Growth and Agglomeration in the Heterogeneous Space: A Generalized Ak Approach

R. Boucekkine, G. Fabbri, S. Federico and F. Gozzi

Discussion Paper 2017-6

Institut de Recherches Économiques et Sociales
de l'Université catholique de Louvain

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GROWTH AND AGGLOMERATION IN THE HETEROGENEOUS SPACE: A GENERALIZED AK APPROACH

RAOUF BOUCEKKINE*, GIORGIO FABBRI♯, SALVATORE FEDERICO♭, AND FAUSTO GOZZI♮

Abstract. We provide with an optimal growth spatio-temporal setting with capital accumulation and diffusion across space in order to study the link between economic growth triggered by capital spatio-temporal dynamics and agglomeration across space. We choose the simplest production function generating growth endogenously, the AK technology but in sharp contrast to the related literature which considers homogeneous space, we derive optimal location outcomes for any given space distributions for technology (through the productivity parameter A) and population. Beside the mathematical *tour de force*, we ultimately show that agglomeration may show up in our optimal growth with linear technology, its exact shape depending on the interaction of two main effects, a population dilution effect versus a technology space discrepancy effect.

Key words: Growth, agglomeration, heterogeneous and continuous space, capital mobility, infinite dimensional optimal control problems

Journal of Economic Literature Classification: R1; O4; C61.

Date: March 28, 2017.

*Corresponding author. Aix-Marseille University (Aix-Marseille School of Economics and IMéRA), CNRS and EHESS.
♯Aix-Marseille University (Aix-Marseille School of Economics), CNRS and EHESS.
♭Università degli Studi di Siena, Dipartimento di Economia Politica e Statistica.
♮Dipartimento di Economia e Finanza, LUISS *Guido Carli*, Roma.

1. Introduction

Economic growth models with a spatial dimension have been already formulated in the context of the New Economic Geography stream but, as observed by Desmet et Rossi-Hansberg (2010) in an illuminating survey (see also Nijkamp and Poot, 1998) , they use to disregard intertemporal optimization individual behaviors and even capital accumulation. A paradigmatic example of such a growth modelling strategy can be seen in Fujita and Thisse (2002), chapter 11. In this chapter, endogenous growth in driven by the manufacturing sector through horizontal differentiation *à la* Grossman-Helpman while skilled labor is the unique mobile factor.[1] Consumers do not save nor do they decide about schooling (no human capital accumulation). Indeed, with some notable exceptions (see for example the infrastructure location model developed by Martin and Rogers, 1995), the New Economic Geography has roughly left in the dark not only capital accumulation (over time) but also capital mobility through space.

This paper is concerned with the relationship between agglomeration and economic growth. As outlined by Fujita and Thisse (2002), "...in a world of globalization, agglomeration may well be the territorial counterpart of economic growth much in the same way as growth seems to foster inequality among individuals." (page 19). We shall provide a spatio-temporal setting with capital accumulation and diffusion across space showing the link between economic growth triggered by capital spatio-temporal dynamics and agglomeration across space.[2] In line with Boucekkine et al. (2013), we choose the simplest production

[1]A more elaborate modelling of labor mobility and migrations can be found in Mossay (2003).

[2]The spatio-temporal setting is analogous to Brito's (2004) framework. In the latter, production uses a neoclassical production function at any location, output is used for *in situ* consumption and investment while the net trade flow depends on the differentials of the spatially distributed capital stock, consistently with recent empirical results by Comin et al. (2012). Only a limited characterization of optimal solutions is possible in this case, see also Boucekkine et al. (2009).

function generating growth endogenously, the AK technology. This is essential to get the analytical results gathered in this framework. This said, as explained below, our setting is a sharp generalization of Boucekkine et al. (2013): while in the latter space is homogeneous (same production function and one individual per location), we derive here optimal location outcomes for **any given space distributions for technology and population**. Technology space heterogeneity amounts to discrepancy on parameter A of the AK technology across locations, that's roughly speaking spatial differences in productivity, which can be itself due to a wide variety of pure technological or institutional factors.

In such a framework, we shall prove that capital accumulation and diffusion, and subsequent growth in the spatially heterogeneous economy, do come with agglomeration along the optimal spatio-temporal paths. Notice that here agglomeration occurs for different reasons than those usually invoked in the New Economic Geography. First, and trivially, capital accumulation and mobility is the dynamic engine of agglomeration in our story, and it is little doubtful that in real economies capital is more mobile than labor (see Aslund and Dabrowski, 2008, for a series of studies on this issue, especially in the European case). Second, we do not have increasing returns in our setting (the production function is linear) nor do we impose monopolistic competition (optimal growth setting). Third, using Krugman's terminology (1993), we do look for **first nature causes** for agglomeration as the technology and demographic distributions are exogenously given, and not for the **second nature causes** typically invoked in the New Economic Geography (like economies of scale or knowledge spillovers).

On the technical side, generalizing Boucekkine et al. (2013) approach to heterogeneous space is a daunting task. We have been able however to find a way to undertake it. More precisely, we are able to

explicitly identify the maximal welfare (value function) and the optimal consumption profile in terms of technology and population spatial distributions and the initial spatial distribution of capital (Theorem 3.2). We also single out the partial differential equation which delivers the optimal the spatio-temporal capital dynamics and study the asymptotic convergence properties associated. Ultimately we are able to describe the long-run profile of the capital distribution in an explicit way by a suitable series of spatial functions (Theorem 3.3).[3] As a particular case, considering uniform distributions for both technology and population leads exactly to Boucekkine et al.'s uniform convergence results. We can therefore study the robustness of the asymptotic convergence to uniform spatial distributions to population and technology space dependence.

The numerical analysis and the discussion provided in the last section of the paper allow to identify on an adequately calibrated version of the model the two main effects at work when the space distributions of technology and population are heterogeneous. On the one hand we have a **technological space discrepancy effect**: The planner has the incentive to favor the concentration of the capital in the areas where it is more productive so that she will tend to promote (relatively more) investment in areas where technology is better and to privilege consumption in technologically lagged regions. On the other hand we have a **population effect**: the Benthamite form of the functional considered (the utility of each individual is weighted exactly in the same way, regardless of the position and of the population size in the location) induces the planner to guarantee an adequate level of per capita

[3]The results are obtained thanks to the dynamic programming in infinite dimensions and to the main methodological novelty of the present work with respect to the existing literature in spatial growth models: the use of the spectrum and the eigenfunctions of an appropriate Sturm-Liouville operator $\mathcal{L}$, the one associated to the (linear) zero consumption problem. A precise description of the techniques we use, together with a complete proof of all the analytical results, is given in Appendix A.

consumption across space so that areas with higher population get also a higher aggregate consumption and therefore a lower investment. The simulations show how the two effects work separately and then how they interact.

The paper proceeds as follows. Section 2 is devoted to description of the model. Section 3 presents the main analytical results. Section 4 concerns numerical simulations and associated remarks. Section 5 concludes. Appendix A provides the proofs of the analytical results.

2. The model

We study a spatial economy developing on the unit circle S^1 in the plane[4]:

$$S^1 := \{(\sin\theta, \cos\theta) \in \mathbb{R}^2 \,:\, \theta \in [0, 2\pi)\}.$$

We suppose that, for all time $t \geq 0$ and any point in the space $\theta \in [0, 2\pi)$, the production is a linear function of the employed capital:

$$Y(t, \theta) = A(\theta)K(t, \theta),$$

where $K(t, \theta)$ and $Y(t, \theta)$ represent, respectively, the aggregate capital and output at the location θ at time t while $A(\theta)$ is the exogenous location-dependent technological level. In the model there is no state intervention and then, at any time, the local production is split into investment in local capital and local consumption so that, once we include a location-dependent depreciation rate $\delta(\theta)$ and the net trade

[4]The functions over S^1 can be clearly identified with 2π-periodic functions over $\mathbb{R}$. We shall confuse these functions, as well as the point $\theta \in [0, 2\pi)$ with the corresponding point $(\sin\theta, \cos\theta) \in S^1$. Hence, given a function $f : S^1 \to \mathbb{R}$, the derivatives with respect to $\theta \in S^1$ will be intended through the identification of functions defined on S^1 with 2π-periodic functions defined on $\mathbb{R}$.

balance $\tau(t,\theta)$, we get the following accumulation law of capital:

$$\begin{aligned}\frac{\partial K}{\partial t}(t,\theta) &= I(t,\theta) - \delta(\theta)K(t,\theta) - \tau(t,\theta)\\ &= Y(t,\theta) - C(t,\theta) - \delta(\theta)K(t,\theta) - \tau(t,\theta)\\ &= (A(\theta) - \delta(\theta))K(t,\theta) - C(t,\theta) - \tau(t,\theta).\end{aligned}$$

We can always include the depreciation rate $\delta(\theta)$ in the coefficient $A(\theta)$ so the previous equation simply becomes

$$\frac{\partial K}{\partial t}(t,\theta) = A(\theta)K(t,\theta) - C(t,\theta) - \tau(t,\theta).$$

Following the idea of Brito (2004) (and then used by all the papers in the related stream of literature, see for instance Brock and Xepapadewas, 2008 and Fabbri, 2016, and the references therein), given $0 \leq \theta_1 < \theta_2 < 2\pi$, the net trade balance over the region (θ_1, θ_2) is given by the balance of the flow of capital, at time t, at the boundaries θ_1 and θ_2:

$$\int_{\theta_1}^{\theta_2} \tau(t,\theta)d\theta = \frac{\partial K}{\partial \theta}(t,\theta_1) - \frac{\partial K}{\partial \theta}(t,\theta_2).$$

The last expression holds for any choice of θ_1 and θ_2 and it also equals the quantity $\int_{\theta_1}^{\theta_2} -\frac{\partial^2 K}{\partial \theta^2}(t,\theta)d\theta$ so, letting θ_2 to θ_1, we get, for any $\theta \in [0, 2\pi)$, $\tau(t,\theta) = -\frac{\partial^2 K}{\partial \theta^2}(t,\theta)$. The capital evolution law reads then as

$$\frac{\partial K}{\partial t}(t,\theta) = \frac{\partial^2}{\partial \theta^2}K(t,\theta) + A(\theta)K(t,\theta) - C(t,\theta).$$

If, for any (t,θ), we finally express the total consumption $C(t,\theta)$ as the product of the per-capita consumption[5] $c(t,\theta)$ and the time-independent exogenous (density of) population $N(\theta)$, we obtain

(1)
$$\begin{cases}\frac{\partial K}{\partial t}(t,\theta) = \frac{\partial^2}{\partial \theta^2}K(t,\theta) + A(\theta)K(t,\theta) - c(t,\theta)N(\theta), \quad t > 0,\ \theta \in S^1,\\ K(0,\theta) = K_0(\theta),\ \theta \in S^1,\end{cases}$$

[5]We suppose resources and consumption are equally distributed among the population of a certain location.

where K_0 denotes the initial distribution of capital over the space S^1. We suppose that the policy maker operates to maximize the following intertemporal constant relative risk aversion functional:

$$\int_0^\infty e^{-\rho t} \int_0^{2\pi} \frac{c(t,\theta)^{1-\sigma}}{1-\sigma} N(\theta) d\theta dt, \tag{2}$$

where $\rho > 0$ and $\sigma \in (0,1) \cup (1,\infty)$ are given constant and the constraints

$$c(t,\theta) \geq 0, \qquad \text{and} \qquad K(t,\theta) \geq 0 \tag{3}$$

are imposed. This is indeed a Benthamite functional in the following sense: at any time t, the planner linearly weights the per-capita utility at any location using the population density. In other terms, the consumption/utility of all the people in the economy matters in the same way in the target. This fact will have a certain importance in the following.

The described model is a strict generalization of that considered by Boucekkine et al. (2013) because we consider here a technological level $A(\theta)$ and a population density $N(\theta)$ depending on the location θ. In other words here A and N are functions $A, N \colon S^1 \to \mathbb{R}$ instead of just two space-independent constants.

3. Main analytical results

The model presented in the previous section is, mathematically speaking, an optimal control problem with state equation (1), objective functional (2) and pointwise constraints (3). In this section we present the two main analytical results of this paper. The first characterizes the optimal strategies of the optimal control problem (1)-(2)-(3) while the second studies the long run behavior of the optimal capital path. As our results will be expressed in terms of the eigenvalues and the

eigenfunctions of a suitable Sturm-Liouville problem, we begin our exposition by recalling the definitions of these concepts and some related results. In what follows we will avoid all mathematical difficulties which are unnecessary at this stage, hence many concepts will be expressed in an informal way: the reader interested in the complete mathematical setting can find precise definitions, statements and proofs in the technical Appendix A.

We consider the differential operator associated to the zero-consumption diffusion dynamics of (1), namely

$$\mathcal{L}u(\theta) := \frac{\partial^2}{\partial\theta^2}u(\theta) + A(\theta)u(\theta). \tag{4}$$

The operator $\mathcal{L}$ is well defined on regular enough functions $\phi\colon S^1 \to \mathbb{R}$. A non identically zero regular function $\phi\colon S^1 \to \mathbb{R}$ is called *eigenfunction* of $\mathcal{L}$ if there exists a real number (*eigenvalue*) λ such that $\mathcal{L}\phi = \lambda\phi$. It can be proved (see Theorems 2.4.2 and 2.5.1 by Brown et al., 2013) that there is a countable discrete set of eigenvalues $\{\lambda_n\}_{n\geq 0}$ which can be ordered in decreasing way. The highest eigenvalue, λ_0, is associated to a unique eigenfunction (i.e. its multiplicity is 1) and this is the only eigenfunction without zeros. Eigenfunctions are defined up to a multiplicative factor; we denote by $\mathbf{e}_0$ the unique eigenfunction corresponding to the eigenvalue λ_0 such that $\mathbf{e}_0(\theta) > 0$ for each $\theta \in S^1$ and $\int_0^{2\pi} \mathbf{e}_0^2(\theta)d\theta = 1$. It can be proved (see again Theorems 2.4.2 and 2.5.1 of Brown et al., 2013) that the multiplicity of any other eigenvalue is either 1 or 2, that $\lambda_n \to -\infty$, as $n \to \infty$, and that there exists an orthonormal basis of $L^2(S^1)$ (see (11) for its definition) of eigenfunctions $\{\mathbf{e}_n\}_{n\geq 0}$ corresponding to the sequence of eigenvalues[6] $\{\lambda_n\}_{n\geq 0}$.

We have now collected the elements we need to describe the solution of the model and we can proceed by presenting it. We will work under

[6]In the sequence $\{\lambda_n\}_{n\geq 0}$ a certain value appears once, respectively twice, if its multiplicity is 1, respectively 2.

the following spatial counterpart of the usual assumption needed in the standard one-dimensional AK model to ensure the finiteness of the intertemporal utility[7].

Hypothesis 3.1. *The discount rate satisfies*

$$\rho > \lambda_0(1-\sigma). \tag{5}$$

We can now state the first important result on optimal spatio-temporal capital dynamics together with the optimal consumption strategy across time and space.

Theorem 3.2. *Let Hypothesis 3.1 hold. Assume that $A, N\colon S^1 \to \mathbb{R}^+$ are bounded and not identically zero, denote by α_0 the value*[8]

$$\alpha_0 := \left(\frac{\sigma}{\rho-\lambda_0(1-\sigma)}\int_0^{2\pi} \mathbf{e}_0(\theta)^{-\frac{1-\sigma}{\sigma}}\mathbf{N}(\theta)d\theta\right)^{\frac{\sigma}{1-\sigma}}, \tag{6}$$

and by $\boldsymbol{\beta}$ the function $\alpha_0\mathbf{e}_0$. Then the optimal evolution $K^*(t,\theta)$ of the capital density is given by the unique solution of the following PDE:

$$\begin{cases} \frac{\partial K}{\partial t}(t,\theta) = \frac{\partial^2}{\partial\theta^2}K(t,\theta) + A(\theta)K(t,\theta) - \left(\int_0^{2\pi}\boldsymbol{\beta}(\eta)K(\eta)d\eta\right)(\boldsymbol{\beta}(\theta))^{-1/\sigma}N(\theta) \\ K(0,\theta) = K_0(\theta),\ \theta\in S^1. \end{cases} \tag{7}$$

Moreover the optimal consumption strategy $c^*(t,\theta)$ is given, as a feedback function of the current optimal state trajectory, as:

$$c^*(t,\theta) = \left(\int_0^{2\pi}\boldsymbol{\beta}(\eta)K^*(t,\eta)d\eta\right)(\boldsymbol{\beta}(\theta))^{-1/\sigma}, \qquad t\geq 0,\ \theta\in S^1. \tag{8}$$

Finally $c^*(t,\theta)$ can also be expressed explicitly in terms of the initial capital density $K_0(\theta)$ as

$$c^*(t,\theta) = \left(\int_0^{2\pi}\boldsymbol{\beta}(\eta)K_0(\eta)d\eta\right)e^{gt}(\boldsymbol{\beta}(\theta))^{-1/\sigma},$$

[7]The assumption that we will make on A will imply that λ_0 is positive (see Remark A.6). Hence, the condition (5) is obviously verified when $\sigma > 1$ (that is the case for reasonable calibrations of the model, see Section 4).

[8]This number is well defined and strictly positive thanks to (5).

where g is the growth rate of the economy, given by

$$g := \frac{\lambda_0 - \rho}{\sigma}. \tag{9}$$

Proof. See Appendix A and in particular Corollary A.5. □

Once we compare the optimal consumption profile described in the previous theorem with the counterpart under space homogeneity (Boucekkine et al., 2013), we can immediately figure out the crucial role of a location-dependent technology (via coefficient A). Indeed under homogeneous space, the (per-capita and aggregate) optimal consumption level is always equal across locations while here the expression of the optimal consumption is given by the space-independent term $\left(\int_0^{2\pi} \boldsymbol{\beta}(\eta) K_0(\eta) d\eta\right) e^{gt}$ and by the space-dependent term $(\boldsymbol{\beta}(\theta))^{-1/\sigma} = (\alpha_0 \mathbf{e}_0)^{-1/\sigma}$. The latter depends on $A(\cdot)$ both *via* α_0 and $\mathbf{e}_0$ and on $N(\cdot)$ *via* α_0. This fact is interesting from a theoretical point of view since *a priori* one might guess that the egalitarian nature of the Benthamite functional could be enough to guarantee equalization of individual utility across space. On the contrary the structural conditions of the economy can lead the planner to diversify per-capita consumption across locations (first nature causes). As we will see in Section 4 the differentiation does not always go in the expected way.

Notice for now that by the expression of the optimal consumption, we get the following expression for optimal social welfare:

$$V(K_0) = \frac{\alpha_0^{1-\sigma} \left(\int_0^{2\pi} K_0(\theta) \mathbf{e}_0(\theta) d\theta\right)^{1-\sigma}}{1-\sigma}.$$

Differently from the homogeneous space case where maximal welfare only depends on aggregate capital, here the stock of capital in different locations enter the optimal welfare expression with different weights. Roughly speaking (see Section 4 for numerical examples) the spatial function $\mathbf{e}_0$ tends to be larger in the regions where A is bigger. So,

for a given amount of initial aggregate capital, welfare will be higher if capital is more accumulated in the more productive locations. Finally observe that this property holds true irrespectively of the population distribution as one can realize by rewriting the expression of $V(K_0)$ above and disentangling the contributions of population and capital initial densities:

$$\left(\frac{\sigma}{\rho-\lambda_0(1-\sigma)}\int_0^{2\pi}\mathbf{e}_0(\theta)^{-\frac{1-\sigma}{\sigma}}\mathbf{N}(\theta)d\theta\right)^{\sigma}\frac{\left(\int_0^{2\pi}K_0(\theta)\mathbf{e}_0(\theta)d\theta\right)^{1-\sigma}}{1-\sigma}.$$

The importance of heterogeneous technology and population distributions is also essential in our second result describing the long-run profile of the detrended optimal capital: while in case of space-constant A and N the space-distribution of the wealth always converges (under the hypotheses of Theorem 3.3) to a uniform profile, here an articulated expression, depending on the whole technological and human population distributions, arises.

Theorem 3.3. *Let hypotheses of Theorem 3.2 hold and suppose that*

$$g > \lambda_1 \tag{10}$$

where g is defined in (9) and λ_1 is the second eigenvalue of the problem considered above. Define the detrended optimal path $K_g(t,\theta) := e^{-gt}K(t,\theta)$, for $t \geq 0$. Then

$$K_g(t,\theta) \overset{t\to\infty}{\longrightarrow} \int_0^{2\pi}K_0(\eta)\boldsymbol{\beta}(\eta)d\eta\left(\frac{\mathbf{e}_0(\theta)}{\alpha_0}+\sum_{n\geq 1}\frac{\beta_n}{\lambda_n-g}\mathbf{e}_n(\theta)\right)$$

where, for $n \geq 1$,

$$\beta_n := \int_0^{2\pi}(\boldsymbol{\beta}(\eta))^{-1/\sigma}N(\eta)\mathbf{e}_n(\eta)d\eta.$$

Proof. See Appendix A and in particular Proposition A.7. □

4. Numerical exercises

The explicit representation of the long-run configuration of the economy given in Theorem 3.3 can be used to undertake a numerical analysis of the system in some specific cases of interest.[9]

First we calibrate the model. In all the simulations we choose the discounting parameter ρ equal to 3% (consistent e.g. with the data of Lopez, 2008) and the inverse of the elasticity of intertemporal substitution σ equal to 5 (here it is also the constant relative risk aversion of the utility function so its value is coherent with those found e.g. by Barsky et al., 1997). In all our simulations we use the non-uniform technological distribution $A(\cdot)$ on $[0, 2\pi]$ having a pick at the point π and attaining lower values in the further locations represented in the first picture of Figure 1. The values of $1/A$ (that is the value of the ratio capital-over-output K/Y that in the model also equals the wealth-over-GDP ratio) is in the range $4 \div 6$ in line, for example, with the values found by Piketty and Zucman (2014).

In the described situation, computing the first eigenvalue of the operator $\mathcal{L}$ defined in (4) and using (9) we get the reasonable value of the global growth rate equal to 3.17%. As a further check we also observe that the (spatial-heterogeneous) saving rate in the long-run varies from 18% to 37% in line for instance with the World Bank data (see e.g. World Bank Group, 2016).

The effect of this non-uniform spatial technological distribution, whenever the population is constant with density everywhere equal to 1, is represented in Figure 1. We can promptly see the effect of the spatial polarization of the capital marginal (and average) productivity

[9]To numerically compute the eigenfunctions e_n we use the package *Chebfun* written for MATLAB. See Birkisson and Driscoll (2011) and Driscoll and Hale (2016) for details on the implementation of the routines on linear differential operators and in particular on eigenfunctions of Sturm-Liouville operators in Chebfun.

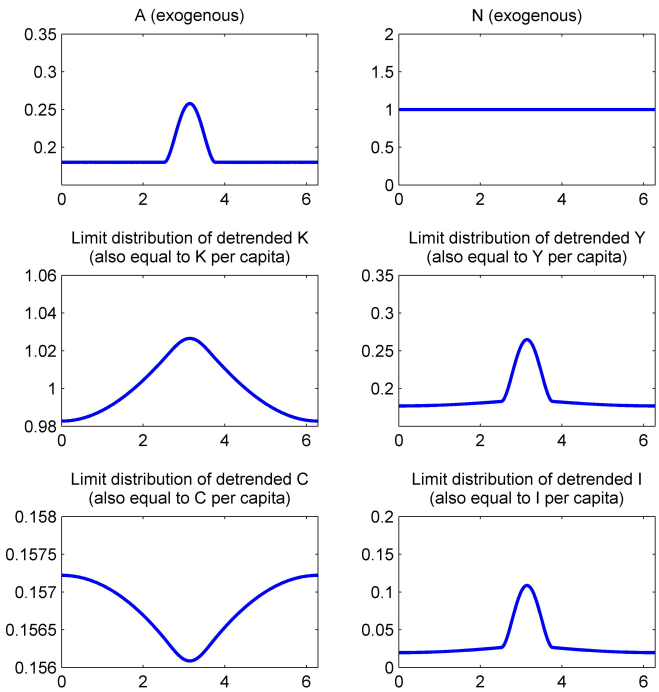

FIGURE 1. The pure technological space discrepancy effect

on capital accumulation in the first picture of the second line of Figure 1. In fact the capital tends to accumulate in the more productive areas while those with lower productivity remain behind: the higher productivity of capital pushes the planner to increase investments and thus savings relatively more in the more productive regions as shown in the second picture of the third line of Figure 1. As a byproduct the planner privileges consumption in peripheral regions but this is a second-order effect of small magnitude as one can see in the first picture of the third line of Figure 1. These are the outcomes of the pure technological space discrepancy effect (or productivity effect) announced in the introduction.

Looking at the (spatial) relative magnitudes in the distributions of A compared to those of the long-run detrended K, we can easily realize that the capital distribution is much less concentrated than the

technological level[10]. We have indeed an endogenous spatial spillover effect that is the combined result of both the capital exogenous diffusivity and the endogenous investment and consumption decisions by the planner.

The difference with respect to the results of Boucekkine at al. (2013) is crystal clear: once we introduce the spatial heterogeneity in capital productivity, the optimal detrended capital does not converge anymore to a spatial-homogeneous distribution. Indeed the homogeneous space case, where all the detrended variables (capital, output, consumption, investment) converge to the spatial-homogeneous configuration, arises as a special case, only if A is constant over the locations.

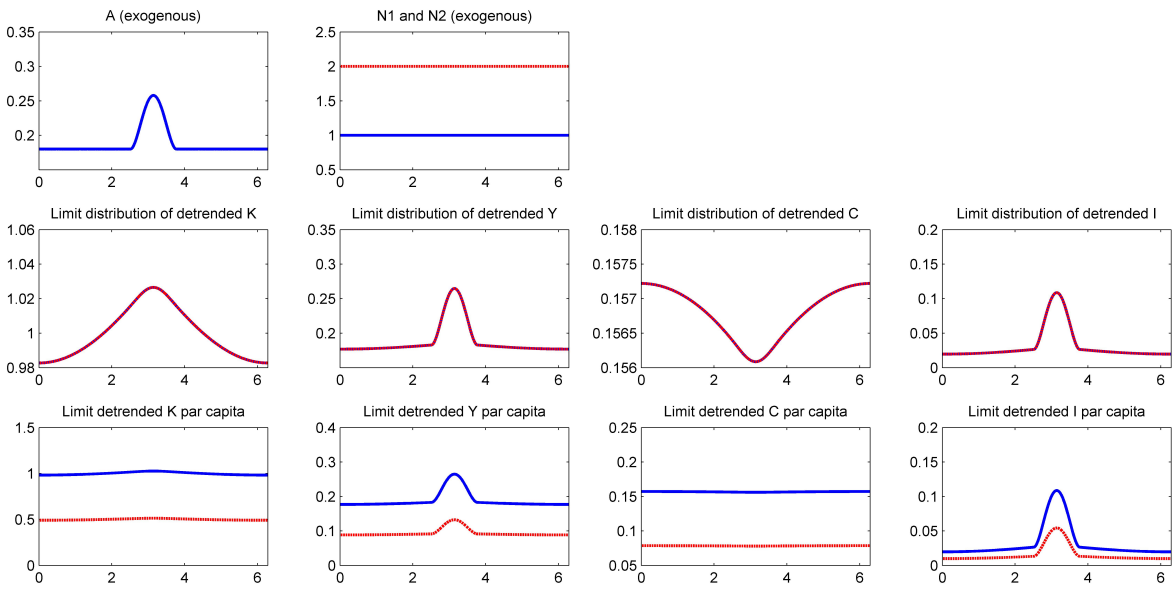

FIGURE 2. The pure dilution effect

Figure 2 emphasizes the pure dilution effect we have in the model. We consider the same technological distribution as in the previous picture and we vary uniformly the population density, more precisely we

[10]Conversely the concentration of the long-run detrended output is more picked because the output has the form $Y = AK$.

double the previous constant population density (in the picture the previous benchmark situation is in blue, with continuous line, while the new profile is in red, dotted line). The effect, in terms of aggregate optimal behavior is zero while per-capita variables are mechanically halved. This effect could be predicted directly from expression (8) taking into account the effect of population distribution on α given by (6). Observe that the pure dilution effect is not due to the homogeneous distribution of the population we use: whatever the initial population distribution, a uniform increase of the population of $n\%$ in the whole space induces a spatial uniform proportional reduction (by a factor $\frac{1}{1+n/100}$) of per capita variables.

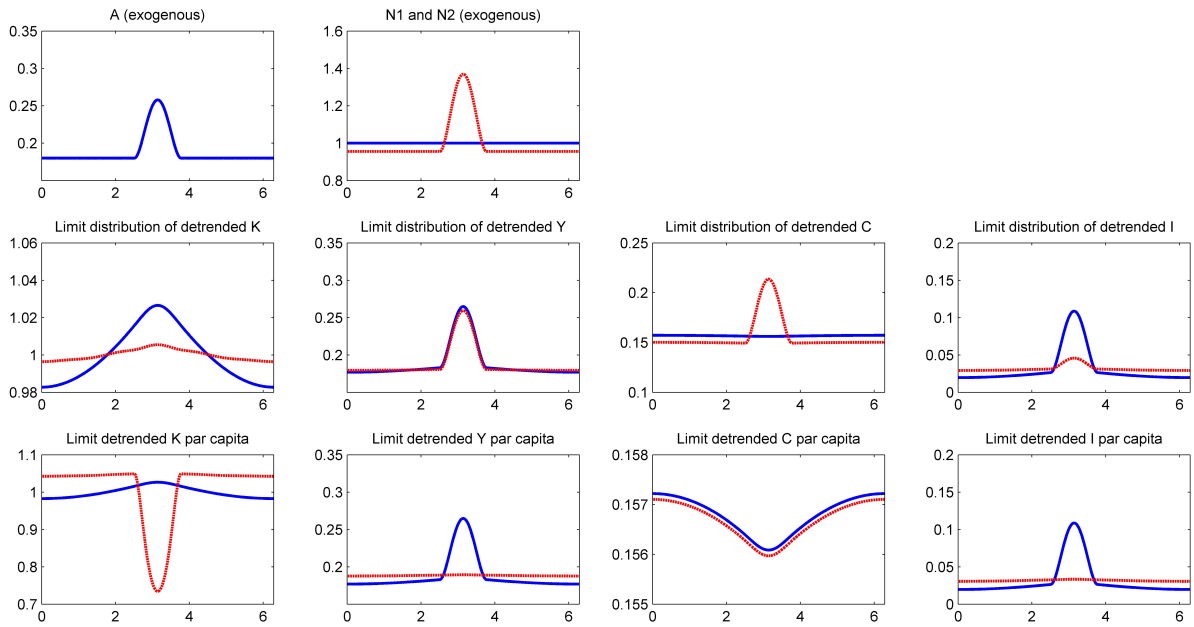

FIGURE 3. Technological discrepancy and population effects both at work

In Figure 3 we consider a concentration of capital productivity and population density in the same areas (a quite frequent configuration) showing how the technological space discrepancy and the population effects combine and can partially offset each other. In the simulation

we keep the same technology distribution as before and we consider two possible population distributions: in the first one (the blue and continuous line in the pictures) we have the same situation as in Figure 1, where the population is uniformly distributed across space with a constant unitary density, while in the second (the red and dotted line in the picture) the population has the same total size but is concentrated in the high productivity zones. In this second case two distinct motivations drive the planner: on one hand, she will tend to invest more in the more productive areas, but on the other, she is tempted to assign a reasonable enough per capita level of consumption in each region (again due to the Benthamite form of the utility functional). The total effect is depicted in the various pictures of Figure 3: the aggregate investment in more productive areas for the second population profile remains relatively higher[11] but the effect is mitigated because aggregate consumption is higher in these areas as well. All in all the distribution of long-run detrended capital is much more uniform in the second case so that capital accumulates relatively more in less productive areas. For this reason the change in the population distribution translates into a case of efficiency loss in the economic system: as one can see (third picture of the third line of Figure 3), per-capita consumption in the new configuration is always smaller that in the original one at any location.

5. Conclusions

In this paper we introduce and study a general spatial model of economic growth. We are able to solve it analytically by using dynamic programming in infinite dimensions. This is made possible thanks to the use of the eigenfunctions of the linear Sturm-Liouville problem related to the consumption-free dynamics of the model. With respect to

[11]This outcome depends on the chosen distribution of the population, a bigger concentration of the population would of course accentuate the population effect.

previous related contributions, our model is more general both for the possibility of studying heterogeneous spatial distributions of technology and for allowing for non-homogeneous spatially distributed population. The numerical exercises allow to identify two opposing effects: technological space discrepancy versus population dilution effect. We show that the shape of agglomeration triggered by growth depends pretty much on the relative strengths of the two latter effects. In summary, our setting delivers an agglomeration theory entirely based on optimal spatio-temporal capital dynamics for any given technology and population space distributions (first nature causes) which sharply departs from the agglomeration theories put forward in the New Economic Demography literature which mostly disregards capital accumulation and focuses on second nature causes.

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Appendix A. Proofs of the analytical results

In order to use the infinite dimensional dynamic programming to prove Theorems 3.2 and 3.3 we first need to recall some preliminary concepts and results.

A.1. **The infinite dimensional setting.** We can represent (1) as an abstract dynamical system in infinite-dimension. Some steps are needed to describe this construction. Consider the space[12]

$$\mathcal{H} := L^2(S^1) := \left\{ f : S^1 \to \mathbb{R} \text{ measurable } \Big| \int_0^{2\pi} |f(\theta)|^2 d\theta < \infty \right\}. \tag{11}$$

This is a Hilbert space when endowed with the inner product $\langle f, g\rangle := \int_0^{2\pi} f(\theta)g(\theta)d\theta$, inducing the norm $\|f\| = \int_0^{2\pi} |f(\theta)|^2 d\theta$. We will also use the following spaces of real functions defined on S^1:

$$L^\infty(S^1) := \{f \in \mathcal{H} \mid |f| \leq C \text{ for some } C > 0\},$$
$$H^1(S^1) := \{f \in \mathcal{H} \mid \exists\, f' \text{ in weak sense and belongs to } \mathcal{H}\},$$
$$H^2(S^1) := \{f \in \mathcal{H} \mid \exists\, f' \text{ in weak sense and belong to } H^1(S^1)\}.$$

Suppose from now that that the coefficients of the state equation satisfy the following conditions:

$$A \in L^\infty(S^1), \qquad N \in L^\infty(S^1). \tag{12}$$

The differential operator

$$\mathcal{L}u := \frac{\partial^2 u}{\partial \theta^2} + A(\cdot)u, \qquad u \in H^2(S^1)$$

is well defined and $\mathcal{H}$-valued. It is also self-adjoint, i.e.

$$\mathcal{L}^* = \mathcal{L}. \tag{13}$$

The operator $\mathcal{L}$ is the sum of the Laplacian operator on S^1 with the bounded operator $\mathbf{A} : \mathcal{H} \to \mathcal{H},\ u \mapsto A(\cdot)u$. The Laplacian operator is closed on the domain $H^2(S^1)$ and generates a C_0-semigroup on the space $\mathcal{H}$. Hence, as $\mathbf{A}$ is bounded, we deduce that also $\mathcal{L}$ is closed on the domain

$$D(\mathcal{L}) := H^2(S^1)$$

and generates a C_0-semigroup on the space $\mathcal{H}$. From now on, in order to avoid confusion, we will denote the elements of $\mathcal{H}$ by bold letters. With this convention, we can formally rewrite (1) as an abstract dynamical system in the space $\mathcal{H}$:

$$\begin{cases} \mathbf{K}'(t) = \mathcal{L}\mathbf{K}(t) - \mathbf{c}(t)\mathbf{N}, & t \in \mathbb{R}^+, \\ \mathbf{K}(0) = \mathbf{K}_0 \in \mathcal{H}, \end{cases} \tag{14}$$

[12]The precise definition of $L^2(S^1)$ includes the quotient with respect to the equivalence relation of being equal almost everywhere. We do not provide it here as this does not affect our setting and results.

with the formal equalities $\mathbf{K}(t)(\theta) = K(t,\theta)$, $[\mathbf{c}(t)\mathbf{N}](\theta) = c(t,\theta)N(\theta)$ and we will read the original system as (14).[13]

By general theory of semigroups (see Proposition 3.1 and 3.2, Section II-1, of Bensoussan et al., 2007, also considering (13)), given $\mathbf{c} \in L^1_{loc}(\mathbb{R}^+;\mathcal{H})$, there exists a unique (weak) solution $\mathbf{K}^{\mathbf{K}_0,\mathbf{c}} \in L^1_{loc}(\mathbb{R}^+;\mathcal{H})$ to (14) in the following sense: for each $\boldsymbol{\varphi} \in D(\mathcal{L})$ the function $t \mapsto \langle \mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(t), \boldsymbol{\varphi}\rangle$ is locally absolutely continuous and

$$\begin{cases} \frac{d}{dt}\langle \mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(t), \boldsymbol{\varphi}\rangle = \langle \mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(t), \mathcal{L}\boldsymbol{\varphi}\rangle - \langle \mathbf{c}(t)\mathbf{N}, \boldsymbol{\varphi}\rangle, \quad a.e.\ t \in \mathbb{R}^+, \\ \mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(0) = \mathbf{K}_0 \in \mathcal{H}. \end{cases} \tag{15}$$

Consider the positive cone in $\mathcal{H}$, i.e. the set

$$\mathcal{H}^+ := \{\mathbf{K} \in \mathcal{H} \mid \mathbf{K}(\cdot) \geq 0\},$$

the positive cone in $\mathcal{H}$ without the zero function, i.e. the set

$$\mathcal{H}^+_0 := \{\mathbf{K} \in \mathcal{H} \mid \mathbf{K}(\cdot) \geq 0 \text{ and } \mathbf{K}(\cdot) \not\equiv 0\},$$

and define the set of admissible strategies as[14]

$$\mathcal{A}(\mathbf{K}_0) := \{\mathbf{c} \in L^1_{loc}(\mathbb{R}^+;\mathcal{H}^+) \mid \mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(t) \in \mathcal{H}^+_0 \quad \forall t \geq 0\}.$$

Then we can rewrite the original optimization problem as the one of maximizing the objective functional

$$J(\mathbf{K}_0;\mathbf{c}) := \int_0^\infty e^{-\rho t}\mathcal{U}(\mathbf{c}(t))dt, \tag{16}$$

over all $\mathbf{c} \in \mathcal{A}(\mathbf{K}_0)$ where

$$\mathcal{U} : \mathcal{H}^+ \to \mathbb{R}^+, \quad \mathcal{U}(\mathbf{c}) := \int_0^{2\pi} \frac{\mathbf{c}(\theta)^{1-\sigma}}{1-\sigma}\mathbf{N}(\theta)d\theta.$$

In the following we call $(\mathbf{P})$ this problem and, as usual, we define the associated value function as

$$V(\mathbf{K}_0) := \sup_{\mathbf{c}\in\mathcal{A}(\mathbf{K}_0)} J(\mathbf{K}_0;\mathbf{c}). \tag{17}$$

A.2. **HJB equation.** Through the dynamic programming approach we associate to the problem $(\mathbf{P})$ the following Hamilton-Jacobi-Bellman (HJB) equation in $\mathcal{H}$ (which "should be" satisfied by the value function):

$$\rho v(\mathbf{K}) = \langle \mathbf{K}, \mathcal{L}\nabla v(\mathbf{K})\rangle + \sup_{\mathbf{c}\in\mathcal{H}^+} \{\mathcal{U}(\mathbf{c}) - \langle \mathbf{c}\mathbf{N}, \nabla v(\mathbf{K})\rangle\}. \tag{18}$$

An explicit solution of this equation can be given in a suitable half-space of $\mathcal{H}$ as shown by the following proposition.

Proposition A.1. *Let (5) and (12) hold. The function*

$$v(\mathbf{K}) = \frac{\langle \mathbf{K}, \alpha_0\mathbf{e}_0\rangle^{1-\sigma}}{1-\sigma}, \quad \mathbf{K} \in \mathcal{H}^+_{\mathbf{e}_0}, \tag{19}$$

[13]The correspondence between the concept of solution to the abstract dynamical system in $\mathcal{H}$ that we introduce below (weak solution) and the solution of (1) can be argued as in Proposition 3.2, page 131, of Bensoussan et al. (2007).

[14]In this formulation we require the slightly sharper state constraint $\mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(t) \in \mathcal{H}^+_0$ in place of the wider (original) one $\mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(t)(\cdot) \geq 0$ almost everywhere. This is without loss of generality: indeed, if $\mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(t) \equiv 0$ at some $t \geq 0$, the unique admissible (hence the optimal) control from t on is the trivial one $\mathbf{c}(\cdot) \equiv 0$, so we know how to solve the problem once we fall into this state and there is no need to define the Hamilton-Jacobi-Bellman equation at this point. The reason to exclude the zero function from the set $\mathcal{H}^+$ and considering the set $\mathcal{H}^+_0$ is allowing a well-definition of the Hamilton-Jacobi-Bellman equation.

where

$$\mathcal{H}^+_{\mathbf{e}_0} := \{\mathbf{K} \in \mathcal{H} \mid \langle \mathbf{K}, \mathbf{e}_0 \rangle > 0\}. \tag{20}$$

and

$$\alpha_0 := \left(\frac{\sigma}{\rho - \lambda_0(1-\sigma)} \int_0^{2\pi} \mathbf{e}_0(\theta)^{-\frac{1-\sigma}{\sigma}} \mathbf{N}(\theta) d\theta \right)^{\frac{\sigma}{1-\sigma}}, \tag{21}$$

is a classical solution[15] *of* (18) *over* $\mathcal{H}^+_{\mathbf{e}_0}$.

Proof. Define the strictly positive cone in $\mathcal{H}$, i.e.

$$\mathcal{H}^{++} := \left\{ f : S^1 \to \mathbb{R}^{++} \mid \int_0^{2\pi} |f(\theta)|^2 d\theta < \infty \right\},$$

Setting

$$\mathcal{U}^*(\boldsymbol{\alpha}) := \sup_{\mathbf{c} \in \mathcal{H}^+} \{\mathcal{U}(\mathbf{c}) - \langle \mathbf{c}\mathbf{N}, \boldsymbol{\alpha} \rangle\}, \quad \boldsymbol{\alpha} \in \mathcal{H}^{++},$$

we have

$$\mathcal{U}^*(\boldsymbol{\alpha}) := \sup_{\mathbf{c} \in \mathcal{H}^+} \int_0^{2\pi} \left(\frac{\mathbf{c}(\theta)^{1-\sigma}}{1-\sigma} \mathbf{N}(\theta) - \mathbf{c}(\theta)\mathbf{N}(\theta)\boldsymbol{\alpha}(\theta) \right) d\theta = \int_0^{2\pi} u^*(\mathbf{N}(\theta), \boldsymbol{\alpha}(\theta)) d\theta,$$

where

$$u^*(N, q) := \sup_{c \geq 0} \left\{ \frac{c^{1-\sigma}}{1-\sigma} N - qcN \right\} = \frac{\sigma}{1-\sigma} N q^{-\frac{1-\sigma}{\sigma}}, \quad q > 0, \ N \geq 0,$$

with optimizer

$$c^*(q) = q^{-\frac{1}{\sigma}}, \quad q > 0. \tag{22}$$

Plugging (19) into (18), and using that

$$\nabla v(\mathbf{K}) = \langle \mathbf{K}, \alpha_0 \mathbf{e}_0 \rangle^{-\sigma} \alpha_0 \mathbf{e}_0, \quad \mathbf{K} \in \mathcal{H}^+_{\mathbf{e}_0}, \tag{23}$$

we need to check the equality

$$\begin{aligned} \frac{\rho}{1-\sigma} \langle \mathbf{K}, \alpha_0 \mathbf{e}_0 \rangle^{1-\sigma} &= \langle \mathbf{K}, \mathcal{L}\alpha_0 \mathbf{e}_0 \rangle \langle \mathbf{K}, \alpha_0 \mathbf{e}_0 \rangle^{-\sigma} \\ &\quad + \frac{\sigma}{1-\sigma} \alpha_0^{-\frac{1-\sigma}{\sigma}} \left(\int_0^{2\pi} \mathbf{e}_0(\theta)^{-\frac{1-\sigma}{\sigma}} \mathbf{N}(\theta) d\theta \right) \langle \mathbf{K}, \alpha_0 \mathbf{e}_0 \rangle^{1-\sigma}. \end{aligned} \tag{24}$$

By definition of λ_0 and $\mathbf{e}_0$, we have $\mathcal{L}\mathbf{e}_0 = \lambda_0 \mathbf{e}_0$. So (24) holds by (21). □

For notational reasons we set

$$\boldsymbol{\beta} := \alpha_0 \mathbf{e}_0,$$

so we can rewrite (19) as

$$v(\mathbf{K}) = \frac{\langle \mathbf{K}, \boldsymbol{\beta} \rangle^{1-\sigma}}{1-\sigma}, \quad \mathbf{K} \in \mathcal{H}^+_{\mathbf{e}_0}. \tag{25}$$

Finally, from the definition of $\boldsymbol{\beta}$ and (21) we get the following identity that will be useful in the next subsection

$$\left(\int_0^{2\pi} \boldsymbol{\beta}(\theta)^{-\frac{1-\sigma}{\sigma}} \mathbf{N}(\theta) d\theta \right) = \frac{\rho - \lambda_0(1-\sigma)}{\sigma}. \tag{26}$$

A.3. **Solution of the optimal control problem via dynamic programming in infinite dimensions.** Proposition A.1 suggests to consider a different set of

[15]By a classical solution of (18) in an open subset $\mathcal{H}_1$ of $\mathcal{H}$ we mean a function $\psi : \mathcal{H}_1 \to \mathbb{R}$ which is C^1 in its domain and which verifies (18) at every point $\mathbf{K} \in \mathcal{H}_1$.

admissible controls, i.e.

$$\mathcal{A}^+_{\mathbf{e}_0}(\mathbf{K}_0) := \{\mathbf{c} \in L^1_{loc}(\mathbb{R}^+; \mathcal{H}^+) \mid \mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(t) \in \mathcal{H}^+_{\mathbf{e}_0} \quad \forall t \geq 0\}.$$

Since $\mathcal{H}^+_0 \subseteq \mathcal{H}^+_{\mathbf{e}_0}$, we have also $\mathcal{A}(\mathbf{K}_0) \subseteq \mathcal{A}^+_{\mathbf{e}_0}(\mathbf{K}_0)$. We define an auxiliary problem associated to this new relaxed constraint, i.e.

$$(\tilde{\mathbf{P}}) \qquad \text{Maximize } J(\mathbf{K}_0; \mathbf{c}) \text{ over } \mathbf{c} \in \mathcal{A}^+_{\mathbf{e}_0}(\mathbf{K}_0) \tag{27}$$

The value function of the problem $(\tilde{\mathbf{P}})$ is

$$\tilde{V}(\mathbf{K}_0) := \sup_{\mathbf{c} \in \mathcal{A}^+_{\mathbf{e}_0}(\mathbf{K}_0)} J(\mathbf{K}_0; \mathbf{c}). \tag{28}$$

Clearly we have the inequality

$$\tilde{V} \geq V \quad \text{over } \mathcal{H}^+_0. \tag{29}$$

The reason to consider the relaxed state constraint $\mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(\cdot) \in \mathcal{H}^+_{\mathbf{e}_0}$, in place of the stricter original one $\mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(\cdot) \in \mathcal{H}^+_0$, is that the former is somehow the "natural" one from the mathematical point of view and allows an explicit solution. On the other hand, the real constraint is still $\mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(\cdot) \in \mathcal{H}^+$, so we need to establish a relationship between the solutions of the two problems $(\mathbf{P})$ and $(\tilde{\mathbf{P}})$. Our approach relies on the following obvious result.

Lemma A.2. *If $\mathbf{c}^*$ is an optimal control for $(\mathbf{P})$ and $\mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(\cdot) \in \mathcal{H}^+_0$ (i.e. the solution of the optimization problem with relaxed state constraint actually satisfies the stricter one), then $\mathbf{c}^*$ is optimal also for $(\mathbf{P})$.*

We focus on the solution to $(\tilde{\mathbf{P}})$. Considering (22), the feedback map associated to the function v defined in (25) results in

$$\mathcal{H}^+_{\mathbf{e}_0} \to \mathcal{H}^+_0, \quad \mathbf{K} \mapsto \langle \boldsymbol{\beta}, \mathbf{K} \rangle \boldsymbol{\beta}^{-\frac{1}{\sigma}}, \tag{30}$$

where $\boldsymbol{\beta}^{-\frac{1}{\sigma}}(\theta) := (\boldsymbol{\beta}(\theta))^{-\frac{1}{\sigma}}$. By using the same results invoked for equation (14) above we find that the associated closed loop equation

$$\begin{cases} \mathbf{K}'(t) = \mathcal{L}\mathbf{K}(t) - \langle \boldsymbol{\beta}, \mathbf{K}(t) \rangle \boldsymbol{\beta}^{-\frac{1}{\sigma}} \mathbf{N}, \\ \mathbf{K}(0) = \mathbf{K}_0 \in \mathcal{H}^+_0, \end{cases} \tag{31}$$

admits a unique weak solution, i.e. there exists a unique function $\mathbf{K}^{\mathbf{K}_0,*} \in L^1_{loc}(\mathbb{R}^+; \mathcal{H})$ such that the function $t \mapsto \langle \mathbf{K}^{\mathbf{K}_0,*}(t), \boldsymbol{\varphi} \rangle$ is absolutely continuous for every $\boldsymbol{\varphi} \in D(\mathcal{L})$ and

$$\begin{cases} \frac{d}{dt} \langle \mathbf{K}^{\mathbf{K}_0,*}(t), \boldsymbol{\varphi} \rangle = \langle \mathbf{K}^{\mathbf{K}_0,*}(t), \mathcal{L}\boldsymbol{\varphi} \rangle - \langle \boldsymbol{\beta}, \mathbf{K}^{\mathbf{K}_0,*}(t) \rangle \langle \boldsymbol{\varphi}, \boldsymbol{\beta}^{-\frac{1}{\sigma}} \mathbf{N} \rangle, \quad a.e.\ t \in \mathbb{R}^+, \\ \mathbf{K}^{\mathbf{K}_0,*}(0) = \mathbf{K}_0 \in \mathcal{H}^+_0. \end{cases} \tag{32}$$

Consider (26) and set

$$g := \lambda_0 - \int_0^{2\pi} \mathbf{N}(\theta) \boldsymbol{\beta}(\theta)^{-\frac{1-\sigma}{\sigma}} d\theta = -\frac{\rho - \lambda_0}{\sigma}. \tag{33}$$

Taking $\boldsymbol{\varphi} = \boldsymbol{\beta}$ in (32), we get

$$\langle \mathbf{K}^{\mathbf{K}_0,*}(t), \boldsymbol{\beta} \rangle = \langle \boldsymbol{\beta}, \mathbf{K}_0 \rangle e^{gt}, \qquad t \geq 0, \tag{34}$$

Hence

$$\mathbf{K}_0 \in \mathcal{H}^+_{\mathbf{e}_0} \;\Rightarrow\; \mathbf{K}^{\mathbf{K}_0,*}(t) \in \mathcal{H}^+_{\mathbf{e}_0}.$$

So the control

$$\mathbf{c}^*(t) := \langle \boldsymbol{\beta}, \mathbf{K}(t) \rangle \boldsymbol{\beta}^{-\frac{1}{\sigma}} = \langle \boldsymbol{\beta}, \mathbf{K}_0 \rangle \boldsymbol{\beta}^{-\frac{1}{\sigma}} e^{gt}, \quad t \geq 0, \tag{35}$$

belongs to $\mathcal{A}^+_{\mathbf{e}_0}(\mathbf{K}_0)$.

Lemma A.3. *For each* $\mathbf{c} \in \mathcal{A}^+_{\mathbf{e}_0}(\mathbf{K}_0)$ *we have*

$$\langle \mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(t), \boldsymbol{\beta}\rangle \leq \langle \boldsymbol{\beta}, \mathbf{K}_0\rangle e^{\lambda_0 t}, \quad \forall t \geq 0.$$

Proof. Denote by $\mathbf{0}$ the null control, i.e. the control $\mathbf{c}(t)(\theta) = 0$ for each $(t,\theta) \in \mathbb{R}^+ \times S^1$. Then (15) yields $\langle \mathbf{K}^{\mathbf{K}_0,\mathbf{0}}(t), \boldsymbol{\beta}\rangle = \langle \boldsymbol{\beta}, \mathbf{K}_0\rangle e^{\lambda_0 t}$ for every $t \geq 0$. On the other hand, as $\boldsymbol{\beta}(\theta) > 0$ for each $\theta \in S^1$, standard comparison applied to the ODE (15) yields

$$\langle \mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(\cdot), \boldsymbol{\beta}\rangle \leq \langle \mathbf{K}^{\mathbf{K}_0,\mathbf{0}}(\cdot), \boldsymbol{\beta}\rangle, \tag{36}$$

and the claim follows. □

Theorem A.4. *Let (5) and (12) hold. Let* $\mathbf{K}_0 \in \mathcal{H}^+_{\mathbf{e}_0}$ *and let* v *be the function defined in* (25)*. Then* $v(\mathbf{K}_0) = \tilde{V}(\mathbf{K}_0)$ *and the control* $\mathbf{c}^*$ *defined in* (35) *is optimal for* $(\tilde{\mathbf{P}})$ *starting from the initial state* $\mathbf{K}_0$*; i.e.* $J(\mathbf{K}_0; \mathbf{c}^*) = \tilde{V}(\mathbf{K}_0)$.

Proof. The fact that $\mathbf{c}^* \in \mathcal{A}^+_{\mathbf{e}_0}(\mathbf{K}_0)$ has been already observed in the discussion preceding Lemma A.3. We prove now the optimality. By the usual arguments employed to prove Verification Theorem with a Dynamic Programming approach, using the fact that v is a solution to (18) on $\mathcal{A}^+_{\mathbf{e}_0}(\mathbf{K}_0)$ one gets, for every $\mathbf{c} \in \mathcal{A}^+_{\mathbf{e}_0}(\mathbf{K}_0)$,

$$\begin{aligned} e^{-\rho t} v(\mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(t)) - v(\mathbf{K}_0) = &- \int_0^t e^{-\rho s}\mathcal{U}(\mathbf{c}(s))ds \\ &+ \int_0^t e^{-\rho s}\{\mathcal{U}(\mathbf{c}(s)) - \langle \mathbf{c}(s)\mathbf{N}, \nabla v(\mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(s))\rangle - \mathcal{U}^*(\nabla v(\mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(s))\}ds \end{aligned} \tag{37}$$

We pass (37) to the limit for $t \to \infty$.

- We use (5) and Lemma A.3 in the left hand side;
- we use monotone convergence in the right hand side, as, by definition of $\mathcal{U}^*$, the integrand is nonpositive.

Hence, we get the so called *fundamental identity*, valid for each $\mathbf{c} \in \mathcal{A}^+_{\mathbf{e}_0}(\mathbf{K}_0)$:

$$\begin{aligned} v(\mathbf{K}_0) = &J(\mathbf{K}_0; \mathbf{c}) \\ &+ \int_0^\infty e^{-\rho s}\big\{\mathcal{U}^*(\nabla v(\mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(s)) - \big(\mathcal{U}(\mathbf{c}(s)) - \langle \mathbf{c}(s)\mathbf{N}, \nabla v(\mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(s))\rangle\big)\big\}ds. \end{aligned} \tag{38}$$

From (38), by definition of $\mathcal{U}^*$ we first get $v(\mathbf{K}_0) \geq \tilde{V}(\mathbf{K}_0)$. Then, observing that the integrand in (38) vanishes when $\mathbf{c} = \mathbf{c}^*$, we obtain $v(\mathbf{K}_0) = J(\mathbf{K}_0; \mathbf{c}^*)$. The claim follows. □

From Theorem A.4 and Lemma A.2, we get our first main result corresponding to Theorem 3.2.

Corollary A.5. *Let (5) and (12) hold. Let* $\mathbf{K}_0 \in \mathcal{H}^+_0$*, let* $\mathbf{c}^*$ *be the control defined in* (35) *and assume that* $\mathbf{c}^* \in \mathcal{A}(\mathbf{K}_0)$*. Then* $v(\mathbf{K}_0) = V(\mathbf{K}_0)$ *and* $\mathbf{c}^*$ *is optimal for* $(\mathbf{P})$.

Remark A.6. *The following estimates on* λ_0 *can be obtained from its representation provided in Section 2.10 of Brown et al. (2013):*

$$\frac{1}{2\pi}\int_0^{2\pi} A(\theta)d\theta \;\leq\; \lambda_0 \;\leq\; \sup_{S^1}|A|. \tag{39}$$

The lower bound in particular assures, given the positivity of $A(\cdot)$*, the positivity of* λ_0*. The upper bound is useful to check (5),*

Theorem 2.9.3 of Brown et al. (2013) also gives the following estimates for the second eigenvalue:

$$\lambda_1 \ \leq \sup_{S^1} A - 1,$$

useful to check (10).

The study of the convergence of the transitional dynamics to a stationary state gives the following claim corresponding to Theorem 3.3.

Proposition A.7. *Let (5), (10) and (12) hold. Define the detrended optimal path*

$$\mathbf{K}_g^{\mathbf{K}_0,\mathbf{c}^*}(t) := e^{-gt}\mathbf{K}^{\mathbf{K}_0,\mathbf{c}^*}(t), \qquad t \geq 0.$$

Then

$$\mathbf{K}_g^{\mathbf{K}_0,\mathbf{c}^*}(t) \ \overset{t\to\infty}{\longrightarrow} \ \langle \mathbf{K}_0, \boldsymbol{\beta}\rangle \left(\alpha_0^{-1}\mathbf{e}_0 + \sum_{n\geq 1} \frac{\beta_n}{\lambda_n - g}\mathbf{e}_n \right), \ \textit{in } L^2(S^1),$$

where $\beta_n := \langle \mathbf{e}_n, \boldsymbol{\beta}^{-\frac{1}{\sigma}}\mathbf{N}\rangle$ *for* $n \geq 1$.

Proof. As $\mathbf{K}^{\mathbf{K}_0,\mathbf{c}^*}(\cdot)$ is a weak solution of (31), $\mathbf{K}_g^{\mathbf{K}_0,\mathbf{c}^*}(\cdot)$ is a weak solution of

$$\begin{cases} \mathbf{K}'(t) = \mathcal{L}\mathbf{K}(t) - g\mathbf{K}(t) - \langle \boldsymbol{\beta}, \mathbf{K}(t)\rangle \boldsymbol{\beta}^{-\frac{1}{\sigma}}\mathbf{N} \\ \mathbf{K}(0) = \mathbf{K}_0 \in \mathcal{H}_0^+, \end{cases}$$

i.e., for every $\boldsymbol{\varphi} \in D(\mathcal{L})$,

$$\begin{cases} \frac{d}{dt}\langle \mathbf{K}_g^{\mathbf{K}_0,\mathbf{c}^*}(t), \boldsymbol{\varphi}\rangle = \langle \mathbf{K}_g^{\mathbf{K}_0,\mathbf{c}^*}(t), (\mathcal{L} - g)\boldsymbol{\varphi}\rangle - \langle \boldsymbol{\beta}, \mathbf{K}_g^{\mathbf{K}_0,\mathbf{c}^*}(t)\rangle \langle \boldsymbol{\varphi}, \boldsymbol{\beta}^{-\frac{1}{\sigma}}\mathbf{N}\rangle \\ \mathbf{K}_g^{\mathbf{K}_0,\mathbf{c}^*}(0) = \mathbf{K}_0 \in \mathcal{H}_0^+. \end{cases} \tag{40}$$

As already recalled in Section 3 there exists an orthonormal basis of $L^2(S^1)$ of eigenfunctions $\{\mathbf{e}_n\}_{n\geq 0}$ corresponding to the sequence of eigenvalues $\{\lambda_n\}_{n\geq 0}$ so we have the Fourier series expansion

$$\mathbf{K}_g^{\mathbf{K}_0,\mathbf{c}^*}(t) = \sum_{n\geq 0} K_{g,n}(t)\mathbf{e}_n, \quad \text{where } K_{g,n}(t) := \langle \mathbf{K}_g^{\mathbf{K}_0,\mathbf{c}^*}(t), \mathbf{e}_n\rangle, \qquad n \geq 0.$$

We compute now the Fourier coefficients $K_{g,n}(t)$.

- When $n = 0$, we already know from (34)
$$K_{g,0}(\cdot) \equiv \langle \mathbf{K}_0, \mathbf{e}_0\rangle = \alpha_0^{-1}\langle \mathbf{K}_0, \boldsymbol{\beta}\rangle.$$
- When $n \geq 1$, we have, taking $\boldsymbol{\varphi} = \mathbf{e}_n$ in (40),
$$K'_{g,n}(t) = (\lambda_n - g)K_{g,n}(t) - \langle \mathbf{K}_0, \boldsymbol{\beta}\rangle \beta_n.$$
So, we can explicitly express the Fourier coefficients as:
$$K_{g,n}(t) = \langle \mathbf{K}_0, \mathbf{e}_n\rangle e^{(\lambda_n - g)t} + \langle \mathbf{K}_0, \boldsymbol{\beta}\rangle \frac{\beta_n}{\lambda_n - g}(1 - e^{(\lambda_n - g)t}).$$

Considering (see Remark A.6) that $\lambda_n \leq \lambda_1 < g$ for every $n \geq 1$, we have the convergence

$$K_{g,n}(t) \overset{t\to\infty}{\longrightarrow} \langle \mathbf{K}_0, \boldsymbol{\beta}\rangle \frac{\beta_n}{\lambda_n - g}, \qquad \text{uniformly in } n \geq 1.$$

The claim follows. □

R. Boucekkine, Aix-Marseille University (Aix-Marseille School of Economics and IMéRA), CNRS and EHESS. 2, Place Le Verrier, 13004 Marseille, France.
E-mail address: raouf.boucekkine@univ-amu.fr

G. Fabbri, Aix-Marseille University (Aix-Marseille School of Economics), CNRS and EHESS. 2 Rue de la Charité, 13002 Marseille, France.
E-mail address: giorgio.fabbri@univ-amu.fr

S. Federico, Università degli Studi di Siena, Dipartimento di Economia Politica e Statistica. Piazza San Francesco 7-8, 53100 Siena, Italy.
E-mail address: salvatore.federico@unisi.it

F. Gozzi, Dipartimento di Economia e Finanza, Libera Università degli Studi Sociali *Guido Carli*, Roma, Italy.
E-mail address: fgozzi@luiss.it

Institut de Recherches Économiques et Sociales
Université catholique de Louvain

Place Montesquieu, 3
1348 Louvain-la-Neuve, Belgique

ISSN 1379-244X D/2017/3082/6