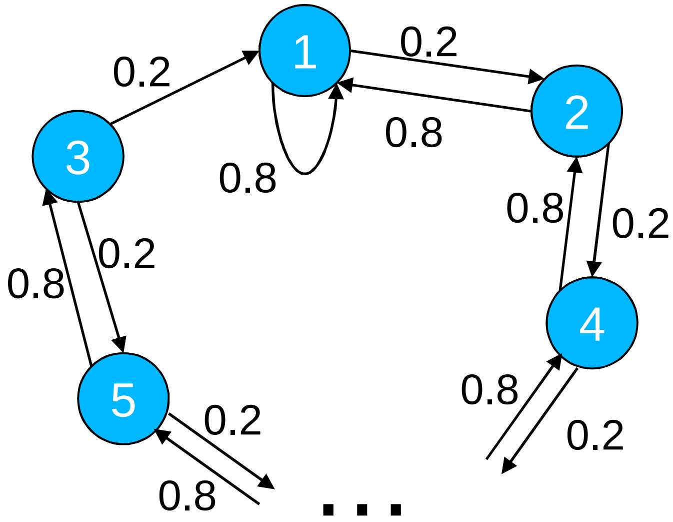
Example: biased ring

G_n:



The perturbed biased ring

G'_n:

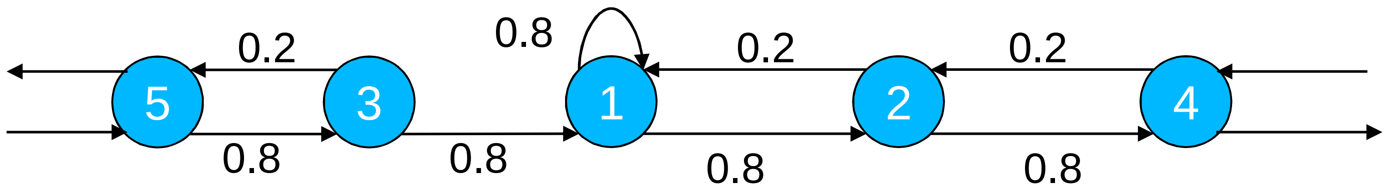


$\pi_n(1)$ converges to a non zero value

) Not democratic

The perturbed biased ring

Limit chain: not connected!



Is it enough to ask for a irreducible limit?

Perturbation that preserve democracy: main result

- Theorem:

  A sequence of random walks on undirected, bounded-degree graphs with weights from a finite set
  - is democratic
  - remains so under finite perturbations which preserve irreducibility of the limit chain.

- The perturbed chains are not necessarily reversible anymore

Tools

- $\pi^{-1}(x)$= first return time of node x

  democracy= smallest first return time grows to infinity

- We can permute the nodes and converge to a different limit chain.

  Look at all limit chains: do they have a stationary distribution?

Conclusions

- 'Combinatorial perturbation' of Markov chains/consensus
- Democracy = every agent has negligible influence
- Robust to local irreducibility-preserving perturbations for reversible, bounded-degree random walks
- Open problems:
  - Lift assumptions
  - Slowly growing set of rogue agents
  - Finitary versions
  - Different norms
- Thanks: S. Zampieri, J. Hendrickx, F. Garin, G. Como