The value of biodiversity as an insurance device

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The value of biodiversity as an insurance device*

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Abstract

This paper presents a benchmark endogenous growth model including biodiversity preservation dynamics. Producing food requires land, and increasing the share of total land devoted to farming mechanically reduces the share of land devoted to biodiversity conservation. However, safeguarding a greater number of species guarantee better ecosystem services – pollination, flood control, pest control, etc., which in turn ensure lower volatility of agricultural productivity. The optimal conversion/preservation rule is explicitly characterized, as well as the value of biological diversity, in terms of the welfare gain from biodiversity conservation. The Epstein-Zin-Weil specification of the utility function allows us to disentangle the effects of risk aversion and aversion to fluctuations. A two-player game extension of the model highlights the effect of volatility externalities and the Paretian sub-optimality of the decentralized choice.

KEY WORDS: Biodiversity, stochastic endogenous growth, insurance value, recursive preferences.

JEL CLASSIFICATION: Q56, Q58, Q10, Q15, O13, O20, C73.

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1 Introduction

From 1999 to 2008, 48 000 square kilometers of wildland were turned into cropland. By 2013, cropland covered 12% of earth's ice-free surface; annually, more than 10% of the world net primary production is turned into crops (Phalan *et al.* [24]). The ensuing destruction of natural habitats causes species extinctions and thus biodiversity loss. This destruction of habitats and natural ecosystems for agricultural purposes, useful and inevitable as it may appear, is increasingly questioned. Natural ecosystems guarantee a wide range of goods and services, such as control of the local climate, clean water provision, flood control, maintenance of soil fertility, pollination, or pest control. In the debate about the trade-off between food production and biodiversity conservation, it is crucial to recognize the ecological and economic determinants of the value of ecosystem services. Ironically, a growing body of evidence shows that one of these determinants is the feedback effect that biodiversity destruction exerts on agricultural productivity. Biodiversity destruction negatively impacts the climatic, hydrological and, more generally, ecological environment, which may in turn negatively affect the mean level and/or the variability of agricultural productivity over time (Fuglie and Nin-Pratt [12], de Mazancourt *et al.* [6]). To put things differently, biodiversity conservation can provide beneficial services that enhance mean agricultural productivity and reduce its variability. We are interested in the second of these effects, namely the ability of a high level of biodiversity to dampen fluctuations in agricultural productivity around its trend. Thus biodiversity acts as a form of insurance, a natural insurance against fluctuations in agricultural production, and this is likely a main determinant of its overall value.

The insurance value of biodiversity has been analyzed in a series of groundbreaking studies by Baumgärtner [3], Quaas and Baumgärtner [25], [26], Baumgärtner and Quaas [4] and Baumgärtner and Strunz [5]. These studies, however, rely on static models under partial equilibrium.

We provide the first analysis of biodiversity as insurance in a stochastic dynamic setup. Precisely, we study a dynamic problem of optimal land conversion in a stylized stochastic endogenous growth model where (1) increasing the share of land devoted to farming allows agricultural production to be increased at the expense of biodiversity; (2) agricultural productivity evolves stochastically around an exogenous deterministic trend, with a volatility that negatively depends on biodiversity.

To better study insurance issues in a dynamic context, we follow the approach pioneered by Epstein and Zin (Epstein and Zin [10], [11], Duffie and Epstein [7]) and represent preferences by a recursive utility function. In this way, we are able to separate aversion to intertemporal fluctuations from relative risk aversion. The risk aversion parameter quantifies the preference for certain rather than uncertain outcomes, and it only makes sense in a stochastic context (but even in static models). Conversely, aversion to intertemporal fluctuations, i.e. the inverse of the elasticity of intertemporal substitution, measures the propensity to smooth consumption over time, and is a fundamental parameter in deterministic dynamic models as well. In typical endogenous growth models, when the growth rate is positive, aversion to intertemporal fluctuations takes the form of a willingness to increase the level of the consumption at the expense of its growth rate. Several studies[1] in the field of natural resources and the environment even prove that these two logically distinct concepts cannot satisfyingly be embodied in a single parameter as they can be when intertemporal additive expected utility is considered.

We develop our approach in two stages. We first characterize the optimal allocation of land to biodiversity conservation. We then consider two farmers exploiting common land in the absence of well-defined property rights. We study how the equilibrium allocation of land differs from the social optimum.

We show that the optimal share of land devoted to biodiversity conservation is constant over time. It decreases with the social discount rate and increases with risk aversion. It

[1] In a model of reservoir management, Howitt *et al.* [17] show that the intertemporal additive expected utility function does not fit their data, whereas the recursive utility function does. Peltola and Knapp [23] use recursive utility to study forestry management, and Lybbert and McPeak [22] for the trade-off among different livestock for Kenyan pastoralists. They empirically identify the distinct values that should be taken by the intertemporal elasticity of substitution and risk aversion parameters. Ha-Duong and Treich [16] evaluate policies in a context of global warming; it is shown that the optimal policy responds differently to variation in the intertemporal substitution parameter and in the risk aversion one. The same result is observed by Knapp and Olson [18] for rangeland and groundwater management. They consider the effect of both parameters on the optimal decision rule, showing in particular that when intertemporal substitution has a major effect, risk aversion does not impact the optimal policy. The different roles of the two parameters is also proved in Epaulard and Pommeret [9] in the context of extraction of a non-renewable resource in a continuous time framework.

decreases with aversion to fluctuations when the average trend of agricultural productivity is larger than the discount rate, but increases with aversion to fluctuations in the opposite case.

We then compute the value of biodiversity, defined with reference to Lucas [20], [21] as the welfare gain from biodiversity conservation under the optimal solution compared to a solution where total land conversion is achieved. We study the determinants of this particular value of biodiversity, interpreted as insurance against the volatility of agricultural productivity. We show that it is an increasing function of risk aversion, but that the effect of aversion to fluctuations is ambiguous. In general, and quite intuitively, the signs of the effects of increasing aversion to fluctuations on the share of land devoted to biodiversity conservation and on the value of biodiversity are the same. But when the average trend of agricultural productivity is smaller than the discount rate and both intrinsic agricultural volatility and risk aversion are high, a counterintuitive result may appear: the two signs may be opposite. In this case with poor prospects for the economy, when society becomes more averse to fluctuations it preserves less and less of something that has more and more value: biodiversity.

In the two-player extension of the model, land is a common-property resource that two farmers want to exploit. The behavior of the two farmers is supposed to be coordinated so that the system reaches a Nash equilibrium of the game. We highlight the systematic over-exploitation of the natural resource with respect to the previously described social optimum due to the volatility externalities. We show that the increment in area of land devoted to farming (that is the complement to 1 of the area devoted to biodiversity conservation) does not depend on risk aversion, but is decreasing with aversion to fluctuations and tends to vanish when aversion to fluctuations is very high.

Our analysis contributes to three strands of the literature. First, we contribute to a nascent theoretical literature on the insurance value of biodiversity. As emphasized above, we are interested here not only in the link between biodiversity and destruction of the environment[2], but also in ecosystem variability (climate, water provision, flood control, maintenance of soil fertility, pollination, etc.). This kind of interaction between biodiversity preservation and uncertainty substantially differs from the *portfolio differentiation* arguments used, for instance, by Weitzman [29], where the word "biodiversity" is used to refer to the number of

[2] The two are closely related, being linked by the species-area relationship, see Rosenzweig [27].

cultivated varieties. A series of papers by Baumgärtner, Quaas and Strunz ([3], [25], [26], [4] and [5]) theoretically analyzes the phenomenon in the framework of static models. Naturally the static context is a limitation, because the effects of biodiversity degradation accumulate and spread over time. We propose the first stochastic dynamic model in this literature. In our model, the volatility of agricultural productivity depends on the whole historical path of biodiversity conservation decisions. This dynamic context is essential to being able to speak about elasticity of intertemporal substitution and aversion to fluctuations.

The second strand that our paper contributes to is the small but growing literature on natural resources that disentangles intertemporal substitution and risk aversion. In different contexts, Krautkraemer *et al.* [19], Howitt *et al.* [17], Peltola and Knapp [23], Lybbert and McPeak [22], Ha-Duong and Treich [16] and Knapp and Olson [18] show that recursive utility better fits the data than intertemporal additive expected utility, because intertemporal substitution and risk aversion cannot both be represented by the same parameter. In the specific case of the relation between optimal growth and biodiversity conservation, we show that the optimal allocation of the land responds qualitatively differently to the two parameters.

Finally, our paper contributes to the literature on dynamic games in continuous time. Continuous time stochastic games are common in various areas of economic theory (for example natural resource exploitation, capital accumulation, oligopoly theory), see for instance Van Long [28] or Haunschmied *et al.* [15]. However, to the best of our knowledge this is the first continuous time model in the economic theory literature to use a Nash equilibrium of a game with Epstein-Zin-Weil preferences.

The paper proceeds as follows. In section 2 we present the basics of the model. We explicitly solve it to obtain the optimal land conversion rate in section 3, and we compute the value of biodiversity in section 4. In section 5, we solve the two-player common property resource game and compare its outcome to the optimal solution. Section 6 concludes. All the proofs of the results are in the Appendix.

2 The model

We build a highly stylized model, the simplest we can think of that allows us to compute the value of biodiversity as insurance against agricultural productivity fluctuations. We first present the model and its optimal solution, and then determine of the value of biodiversity.

We consider an agricultural economy. We describe the problem of a planner or of the unique farmer living in this economy. She owns a stock $L = 1$ of land and she has to decide how to allocate it between two possible intended uses: farming and biodiversity preservation.

For $t \geq 0$, we respectively denote by $f(t) \in [0, 1]$ and $1 - f(t) \in [0, 1]$ the share of land used respectively for farming and to maintain biodiversity. According to the well-known species-area curve first proposed in the 1920s by O. Arrhenius [2] and H. Gleason [8], as the number of species is constrained by available land, the level of biodiversity $B(t)$ depends on the area of land left undeveloped:

$$B(t) = g(1 - f(t)) \quad \text{with} \quad g'(\cdot) > 0,$$

where the function $g(\cdot)$ is concave and usually specified as a power function.

We assume that agricultural production at time t is given by:

$$Y(t) = f(t)A(t) \tag{1}$$

where $A(t)$ is the productivity of a unit of land devoted to farming at time t, whose dynamics is described by a stochastic differential equation (SDE). More precisely, given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a real standard Brownian motion $W \colon [0, +\infty) \times \Omega \to \mathbb{R}$, adapted to some filtration $\mathcal{F}_t$, we assume that $A(t)$ is a solution of the following SDE:

$$\begin{cases} \mathrm{d}A(t) = \alpha A(t)\,\mathrm{d}t + \sqrt{f(t)}\sigma A(t)\,\mathrm{d}W(t) \\ A(0) = A_0. \end{cases} \tag{2}$$

In such an expression, $\alpha \in \mathbb{R}$ represents some (fixed and exogenous) parameter of technological progress in farming activities (it can be equal to 0). The term $\sqrt{f(t)}\sigma$ measures the volatility of agricultural productivity. The exogenous part $\sigma > 0$ represents the intrinsic volatility, due for instance to weather (floods, droughts, etc.). Total volatility decreases as the land devoted to biodiversity preservation, and then biodiversity itself, increases[3]. It is in this sense that biodiversity appears in the model as insurance against adverse outcomes.

We suppose that at each time $t \geq 0$ all the production is consumed, so that:

$$C(t) = Y(t). \tag{3}$$

This assumption is not innocuous. It implies that there is no precautionary saving: the economy cannot store or save part of its agricultural production to hedge against the risk of poor future productivity.

The planner maximizes, over the set of the $[0,1]$-valued $\mathcal{F}_t$-adapted processes, an aggregate social welfare criterion in the form of an infinite horizon, continuous time, Epstein-Zin-Weil utility function characterized by a constant relative risk aversion of θ (positive and different from 1), an intertemporal elasticity of $\phi^{-1} > 0$ and a discount rate of $\rho > 0$. Recall that the case $\theta = \phi$ corresponds to the usual time additive expected utility function. The inverse of the intertemporal elasticity of substitution ϕ can also be interpreted as a measure of aversion to fluctuations, as an agent with a high ϕ prefers to smooth consumption over time[4].

As proved in Example 3 page 367 of Duffie and Epstein [7], the corresponding aggregator can be written as:

$$F(C,V) = \frac{\rho}{1-\phi}(1-\theta)V\left[\left(\frac{C}{\left((1-\theta)V\right)^{\frac{1}{1-\theta}}}\right)^{1-\phi} - 1\right]. \tag{4}$$

We denote by $V(A_0)$ the value function of the described problem.

Remark 2.1. *In an informal way, as, for instance, in Epaulard and Pommeret [9], we could represent the Epstein-Zin-Weil preferences using an infinitesimal representation, having the advantage of being easily linked to the definition of discrete-time recursive utility. In this context the utility at period t depends on current consumption as well as on the certain equivalent $\overline{U}$ of future utility:*

$$U(t) = \left[\frac{C(t)^{1-\phi}}{1-\phi} + e^{-\rho dt}\frac{\overline{U}(t+dt)^{1-\phi}}{1-\phi}\right]$$

[3] We use a square root function only to simplify the computations, but the method we use works for more general concave functions.

[4] If $\theta > \phi$, individuals are more risk-averse than concerned about consumption smoothing. In this case (Gollier [13]), an agent is said to have Preferences for an Early Resolution of Uncertainty (PERU), notably the case among poor, vulnerable populations (Lybbert and MacPeak [22]).

being $\overline{U}(t+dt)$ *the quantity:*

$$\overline{U}(t+dt) = \left[E\left(U\left(t+dt\right)^{1-\theta}\right)\right]^{\frac{1}{1-\theta}}.$$

3 The optimal land conversion rate

The following conditions, which we will always suppose to be verified in the following, will be shown to be necessary to ensure that the value function remains finite and to explicitly express its value.

Hypothesis 3.1. *The parameters satisfy the following conditions:*

$$\phi \neq 1, \qquad \rho > \alpha(1-\phi), \qquad \frac{1-\theta}{1-\phi} > 0. \tag{5}$$

Observe in particular that the second inequality in (5) is always satisfied if $\phi > 1$, and that the third inequality requires that either $\theta, \phi < 1$ or $\theta, \phi > 1$. If $\theta = \phi$, as in the expected utility case, this third condition is always satisfied. Under Hypothesis 3.1 the value of the positive constant

$$\frac{\rho - \alpha(1-\phi)}{\frac{\sigma^2}{2}\theta\phi}$$

will be important to distinguish between interior and corner solutions. In the two following propositions we will see what happens when this constant is greater or smaller than 1. We begin by describing the dynamics of the system in the interior solution case.

Proposition 3.2. *Let Hypothesis 3.1 be satisfied. Assume that:*

$$\frac{\rho - \alpha(1-\phi)}{\frac{\sigma^2}{2}\theta\phi} < 1. \tag{6}$$

Then the value function of the problem can be written explicitly. It is equal to:

$$V(A) = \frac{1}{1-\theta}\beta A^{1-\theta} \tag{7}$$

where

$$\beta = \left[\frac{\rho}{\frac{\sigma^2}{2}\theta}\left(\frac{\rho - \alpha(1-\phi)}{\frac{\sigma^2}{2}\theta\phi}\right)^{-\phi}\right]^{\frac{1-\theta}{1-\phi}}. \tag{8}$$

The optimal control is constant and deterministic, and it is given by:

$$f^*(t) = f^* := \frac{\rho - \alpha(1-\phi)}{\frac{\sigma^2}{2}\theta\phi}, \qquad \textit{for any } t \geq 0. \tag{9}$$

Finally, (5) guarantees the respect of the transversality condition.

Proof. See Appendix. □

Lemma 3.3. *The optimal conversion rate f^* is an increasing function of the discount rate ρ, a decreasing function of the intrinsic volatility of agricultural productivity σ, and a decreasing function of risk aversion θ. It is also a decreasing function of aversion to fluctuations ϕ if $\alpha - \rho < 0$, but an increasing function of aversion to fluctuations if $\alpha - \rho > 0$.*

Proof. Straightforward derivations of (9) give the results. □

The first three results are consistent with intuition. The higher the discount rate, that is, the more impatient a society is, the less it cares about the future and the less it wants to insure against future uncertainty. Such a society has a strong incentive to convert a large part of its land to agriculture to enjoy food consumption now. Conversely, the higher the intrinsic volatility and the more risk-averse a society is, the more it wants to insure against future uncertainty.

Things are more complex regarding the effect of the society's aversion to fluctuations. When the difference between the trend of agricultural productivity and the discount rate is negative, future prospects are – on average – rather poor, and at the same time the society is impatient. Both effects lead to better present than future outcomes. A society averse to fluctuations logically wants to counteract these forces, and is thus willing to insure against adverse outcomes in the future by conserving more biodiversity. The opposite occurs when $\alpha - \rho > 0$.

Notice than when $\alpha > \rho$, increasing both risk aversion and aversion to fluctuations has an ambiguous effect on the optimal conversion rate, since the two parameters characterizing preferences act in opposite directions.

In the situation described in Proposition 3.2 a complete description of the optimal dynamics of the system can be provided. What we have is the following corollary of the previous result.

Corollary 3.4. *Let the assumptions of Proposition 3.2 be satisfied. Then the optimal evolution of $A(t)$ and $C(t)$ are respectively:*

$$A(t) = A_0 \exp\left[\left(\alpha - \frac{\sigma^2}{2} f^*\right) t + \sqrt{f^*}\sigma W(t)\right] \tag{10}$$

and:

$$C(t) = f^* A(t).$$

In particular

$$\mathbb{E}\left[A(t)\right] = A_0 e^{\alpha t}, \qquad \mathbb{V}ar\left[A(t)\right] = A_0^2 e^{2\alpha t}\left(e^{\sigma^2 f^* t} - 1\right). \tag{11}$$

The dynamics of the optimal land productivity described in (10) is a geometric Brownian motion, so that at any time t the distribution of $A(t)$ is log-normal and has, respectively, the expected value and the variance described in (11). Given the expression of the dynamics of A in (2), the growth rate of the expected value of $A(t)$ only depends on the parameter α, while f^* positively impacts its variance.

We can now return to the results of lemma 3.3 and look more closely at their significance. First, the effect of increased risk aversion can be seen as a form of precautionary saving effect. Indeed, increasing θ has the consequence of reducing f^* which, on the one hand, decreases the (certain) consumption level today but, on the other hand, increases the average value of the consumption growth rate[5] that is given by $\mathbb{E}[g] = \alpha - \frac{\sigma^2}{2} f^*$.

Second, increasing society's aversion to fluctuations also generates a level effect and a growth effect. The situation is similar to that of the standard deterministic benchmark AK growth model (see for instance Acemoglu [1], Section 11.1), with its linear production function characterized by a technological level A that corresponds to parameter α of our model, and its discount rate ρ. In the case of the AK model, the parameter ϕ appearing in the instantaneous utility $u = \frac{C(t)^{1-\phi}}{1-\phi}$ cannot be interpreted as risk aversion, since the model is deterministic. ϕ is actually the inverse of the elasticity of intertemporal substitution, that is, aversion to fluctuations. In that context, the effect of increasing the aversion to fluctuations parameter ϕ depends on the sign of $A - \rho$: if $A - \rho > 0$ then increasing ϕ increases the initial consumption

[5] According to equation (10), the average value of the growth rate is given by $\frac{1}{t}\mathbb{E}\left[\left(\alpha - \frac{\sigma^2}{2} f^*\right) t + \sqrt{f^*}\sigma W(t)\right]$ and should not be confused with the the value of the growth rate of the average value of $C(t)$, which is α, as shown in (11).

$C(0)$ and reduces the positive growth rate, while the opposite happens when $A - \rho < 0$. In both cases, the effect of increasing ϕ is to equalize consumption over time[6]. Our model embodies the same mechanism, and the effect of ϕ depends on the value of $\alpha - \rho$. When it is positive, the bigger ϕ is, the higher the initial consumption $C(0) = f * A(0)$ is, and the lower the average value of the growth rate (one can easily see that, insofar as $\alpha - \rho > 0$, $\frac{d\mathbb{E}[g]}{d\phi} = \frac{d}{d\phi}\left(\frac{\alpha(\theta-1}{\theta} + \frac{\alpha-\rho}{\phi\theta}\right) < 0$). The opposite happens when $\alpha - \rho < 0$.

We now characterize the value function in the corner solution case.

Proposition 3.5. *Let Hypothesis 3.1 be satisfied. Assume that*

$$\frac{\rho - \alpha(1-\phi)}{\frac{\sigma^2}{2}\theta\phi} \geq 1. \tag{12}$$

Then the value function of the problem can be written explicitly. It is equal to:

$$V_C(A) = \frac{1}{1-\theta}\beta_C A^{1-\theta}, \tag{13}$$

where

$$\beta_C := \left[\frac{1}{\rho}\left(\rho - \alpha(1-\phi) + \frac{\sigma^2}{2}\theta(1-\phi)\right)\right]^{-\frac{1-\theta}{1-\phi}}. \tag{14}$$

Moreover the optimal control is constant and deterministic, and it is given by:

$$f^*(t) = f^* := 1, \qquad \textit{for any } t \geq 0.$$

Proof. See Appendix. □

As underlined by the previous results, the structure of the value function is the same in the two cases (i.e. both are homogeneous of degree $1-\theta$), but of course the multiplicative constants differ. A corollary similar to Corollary 3.4 can be obtained in the corner case: the optimal dynamics of $A(t)$ is described by (10) where, instead of f^*, we have 1.

4 The value of biodiversity

In our model, the value of biodiversity comes from its ability to insure society against fluctuations in agricultural productivity. Our aim here is to make this property more precise, so

[6] Since the optimal consumption in the AK model is simply exponential, the sole "fluctuation" that can be reduced in order to homogenize consumption over time is measured by the absolute value of its growth rate.

that the value of biodiversity can be properly quantified. To do so, we build on the famous papers by Lucas [20], [21] on the welfare cost of fluctuations.

Following Lucas [20], [21], we define and compute the value from biodiversity as the welfare gain of biodiversity conservation. According to Lucas, the welfare cost of fluctuations is the percentage increase in consumption needed at all dates to compensate the representative agent for the presence of fluctuations, i.e. to make him indifferent between the actual consumption path, subject to fluctuations, and the corresponding deterministic consumption path. In the same spirit, we define here the welfare gain from biodiversity conservation as the percentage increase in consumption that a society considers acceptable compensation for exchanging the optimal situation (biodiversity level of $1-f^*$) for a situation with nil biodiversity and all land used for farming. It is thus defined as follows:

Definition 4.1. *The welfare gain from biodiversity conservation (i.e. the* value *of biodiversity) is the percentage increase in consumption society is willing to accept at all dates to give up the optimal level of biodiversity in favour of no biodiversity at all.*

The value function in the no-biodiversity case is denoted by $V_B(A)$ and is characterized in the following proposition. Let λ be the welfare gain defined above. According to Definition 4.1, λ satisfies:

$$V\left(A\right) = V_B((1+\lambda)\,A). \tag{15}$$

Observe that when (12) is satisfied, i.e. when we are at the optimum in the corner case $f^* = 1$, the optimal and the no-biodiversity solution are equivalent, so in this section we assume that (6) is verified, i.e. that we are in the interior case at the optimum. We will also make a technical assumption to be able to characterize the explicit form of the welfare in the no-biodiversity case.

Proposition 4.2. *Let Hypothesis 3.1 and Assumption (6) be satisfied and suppose that:*

$$\rho - \alpha(1-\phi) + \frac{\sigma^2}{2}\theta(1-\phi) > 0. \tag{16}$$

Then the welfare in the no-biodiversity case is given by:

$$V_B(A) = \frac{1}{1-\theta}\beta_B A^{1-\theta} \tag{17}$$

where

$$\beta_B := \left[\frac{1}{\rho}\left(\rho - \alpha(1-\phi) + \frac{\sigma^2}{2}\theta(1-\phi)\right)\right]^{-\frac{1-\theta}{1-\phi}}. \tag{18}$$

Proof. See Appendix. □

Remark 4.3. *Observe that, since (5) holds and we assume that (6) is verified, $V_B(A)$ is always smaller than $V(A)$.*[7] *Indeed, if we denote by $a := \rho - \alpha(1-\phi) > 0$ and $b := \frac{\sigma^2}{2}\theta\phi > 0$, thanks to (6) we have $a < b$. We distinguish two cases: $\phi > 1$ and $\phi < 1$ (we cannot have $\phi = 1$ because of Hypothesis 3.1).*

If $\phi > 1$ the function $(a,b) \mapsto a^{\phi}b^{1-\phi}$ is convex and then

$$\beta^{-\frac{1-\phi}{1-\theta}} = \frac{b}{\rho\phi}\left(\frac{a}{b}\right)^{\phi} = \frac{1}{\rho\phi}a^{\phi}b^{1-\phi} > \frac{1}{\rho\phi}\left(\phi a + (1-\phi)b\right) = \frac{1}{\rho}\left(a + (\frac{1}{\phi} - 1)b\right) = \beta_B^{-\frac{1-\phi}{1-\theta}}. \tag{19}$$

So, thanks to (5) we can conclude that $\beta < \beta_B$ (both are positive). Thanks to (5), when $\phi > 1$, we have $\theta > 1$ thus the factor $\frac{1}{1-\theta}$ is negative and, from the previous relation between β and β_B, we can conclude that $V_B(A) < V(A) < 0$ for any positive A.

Conversely, if $\phi < 1$ the function $(a,b) \mapsto a^{\phi}b^{1-\phi}$ is concave and then $\beta^{-\frac{1-\phi}{1-\theta}} < \beta_B^{-\frac{1-\phi}{1-\theta}}$. So $\beta > \beta_B$ and (now $1-\theta > 0$) $0 < V_B(A) < V(A)$ for any positive A.

Remark 4.4. *From Remark 4.3, since we assume that (5) and (6) are verified, both V and V_B are increasing functions of A, and as we showed that $V_B(A)$ is always smaller than $V(A)$ then λ is always positive.*

Proposition 4.5. *The welfare gain of biodiversity conservation is:*

$$\lambda = f^*\left(\phi + \frac{1-\phi}{f^*}\right)^{\frac{1}{1-\phi}} - 1. \tag{20}$$

Proof. See Appendix. □

Lemma 4.6. *The value of biodiversity λ is an increasing function of the intrinsic volatility of agricultural productivity and of risk aversion. However, the effect of aversion to fluctuations on the value of biodiversity is ambiguous.*

[7] This result already follows from the fact that $V(A)$ is the maximum of the welfare varying the control among a set containing in particular the control $f \equiv 1$, the control chosen in the benchmark. We give in any case a direct argument because the expressions of β and β_B are not, at first glance, immediately comparable.

Proof. See Appendix. □

The first two results are intuitive and fit well with the effects of intrinsic volatility and risk aversion on the conversion rate. Higher intrinsic volatility of agricultural productivity and higher risk aversion result in a lower optimal conversion rate, i.e. more insurance, and a higher value of biodiversity, i.e. a higher welfare gain of biodiversity conservation.

To further investigate the role of aversion to fluctuations, it is useful to examine the case where the optimal conversion rate is very high (f^* close to 1).

Lemma 4.7. *For f^* close to 1,*

$$\lambda \simeq \frac{\phi}{2}(1-f^*)^2 \tag{21}$$

and

$$\frac{d\lambda}{d\phi} \simeq \frac{1}{2}\frac{1-f^*}{\frac{\sigma^2}{2}\theta\phi}\left[\rho-\alpha+\left(\frac{\sigma^2}{2}\theta-\alpha\right)\phi\right]. \tag{22}$$

Proof. See Appendix. □

When the optimal conversion rate is close to 1, the value of biodiversity is proportional to the square of the share of the land optimally devoted to conserving biodiversity.

According to Assumption (6), we have: $\left(\frac{\sigma^2}{2}\theta-\alpha\right)\phi > \rho-\alpha$; likewise, according to Assumption (16), we have: $\left(\frac{\sigma^2}{2}\theta-\alpha\right)\phi < \rho-\alpha+\frac{\sigma^2}{2}\theta$. Then:

$$\frac{1-f^*}{\frac{\sigma^2}{2}\theta\phi}(\rho-\alpha) < \frac{d\lambda}{d\phi} < \frac{1-f^*}{\frac{\sigma^2}{2}\theta\phi}\left[\rho-\alpha+\frac{\sigma^2}{4}\theta\right].$$

The first inequality implies that if $\rho-\alpha>0$, then $\frac{d\lambda}{d\phi}>0$. The second one implies that if $\rho-\alpha+\frac{\sigma^2}{4}\theta<0$, then $\frac{d\lambda}{d\phi}<0$; this requires $\rho-\alpha<0$ and is moreover all the more likely since intrinsic volatility σ and risk aversion θ are small.

At first glance, it may seems as if the signs of the effect of increasing aversion to fluctuations (or in fact any other parameter) on the value of biodiversity and on the share of land devoted to biodiversity conservation should be the same. This is actually what happens when the discount rate ρ is higher than the trend of productivity α. Then, increasing aversion to fluctuations increases both the share of land devoted to biodiversity conservation and the value of biodiversity. It is also what happens when the discount rate ρ is lower than the trend of productivity α, that is when the society is patient and has on average good economic prospects, and when intrinsic volatility and risk aversion are small. Then, increasing aversion to fluctuations decreases both the share of land devoted to biodiversity conservation and the value of biodiversity. There is little need for insurance in these circumstances. Nevertheless, if we look at equation (21), we can see that two effects are at work: on the one hand, a direct effect (the term $\phi/2$) through which increasing ϕ increases λ, and on the other hand, an indirect effect (the term $(1-f^*)^2$) of the same sign as the effect of ϕ on the share of land devoted to biodiversity conservation. When the trend of productivity α is higher than the discount rate ρ, the two effects are discordant. The first effect is all the more likely to prevail since intrinsic volatility and risk aversion are high, as becomes clear if we write equation (22) as:

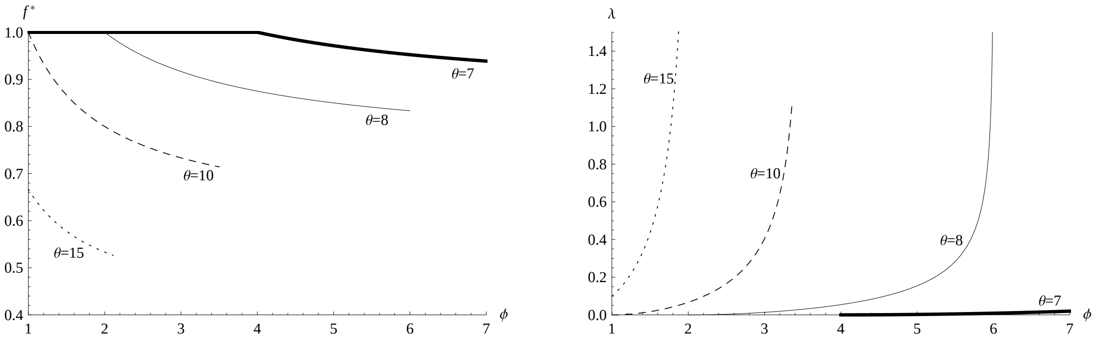
$$\frac{d\lambda/\lambda}{d\phi/\phi} \simeq 1 - \frac{2(\alpha-\rho)}{(\alpha-\rho)\left(\frac{\sigma^2}{2}\theta-\alpha\right)\phi}.$$

In this case, increasing aversion to fluctuations decreases the share of land devoted to biodiversity conservation, while at the same time it increases the value of biodiversity.

Simulations allow us to check that these results hold when f^* is not assumed to be close to 1 (see Figure 1 for a case where $\alpha < \rho$, Figures 2 and 3 for $\alpha > \rho$). They also allow us to exhibit sets of parameters σ, θ and ϕ satisfying the constraints imposed in the model, such that, when $\alpha > \rho$, we have $\frac{d\lambda}{d\phi} > 0$ (see Figure 3). A general observation from these simulations is that the value of biodiversity is very sensitive to aversion to fluctuations, and can become very high when f^* is not close to one, i.e. when it is optimal to allocate a significant share of land to biodiversity conservation.

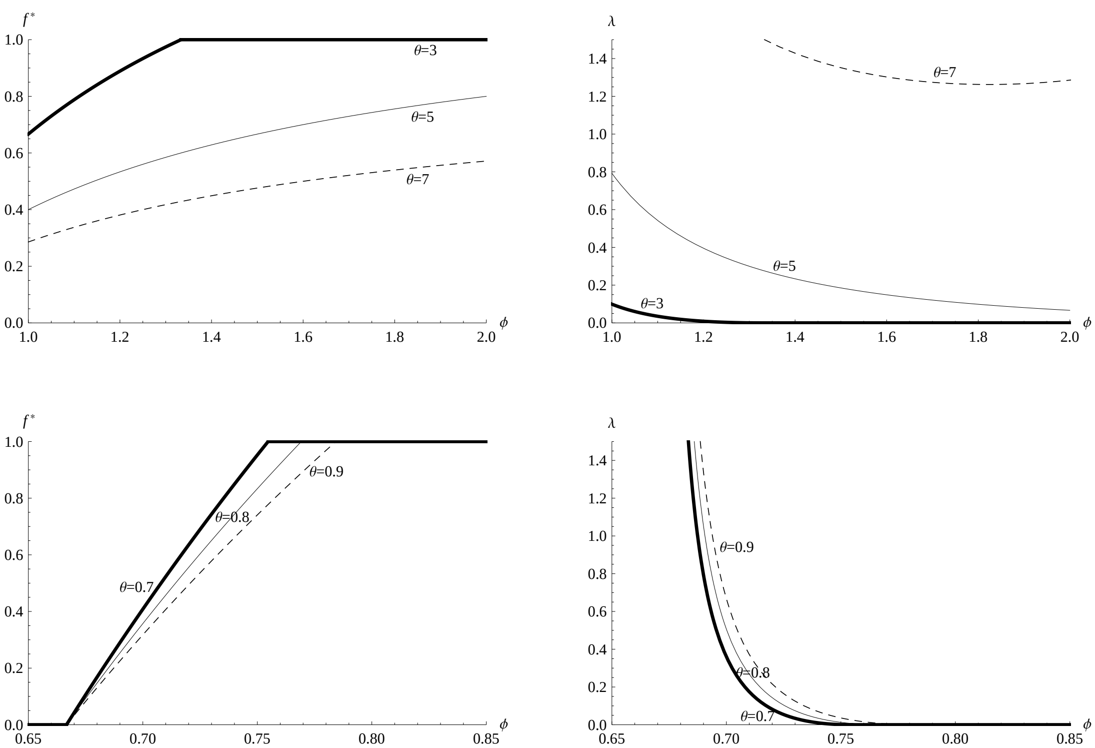
5 The conversion of a common-property resource: volatility externalities at work

We now consider a decentralized version of the model where two farmers can appropriate land, a common property resource, for farming purposes. The two farmers are indexed by $i \in \{1,2\}$. The total amount of land available is still normalized to 1. Farmer i ($i \in \{1,2\}$)



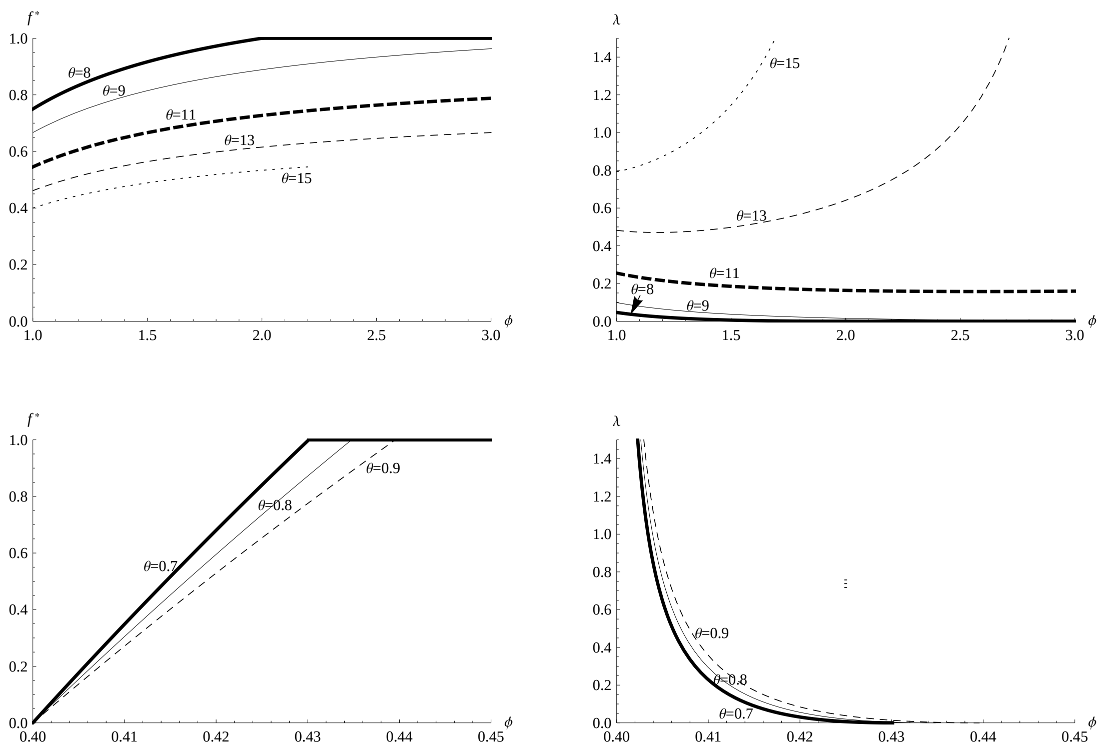
(a) Optimal conversion rate (b) Value of biodiversity

Figure 1: Optimal conversion rate and biodiversity value as a function of aversion to fluctuations for $\alpha = 0.03$, $\rho = 0.05$, $\sigma = 0.1$.



(a) Optimal conversion rate (b) Value of biodiversity

Figure 2: Optimal conversion rate and biodiversity value as a function of aversion to fluctuations for $\alpha = 0.03$, $\rho = 0.01$, $\sigma = 0.1$.



(a) Optimal conversion rate

(b) Value of biodiversity

Figure 3: Optimal conversion rate and biodiversity value as a function of aversion to fluctuations for $\alpha = 0.05$, $\rho = 0.03$, $\sigma = 0.1$.

can appropriate some share $f_i(t)$ of this total amount, although that the following constraint needs to be verified:

$$f_1(t) + f_2\,(t) \le 1, \qquad t \ge 0.$$

So the set of admissible strategies of player i ($i \in \{1,2\}$) depends on the strategy chosen by the other player (denoted "f_{-i}"). More precisely, given $f_{-i}(t)$, it is expressed as

$$\mathcal{U}^i_{f_{-i}} := \left\{ f_i(\cdot)\colon [0,+\infty)\times\Omega \to [0,1] \ : f_i(\cdot) \text{ is } \mathcal{F}_t\text{−progr. meas. and } f_1(t) + f_2\,(t) \le 1 \right\}. \tag{23}$$

Definition 5.1. *A couple* $(f_1(\cdot), f_2(\cdot))$ *of* $[0,1]$*-valued* $\mathcal{F}_t$*-adapted processes is said to be an* admissible couple of strategies *if, for any* $i \in \{1,2\}$, $f_i \in \mathcal{U}^i_{f_{-i}}$.

Given an admissible couple of strategies $(f_1(\cdot), f_2(\cdot))$, the total share of land devoted to farming at time t is $f_1(t) + f_2\,(t)$ while the share of land used to preserve biodiversity at time t is $(1 - f_1(t) - f_2\,(t))$.

We assume that farmers 1 and 2 are potentially heterogeneous in terms of their farming activities (we will assume $\alpha_1 \le \alpha_2$), and in terms of their discount rates denoted ρ_1 and ρ_2. The productivity of a unit of land appropriated by farmer i for farming at time $t \ge 0$ is given by:

$$\begin{cases} \mathrm{d}A_i(t) = \alpha_i A_i(t)\,\mathrm{d}t + \sqrt{f_1\,(t) + f_2\,(t)}\sigma_i A_i(t)\,\mathrm{d}W(t) \\ A_i(0) = A_0^i. \end{cases} \tag{24}$$

The volatility externality comes from the fact that the conversion decisions of the two farmers affect each one's volatility of agricultural productivity.

Given an $\mathcal{F}_t$-adapted $[0,1]$-valued strategy $\overline{f}_2(\cdot)$ for player 2, we will say that $\overline{f}_1(\cdot) \in \mathcal{U}^1_{f_2}$ is a best response of player 1 to $\overline{f}_2(\cdot)$ if it is an optimal strategy (among all the strategies of $\mathcal{U}^1_{f_2}$) for the optimization problem characterized by the state equation (24) (where we consider $i = 1$ and $f_2(\cdot) = \overline{f}_2(\cdot)$) and the Epstein-Zin-Weil utility function with a constant relative risk aversion θ, an elasticity of intertemporal substitution ϕ^{-1} and a discount rate ρ. Similarly, we define a best response of player 2 to some strategy $\overline{f}_1(\cdot)$ of player 1. A couple of $[0,1]$-valued, $\mathcal{F}_t$-adapted processes $(f_1(\cdot), f_2(\cdot))$ is said to be a Nash equilibrium if $f_1(\cdot)$ is a best response to $f_2(\cdot)$ and $f_2(\cdot)$ is a best response to $f_1(\cdot)$.

Proposition 5.2. *Provided that* $\frac{1-\phi}{1-\theta} > 0$, *and that*

$$\rho_1 > \max\left\{\alpha_1(1-\phi), \left(\alpha_1 - \frac{\sigma^2}{2}\theta\right)(1-\phi)\right\}, \tag{25}$$

$$\rho_2 > \max\left\{\alpha_2(1-\phi), \left(\alpha_2 - \frac{\sigma^2}{2}\theta\right)(1-\phi)\right\}, \tag{26}$$

the best response of farmer 1 for a constant and deterministic strategy $f_2(t) = f_2 \in (0,1)$, $t \geq 0$ *of farmer 2 is the constant and deterministic strategy with value*

$$f_1 = \min\left(\frac{\rho_1 - \left(\alpha_1 - \frac{\sigma^2}{2}\theta f_2\right)(1-\phi)}{\frac{\sigma^2}{2}\theta\phi}, 1 - f_2\right) \tag{27}$$

and similarly for farmer 2.

Proposition 5.3. *If, in addition,*

$$\phi > \frac{1}{2} \tag{28}$$

and

$$\frac{\phi}{2\phi-1}\frac{(\rho_1+\rho_2-(\alpha_1+\alpha_2)(1-\phi))}{\frac{\sigma^2}{2}\theta\phi} \in (0,1) \tag{29}$$

then the couple of constant strategies $(f_1(t), f_2(t)) = (\widetilde{f_1}, \widetilde{f_2})$ *for any* $t \geq 0$, *where*

$$\widetilde{f_1} = \frac{\phi}{2\phi-1}\left[\phi\frac{\rho_1-\alpha_1(1-\phi)}{\frac{\sigma^2}{2}\theta\phi} + (1-\phi)\frac{\rho_2-\alpha_2(1-\phi)}{\frac{\sigma^2}{2}\theta\phi}\right] \tag{30}$$

$$\widetilde{f_2} = \frac{\phi}{2\phi-1}\left[\phi\frac{\rho_2-\alpha_2(1-\phi)}{\frac{\sigma^2}{2}\theta\phi} + (1-\phi)\frac{\rho_1-\alpha_1(1-\phi)}{\frac{\sigma^2}{2}\theta\phi}\right] \tag{31}$$

is a Nash equilibrium. It is the unique Nash equilibrium in constant and deterministic strategies.

Proof. See Appendix. □

When the assumptions of Proposition 5.2 are verified the total amount of land devoted to farming is:

$$\widetilde{f} = \frac{\phi}{2\phi-1}\frac{(\rho_1+\rho_2-(\alpha_1+\alpha_2)(1-\phi))}{\frac{\sigma^2}{2}\theta\phi} \tag{32}$$

Denoting by ρ the average discount rate of the two players ($\rho = (\rho_1+\rho_2)/2$) and α the average trend of productivity in the economy ($\alpha = (\alpha_1+\alpha_2)/2$) allows us to write the total amount of land devoted to farming as:

$$\widetilde{f} = \frac{2\phi}{2\phi-1}\frac{\rho-\alpha(1-\phi)}{\frac{\sigma^2}{2}\theta\phi}. \tag{33}$$

If we compare this expression with (9) we observe that (since we suppose that condition (28) is verified) the total area of land devoted to farming in the 2-player case is larger than optimal

$$\widetilde{f} = \frac{2\phi}{2\phi - 1} f^* > f^*. \tag{34}$$

Note that the factor $\frac{2\phi}{2\phi-1}$ measuring the increment in area devoted to farming is decreasing with aversion to fluctuations ϕ and tends to 1 when ϕ is very high. Hence high aversion to fluctuations tends to outweigh the incentive for the two farmers to appropriate too much land for farming.

If we extend the analysis to the corner solution case where condition (29) is violated and $\frac{\phi}{2\phi-1} \frac{\rho_1+\rho_2-(\alpha_1+\alpha_2)(1-\phi)}{\frac{\sigma^2}{2}\theta\phi} \geq 1$, a continuous set of deterministic and constant Nash equilibria arises, any of which means that zero land is devoted to biodiversity conservation.

An analogous N-player version of the game can be studied. In this case, for any choice of parameters, letting N go to infinity induces a no-biodiversity-conservation outcome. It can be seen as the usual situation when a large number of agents interact. This fact suggests that the reference level $f = 1$ adopted in Section 4 is indeed a good benchmark.

6 Conclusion

This paper presents a stylized dynamic model including a particular incentive for biodiversity conservation: its value as insurance against fluctuations in agricultural productivity. Producing food requires land, and increasing the share of total land devoted to farming mechanically reduces the share of land devoted to biodiversity conservation. However, the safeguarding of a greater number of species ensures better ecosystem services – pollination, flood control, pest control, etc. which in turn ensure lower volatility of agricultural productivity. The optimal conversion/conservation rule is explicitly characterized, as well as the value (in terms of the welfare gain from biodiversity conservation) of biological diversity. The Epstein-Zin-Weil specification of the utility function allows us to disentangle the effects of risk aversion and aversion to fluctuations. A two-player game extension of the model highlights the effect of volatility externalities and the Paretian sub-optimality of the decentralized choices.

At least two interesting extensions of our work could be explored.

First, there is the ongoing debate on rent sparing versus rent sharing initiated by Green *et al.* [14]. The question is whether agriculture should be concentrated on intensively farmed land in order to conserve more natural spaces for biodiversity, or whether it should be extensive, less productive, and wildlife-friendly. Our framework can encompass both cases. It would nevertheless be interesting to distinguish between the two management practices and their different consequences in terms both of average agricultural productivity and of volatility.

Second, we consider here that the economy does not have access to financial insurance and that there are no means of crop saving/storage. However, if a financial insurance system and/or storage were available, farmers could insure against adverse outcomes by other means than biodiversity conservation. Quaas and Baumgartner [26] study this problem and show in a static framework that the twos types of insurance (natural and financial) are substitutes. It would be interesting to see whether their result holds in a dynamic framework, and to disentangle the impact of risk aversion and aversion to fluctuations on decision-making.

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Appendix: Proof of the results

Proof of Proposition 3.2. By Proposition 9 and Appendix C of Duffie and Epstein [7]) the value function V can be characterized as the solution of the following Hamilton-Jacobi-Bellman (HJB) equation:

$$0 = \sup_{f\in[0,1]} \left(\alpha A V'(A) + F(fA, V) + \frac{1}{2} f\sigma^2 A^2 V''(A) \right), \tag{35}$$

where V' and V'' are the first and the second derivative of $V(A)$ and $F(C,V)$ is the aggregator defined in (4). This expression can be rewritten as:

$$\rho\frac{1-\theta}{1-\phi}V(A) = \sup_{f\in[0,1]} \left(\alpha A V'(A) + \frac{\rho}{1-\phi}(1-\theta)V(A) \left(\frac{fA}{((1-\theta)V(A))^{\frac{1}{1-\theta}}} \right)^{1-\phi} + \frac{1}{2} f\sigma^2 A^2 V''(A) \right). \tag{36}$$

Our aim is to prove that the function defined in (7) is a solution of such an equation. We try to find a solution of the form

$$V(A) = \frac{1}{1-\theta}\beta A^{1-\theta} \tag{37}$$

for some $\beta > 0$. We have

$$V'(A) = \beta A^{-\theta} \tag{38}$$

and

$$V''(A) = -\theta\beta A^{-\theta-1}.$$

So V of the prescribed form is a solution if and only if:

$$\begin{aligned} 0 = \sup_{f\in[0,1]} \Bigg(-\frac{\rho}{1-\phi}\beta A^{1-\theta} + \frac{\rho}{1-\phi}\beta A^{1-\theta} \left(\frac{fA}{\left(\beta A^{1-\theta}\right)^{\frac{1}{1-\theta}}} \right)^{1-\phi} + \alpha A\beta A^{-\theta} - \frac{1}{2} f\sigma^2 A^2 \theta\beta A^{-\theta-1} \Bigg) \\ = \beta A^{1-\theta} \sup_{f\in[0,1]} \left(-\frac{\rho}{1-\phi} + \frac{\rho}{1-\phi} \left(\frac{fA}{\left(\beta A^{1-\theta}\right)^{\frac{1}{1-\theta}}} \right)^{1-\phi} + \alpha - f\frac{\sigma^2}{2}\theta \right). \end{aligned} \tag{39}$$

The f that maximizes this Hamiltonian is given by

$$f^* = \left(\frac{1}{\rho} \frac{\sigma^2}{2} \theta \beta^{\frac{1-\phi}{1-\theta}} \right)^{-1/\phi} \tag{40}$$

(after finding the expression of β we will be able to show that this expression is indeed always in $(0,1)$). Using this expression in (39) and simplifying $\beta A^{1-\theta}$, we can see that a function of the form (37) is a solution of the HJB equation if and only if:

$$0 = -\frac{\rho}{1-\phi} + \frac{\rho}{1-\phi} \left(\frac{\left(\frac{1}{\rho}\frac{\sigma^2}{2}\theta\beta^{\frac{1-\phi}{1-\theta}} \right)^{-1/\phi}}{\beta^{\frac{1}{1-\theta}}} \right)^{1-\phi} + \alpha - \left(\frac{1}{\rho}\frac{\sigma^2}{2}\theta\beta^{\frac{1-\phi}{1-\phi}} \right)^{-1/\theta} \frac{\sigma^2}{2}\theta, \tag{41}$$

so (after some computations) if and only if:

$$\beta = \left(\frac{\rho \left(\frac{\rho - \alpha(1-\phi)}{\frac{\sigma^2}{2}\theta\phi} \right)^{-\phi}}{\frac{\sigma^2}{2}\theta} \right)^{\frac{1-\theta}{1-\phi}}.$$

Using this expression and (6) it can easily be seen that the expression of f^* given in (40) is always in $(0,1)$ and so the control $f(t) \equiv f^*$ is admissible. According to the general theory (see again Proposition 9 and Appendix C of Duffie and Epstein [7]), being is obtained as the feedback provided by a solution of the HJB of the problem it is the optimal control of the problem.

We now show that condition (5) guarantees respect of the transversality condition.

The term $\frac{\rho(1-\theta)}{1-\phi}V(A)$ appearing in the HJB (36) for the recursive utility corresponds to the standard term $\rho V(A)$ appearing in the standard HJB for separable expected utility functionals. Thus the counterpart of the standard discount $e^{-\rho t}$ is, in the recursive utility setting, given by $e^{-\rho\frac{1-\theta}{1-\phi}t}$ (recall that, as already observed, if we choose $\phi = \theta$, the recursive utility case reduces to the separable expected utility).

Indeed to prove the verification result for our infinite horizon case, the argument needs to follow Proposition 9 of Duffie and Epstein [7] using the function $\tilde{V}(t,A) = e^{-\rho\frac{1-\theta}{1-\phi}t}V(A)$ ($V(A)$ being the value function of our problem, characterized in Proposition 3.2), letting the final time T tend to infinity. Given the sign of $V(A)$ (positive for $\theta \in (0,1)$ and negative for $\theta > 1$) we need to distinguish two cases:

(i) *if* $\theta \in (0,1)$ we need to show that $\lim_{T\to+\infty} \mathbb{E}\left[\tilde{V}(T, A(T))\right] = 0$ for any admissible trajectory $A(\cdot)$. We have that $\mathbb{E}\tilde{V}(T,A(T)) = e^{-\rho\frac{1-\theta}{1-\phi}T}\frac{1}{1-\theta}\beta\mathbb{E}\left[(A(T))^{1-\theta}\right]$. It is obvious that the maximum possible value for $\mathbb{E}\left[(A(T))^{1-\theta}\right]$ is attained when we choose $f(t)$ always equal to 0. In this case we have:

$$e^{-\rho\frac{1-\theta}{1-\phi}T}\frac{1}{1-\theta}\beta\mathbb{E}\left[(A(T))^{1-\theta}\right] = \frac{1}{1-\theta}\beta A_0^{1-\theta}e^{-\rho\frac{1-\theta}{1-\phi}T+\alpha(1-\theta)T} = \frac{1}{1-\theta}\beta A_0^{1-\theta}e^{\left(\frac{1-\theta}{1-\phi}(-\rho+\alpha(1-\phi))\right)T}.$$

So, since $\theta \in (0,1)$, its limit for $T \to \infty$ is equal to zero if and only if $\frac{1-\theta}{1-\phi}(-\rho+\alpha(1-\phi)) < 0$. This condition (which corresponds to the usual condition $\rho > (1-\theta)\alpha$ that obtains in the separable expected utility case) is implied by (5) (multiplying the two conditions).

(ii) *if* $\theta > 1$ we need to show that there exists at least an admissible trajectory $A(\cdot)$ satisfying $\lim_{T\to+\infty} \mathbb{E}\left[\tilde{V}(T, A(T))\right] \neq -\infty$. We consider as an admissible control $f(t) = f^*$ for any $t \geq 0$. We have then:

$$A(T) = A_0 e^{\alpha - \frac{\sigma^2}{2}f^*T + \sqrt{f^*}\sigma W(T)}$$

and then (using the expression for the moments of the log-normal distributions):

$$\mathbb{E}[A(T)^{1-\sigma} = A_0^{1-\sigma}e^{(1-\theta)(\alpha-\theta f^*\frac{\sigma^2}{2}}$$

so that (by writing explicitly the value of f^* given in the text of the proposition):

$$\begin{aligned}\mathbb{E}\left[\tilde{V}(T,A(T))\right] &= e^{-\rho\frac{1-\theta}{1-\phi}T}\frac{1}{1-\theta}\beta\mathbb{E}\left[(A(T))^{1-\theta}\right] = \frac{1}{1-\theta}\beta A_0^{1-\theta}e^{-\rho\frac{1-\theta}{1-\phi}T}e^{\frac{(\alpha-\rho)(1-\theta)}{\phi}T}\\ &= \frac{1}{1-\theta}\beta A_0^{1-\theta}e^{\frac{1-\theta}{1-\phi}(-\rho+\alpha(1-\phi))T}.\end{aligned}$$

Its limit for $T \to \infty$ is equal to zero if and only if $\frac{1-\theta}{1-\phi}(-\rho+\alpha(1-\phi)) < 0$. As before, this condition is implied by (5).

$\square$

Proof of Proposition 3.5. The proof follows the lines of that of Proposition 3.2. Again we have to find a solution of the HJB (35) so we need to prove that, if condition (12) is verified, the following equation is satisfied:

$$0 = \beta_C A^{1-\theta}\sup_{f\in[0,1]}\left(-\frac{\rho}{1-\phi} + \frac{\rho}{1-\phi}\left(\frac{fA}{\left(\beta_C A^{1-\theta}\right)^{\frac{1}{1-\theta}}}\right)^{1-\phi} + \alpha - f\frac{\sigma^2}{2}\theta\right), \tag{42}$$

which, using the expression of β_C given in (14) and simplifying the term $\beta_C A^{1-\theta}$, becomes,

$$
\begin{aligned}
0 &= \sup_{f\in[0,1]}\left[-\frac{\rho}{1-\phi}+\frac{\rho}{1-\phi}\left(\frac{f}{\left(\left(\frac{\rho-\alpha(1-\phi)}{\rho}+\frac{1}{\rho}\frac{\sigma^2}{2}\theta(1-\phi)\right)^{-\frac{1-\theta}{1-\phi}}\right)^{\frac{1}{1-\theta}}}\right)^{1-\phi}+\alpha-f\frac{\sigma^2}{2}\theta\right]\\
&= \sup_{f\in[0,1]}\left[-\frac{\rho}{1-\phi}+\left(f^{1-\phi}\left(\frac{\rho}{1-\phi}-\alpha+\frac{\sigma^2}{2}\theta\right)\right)+\alpha-f\frac{\sigma^2}{2}\theta\right].
\end{aligned}
$$

It can easily be seen that the function $f\mapsto\left(f^{1-\phi}\left(\frac{\rho}{1-\phi}-\alpha+\frac{\sigma^2}{2}\theta\right)\right)-f\frac{\sigma^2}{2}\theta$, defined on $(0,+\infty)$, is increasing in the interval $[0,\bar{f}]$ where

$$
\bar{f}:=\left(\frac{\frac{\sigma^2}{2}\theta}{(\rho-\alpha(1-\phi))+\frac{\sigma^2}{2}\theta(1-\phi)}\right)^{-\theta},
$$

and then decreasing. Since (12) is verified, $\bar{f}\geq 1$ so the maximum of the function restricted to the interval $[0,1]$ is in $f^*=1$. Using this fact we can rewrite (43) as:

$$
0=-\frac{\rho}{1-\phi}+\left(\frac{\rho}{1-\phi}-\alpha+\frac{\sigma^2}{2}\theta\right)+\alpha-\frac{\sigma^2}{2}\theta
$$

which can easily be seen to be true. Concluding as in the proof of Proposition 3.2, we prove that the function defined in (13) is indeed the value function of the problem and that $f^*(t)\equiv f^*:=1$ is the optimal control of the problem. □

Proof of Proposition 4.2. The argument is exactly the same as that we used to prove Proposition 3.2, but here the only possible choice of f is 1 so the HJB equation becomes:

$$
\rho\frac{1-\theta}{1-\phi}V(A)=\left[\alpha AV'(A)+\frac{\rho}{1-\phi}(1-\theta)V(A)\left(\frac{fA}{((1-\theta)V(A))^{\frac{1}{1-\theta}}}\right)^{1-\phi}+f\frac{\sigma^2}{2}A^2V''(A)\right]_{f=1}.
$$

The explicit solution of this equation (which we find by inspection as in the proof Proposition 3.2) is the welfare in the benchmark (it is the function of the problem where only $f=1$ can be chose). Not surprisingly, its form is the same as the value function of the corner case (Proposition 3.5) where the optimal control was always chosen equal to 1. □

Proof of Proposition 4.5. We use the expressions of V and V_C to compute $V(A)=V_C((1+\lambda)A)$. We obtain:

$$
\left(\frac{\rho\left(\frac{\rho-\alpha(1-\phi)}{\frac{\sigma^2}{2}\theta\phi}\right)^{-\phi}}{\frac{\sigma^2}{2}\theta}\right)^{\frac{1-\theta}{1-\phi}}A^{1-\theta}=\left(\frac{\rho-\alpha(1-\phi)}{\rho}+\frac{1}{\rho}\frac{\sigma^2}{2}\theta(1-\phi)\right)^{-\frac{1-\theta}{1-\phi}}(1+\lambda)^{1-\theta}A^{1-\theta},
$$

i.e.

$$\left(f^{*-\phi}\left[f^{*}\phi+(1-\phi)\right]\right)^{\frac{1-\theta}{1-\phi}}=(1+\lambda)^{1-\theta},$$

i.e. (20). $\square$

Proof of Lemma 4.6.

$$\frac{d\lambda}{df^*}=\phi\left(\phi+\frac{1-\phi}{f^*}\right)^{\frac{\phi}{1-\phi}}\left(1-\frac{1}{f^*}\right)<0.$$

Thus:

$$\begin{aligned}
\frac{d\lambda}{d\theta} &= \frac{d\lambda}{df^*}\frac{df^*}{d\theta}=-2\frac{d\lambda}{df^*}\frac{\rho-\alpha(1-\phi)}{\sigma^2\theta^2\phi}>0\\
\frac{d\lambda}{d\sigma^2} &= \frac{d\lambda}{df^*}\frac{df^*}{d\sigma^2}=-2\frac{d\lambda}{df^*}\frac{\rho-\alpha(1-\phi)}{\sigma^4\theta\phi}>0\\
\frac{d\lambda}{d\phi} &= \frac{d\lambda}{df^*}\frac{df^*}{d\phi}+\frac{d\lambda}{d\phi}=-2\frac{d\lambda}{df^*}\frac{\rho-\alpha}{\sigma^2\theta\phi^2}+\\
& f^*\left(\phi+\frac{1-\phi}{f^*}\right)^{\frac{1}{1-\phi}}\left[\frac{1}{(1-\phi)^2}\ln\left(\phi+\frac{1-\phi}{f^*}\right)+\frac{1}{1-\phi}\frac{1-\frac{1}{f^*}}{\phi+\frac{1-\phi}{f^*}}\right].
\end{aligned}$$

$\square$

Proof of Lemma 4.7. It is sufficient to use the Taylor expansion of the expression of λ. The first order term vanishes while the second gives the claim. $\square$

Proof of Proposition 5.2. We start by characterizing the optimal response of player 1 given a certain fixed deterministic constant strategy f_2 of player 2. To do this, we solve the dynamic optimization problem where the value of f_2 is considered fixed (and deterministic). We again use Proposition 9 and Appendix C of Duffie and Epstein [7]) the value function V of player 1's problem can be characterized as the solution of the following Hamilton-Jacobi-Bellman (HJB) equation:

$$0=\sup_{f_1\in[0,1-f_2]}\left(\alpha_1A_1V'(A_1)+F(f_1A_1,V)+\frac{1}{2}(f_1+f_2)\sigma^2A_1^2V''(A_1)\right). \tag{43}$$

where V' and V'' are the first and the second derivative of $V(A)$ and $F(C,V)$ is the aggregator defined in (4). This expression can be rewritten as

$$\begin{aligned}
0 &= \sup_{f_1\in[0,1-f_2]}\left(\alpha_1A_1V'(A_1)+\frac{\rho_1}{1-\phi}(1-\theta)V(A_1)\left[\left(\frac{f_1A_1}{\left((1-\theta)V(A_1)\right)^{\frac{1}{1-\theta}}}\right)^{1-\phi}-1\right]\right.\\
&+ \left.(f_1+f_2)\frac{\sigma^2}{2}A_1^2V''(A_1)\right).
\end{aligned}$$

To prove that the function defined in (7) is a solution of such an equation, we try to find a solution of the form

$$V(A) = \frac{1}{1-\theta}\beta_1 A^{1-\theta} \tag{44}$$

for some $\beta_1 > 0$.

So V of the prescribed form is a solution if and only if:

$$\begin{aligned} 0 &= \sup_{f_1 \in [0,1-f_2]} \left(\frac{\rho_1}{1-\phi}\beta_1 A_1^{1-\theta} \left(\frac{f_1}{(\beta)^{\frac{1}{1-\theta}}} \right)^{1-\phi} - f_1 \frac{\sigma^2}{2} A_1^2 \theta \beta_1 A_1^{-\theta-1} \right) \\ &+ \alpha_1 A \beta_1 A^{-\theta} - \frac{\rho_1}{1-\phi}\beta_1 A_1^{1-\theta} - f_2 \frac{\sigma^2}{2} A_1^2 \theta \beta_1 A_1^{-\theta-1} \\ &= \beta_1 A_1^{1-\theta} \left[\sup_{f_1 \in [0,1-f_2]} \left(\frac{\rho_1}{1-\phi} \left(\frac{f_1}{(\beta)^{\frac{1}{1-\theta}}} \right)^{1-\phi} - f_1 \frac{\sigma^2}{2}\theta \right) + \alpha_1 - \frac{\rho_1}{1-\phi} - f_2 \frac{\sigma^2}{2}\theta \right]. \end{aligned}$$

The interior f_1 that maximizes this Hamiltonian is given by:

$$f_1^* = \left(\frac{1}{\rho_1} \frac{\sigma^2}{2} \theta \beta_1^{\frac{1-\phi}{1-\theta}} \right)^{-1/\phi}. \tag{45}$$

We then replace it to find β_1:

$$0 = \frac{\rho_1}{1-\phi} \left(\frac{\left(\frac{1}{\rho_1}\frac{\sigma^2}{2}\theta\beta_1^{\frac{1-\phi}{1-\theta}} \right)^{-1/\phi}}{\beta_1^{\frac{1}{1-\theta}}} \right)^{1-\phi} - \left(\frac{1}{\rho_1}\frac{\sigma^2}{2}\theta\beta_1^{\frac{1-\phi}{1-\theta}} \right)^{-1/\phi} \frac{\sigma^2}{2}\theta + \alpha_1 - \frac{\rho_1}{1-\phi} - \frac{\sigma^2}{2}\theta, \tag{46}$$

i.e.

$$\frac{\phi}{1-\phi}\left(\frac{1}{\rho_1}\right)^{-1/\phi} \left(\frac{\sigma^2}{2}\theta\right)^{-\frac{(1-\phi)}{\phi}} \left(\beta_1^{\frac{1-\phi}{1-\theta}}\right)^{-1/\phi} = \frac{\rho_1 - [\alpha_1 - f_2\theta\theta](1-\phi)}{1-\phi}, \tag{47}$$

so (after some computations) V of the prescribed form is a solution if and only if:

$$\beta_1 = \left(\frac{\rho_1 \left(\frac{\rho_1 - \left(\alpha_1 - f_2\frac{\sigma^2}{2}\theta\right)(1-\phi)}{\frac{\sigma^2}{2}\theta\phi} \right)^{-\phi}}{\frac{\sigma^2}{2}\theta} \right)^{\frac{1-\theta}{1-\phi}}.$$

Observe that condition (25) guarantees that the quantity $\rho_1 - \left(\alpha_1 - f_2\frac{\sigma^2}{2}\theta\right)(1-\phi)$ is positive for any choice of $f_2 \in [0,1]$

In a similar way, we can find the optimal response of player 2 given a certain fixed deterministic constant strategy f_1 of player 1. We find:

$$f_2^* = \left(\frac{1}{\rho_2} \frac{\sigma^2}{2} \theta \beta_2^{\frac{1-\phi}{1-\theta}} \right)^{-1/\phi} \tag{48}$$

with

$$\beta_2 = \left(\frac{\rho_2 \left(\frac{\rho_2 - \left(\alpha_2 - f_1 \frac{\sigma^2}{2}\theta\right)(1-\phi)}{\frac{\sigma^2}{2}\theta\phi} \right)^{-\phi}}{\frac{\sigma^2}{2}\theta} \right)^{\frac{1-\theta}{1-\phi}}.$$

Thus if we find a couple of values $(f_1,\ f_2)$ that solves

$$\begin{aligned} f_1^* &= \min\left(\frac{\rho_1 - \left(\alpha_1 - f_2^* \frac{\sigma^2}{2}\theta\right)(1-\phi)}{\frac{\sigma^2}{2}\theta\phi}, 1 - f_2^* \right) \\ f_2^* &= \min\left(\frac{\rho_2 - \left(\alpha_2 - f_1^* \frac{\sigma^2}{2}\theta\right)(1-\phi)}{\frac{\sigma^2}{2}\theta\phi}, 1 - f_1^* \right) \end{aligned}$$

the couple of constant deterministic strategies with values $(f_1,\ f_2)$ is a Nash equilibrium.

In particular, if (28) and (29) are verified we find the internal solution characterized by:

$$\begin{aligned} f_1^* \frac{\sigma^2}{2}\theta &= \frac{1}{2\phi - 1}\left[\phi(\rho_1 - \alpha_1(1-\phi)) + (1-\phi)(\rho_2 - \alpha_2(1-\phi))\right] \\ f_2^* \frac{\sigma^2}{2}\theta &= \frac{1}{2\phi - 1}\left[\phi(\rho_2 - \alpha_2(1-\phi)) + (1-\phi)(\rho_1 - \alpha_1(1-\phi))\right]. \end{aligned}$$

$\square$

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