Matching and Saving in Continuous Time: Theory

Ch. Bayer and K. Wälde

Discussion Paper 2010-13

Institut de Recherches Économiques et Sociales
de l'Université catholique de Louvain

Matching and Saving in Continuous Time: Theory

Christian Bayer[(a)] and Klaus Wälde[(b)]
[(a)] TU Berlin and
[(b)] University of Mainz, CESifo and Université catholique de Louvain[1]

April 2010

We analyse optimal saving of risk-averse households when labour income stochastically jumps between two states. The generalized Keynes-Ramsey rule includes a precautionary savings term. A phase diagram analysis illustrates consumption and wealth dynamics within and between states. There is an endogenous lower and upper limit for wealth. We derive the Fokker-Planck equations for the densities of individual wealth and employment status. These equations also characterize the aggregate distribution of wealth and allow us to describe general equilibrium. An optimal consumption path exists and distributions converge to a unique limiting distribution.

JEL Codes: D91, E24, J63, J64
Keywords: matching model, optimal saving, incomplete markets, Poisson uncertainty, Fokker-Planck equations, general equilibrium

1 Introduction

Uncertain labour income is a fact of life. In its simplest conceptual representation, labour income moves stochastically between two states, high and low. Implications of uncertain labour income are often analyzed in continuous-time setups. This is true for search and matching models à la Pissarides (1985), Mortensen and Pissarides (1994), Burdett and Mortensen (1998) and their many applications. They include the analyses e.g. of business cycles as in Shimer (2005), of the effect of match and search frictions on wage distributions as in Moscarini (2005), of mismatch as in Shimer (2007) and of the efficiency of various wage setting mechanisms as in Gautier et al. (2010).

It is standard practice in this literature (for some exceptions see below) to assume strong capital market imperfections implying that households consume their current income. Any labour market transition associated with a labour income jump therefore

[1]Large parts of this paper were written while the authors were working at the Royal Institute of Technology (KTH) in Stockholm and the University of Glasgow, respectively. We are grateful to these institutions for their stimulating research environment. Christian Bayer: TU Berlin, Institute of Mathematics, Straße des 17. Juni 136, 10623 Berlin, bayer@math.tu-berlin.de, Klaus Wälde: University of Mainz, Mainz School of Management and Economics, Jakob-Welder-Weg 4, 55128 Mainz. klaus@waelde.com, www.waelde.com. We are very grateful to Walter Schachermayer and Josef Teichmann for comments and guidance, Michael Graber, Giuseppe Moscarini, Sevi Rodríguez Mora and Carlos Carrillo Tudela for comments and Jeremy Lise for discussions. The second author is indebted to Ken Sennewald and Christian Bauer for earlier collaboration on this topic.

implies a consumption jump – as a result, consumption is much more volatile than in the data. If households were allowed to save, however, they could smooth consumption and would generally self-insure against consumption fluctuations by accumulating wealth. It can be argued that analysing labour market issues without savings is a shortcoming also for many other reasons. One can expect that bargaining and labour supply choices are affected by personal wealth. Analysing the effects of labour market policies is probably biased as well if wealth is not taken into account as wealth should also affect search intensity. In ongoing numerical work, we find that unemployment benefit levels have a strong effect on the distribution of wealth, especially at the lower end. Finally, any normative analysis of optimal unemployment benefit schemes should also take wealth issues into account as social welfare functions or other optimality criteria neglecting wealth tend to be incomplete from a conceptional perspective.

It is the objective of this paper to introduce saving into a matching framework in continuous time. The reason for choosing a continuous-time setup is three-fold. First, a large part of the search and matching literature is in continuous time. It is an advantage if the insights and modelling techniques from this literature can be used with savings as well. Second, there is a rich set of tools available for continuous-time uncertainty which can easily be applied to a matching and saving setup. Finally, continuous-time models generally allow to push the analytical frontier a bit further than discrete time models. More economic intuition can be gained when working e.g. with phase diagrams than when working with abstract concepts only.

The first step in our analysis consists in presenting and solving the maximization problem an individual faces where labour income jumps between two states. For simplicity and in the tradition of this literature, these states are called employment and unemployment even though it could also reflect periods of high and low income. The solution of this maximization problem is described by a generalized Keynes-Ramsey rule where the generalization consists in a precautionary savings term. This term lends itself to intuitive economic interpretation. The Keynes-Ramsey rule provides simple conditions under which there will be (i) consumption and wealth growth in both labour market states, (ii) growth for the employed and decline for the unemployed workers or (iii) decline of consumption and wealth in both labour market states.

In a second step, we provide a phase-diagram analysis of the optimal behaviour of an individual, i.e. of the evolution of wealth and consumption when labour income jumps between being high and low. We can undertake a phase-diagram analysis as in continuous-time deterministic setups as systems with Poisson-uncertainty are piecewise-deterministic systems: between jumps, the system evolves on continuous and differentiable trajectories.

As always, a unique solution to a differential equation system requires as many boundary conditions as differential equations. We derive a boundary condition from borrowing and lending considerations which implies that the highest debt an unemployed worker can ever have is the present value of infinite unemployment benefits. This is the lower limit which is hit by an unemployed worker who dissaves. Once this limit is reached, consumption of the unemployed worker is zero and will remain zero until he finds a new job. Using this boundary condition, the existence of optimal consumption-wealth profiles for both labour market states can be proven.

The third step then asks the natural question about the distribution of wealth and labour market status. Using the Dynkin formula, we obtain the Fokker-Planck equations for the wealth-employment status system. We obtain a two-dimensional partial differential equation system. It describes the evolution of the density of wealth and employment status over time, given some initial condition. When we are interested in long-run properties only, we can set time derivatives equal to zero in the Fokker-Planck equations and obtain an ordinary two-dimensional non-autonomous differential equation system. Boundary conditions can be motivated from our phase diagram analysis. Existence and uniqueness of a stationary long-run distribution of wealth and labour market status and convergence to this long-run distribution can be proven by building on Meyn and Tweedie (1993).

Most of these steps were obtained by assuming an exogenous interest and wage rate. As we want to obtain a true general equilibrium solution, we then close the model by looking at the aggregate distribution of wealth. This allows us to determine an endogenous average wealth level plus an endogenous interest and wage rate.

The conceptional challenges provided by allowing for savings in a matching framework in continuous time are fascinating. In order to focus on these challenges, we remove all features from the standard Pissarides-type matching model which are not essential for understanding saving decisions in such a setup. We therefore work with an island-matching setup in the spirit of Lucas and Prescott (1974) where the wage is competitive. Put simply, we end up with a continuous-time version of Aiyagari (1994), i.e. a decentralized stochastic growth model with uncertainty in the labour income process but aggregate certainty. The focus on island-matching allows us to neglect wage-bargaining issues and the determination of the number of vacancies. Compared to the challenges overcome in our paper, an extension for wage bargaining and an endogenous number of vacancies is relatively straightforward and left for future work.

This paper is related to various strands of the literature. There is a long literature that looks at the effects of labour income uncertainty which is at least partially uninsurable. In a growth model context, one can then ask – inter alia – whether the implied precautionary savings yield a higher per capita capital stock (Huggett, 1993; Aiyagari, 1994; Huggett and Ospina, 2001; Marcet et al., 2007). In a matching setup, savings have been analyzed in the literature starting with Andolfatto (1996) and Merz (1995). In these setups, individuals are fully insured against labour income risk as labour income is pooled in large families. Papers which exploit the advantage of CARA (constant absolute risk aversion) utility functions to jointly analyse saving and matching include Acemoglu and Shimer (1999), Hassler et al. (2005), Shimer and Werning, (2007, 2008) and Hassler and Rodriguez Mora (1999, 2008). These papers often work with closed-form solutions for the consumption-saving decision but cannot always rule out negative consumption levels for poor households.

More recently, a series of papers (Lentz and Tranaes, 2005; Lentz, 2009; Bils et al., 2007, 2009; Nakajima, 2008; Krusell et al., 2008, 2010) allows for individual labour income uncertainty in the presence of saving and search or matching and a CRRA (constant relative risk aversion) utility function. Some of them are not fully general equilibrium as wages are exogenous (Lentz and Tranaes, 2005; Lentz, 2009) or as a small open economy with an exogenous interest rate (but allowing for vacancies, bargaining and match-specific stochastic output) is considered (Bils et al., 2007, 2009). Others are very similar to our setup as they assume competitive wage setting as well (e.g. Krusell et al., 2008),[2] some go beyond our setup and allow for vacancy creation and Nash bargaining where the distribution of wealth implies a distribution of wages (Krusell et al., 2010). All of these models are in discrete time.

Compared to these search, matching and saving models, the main difference of the present paper lies in our strong analytical focus. We aim at characterizing equilibrium properties as much as possible from an analytical perspective. The Keynes-Ramsey rules (the Euler equations) reveal a lot of economic information inter alia on wealth threshold levels crucial for understanding consumption and wealth dynamics. The condition the interest rate has to satisfy such that a stationary general equilibrium consists can easily be seen. The phase diagram analysis illustrates consumption, wealth and employment dynamics in a very clear way. The description of distributions by differential equations allows – once solved numerically – to obtain distributional information more quickly than through simulations.[3]

Technically, this paper builds on earlier work of one of the authors (Wälde, 1999, 2005) who analyzes optimal saving under Poisson uncertainty affecting the return to capital but not labour income.[4] We also use the insights of the long literature on optimal saving under uncertainty in continuous time. Starting with Merton (1969), it includes Turnovsky and Smith (2006), Guo et al. (2005), Bertola et al. (2005), Hassler et al. (2005), Shimer and Werning, (2007, 2008) and Hassler and Rodriguez Mora, 1999, 2008).

The papers which are methodologically closest to ours are Shimer and Werning (2007, 2008) and Lise (2007). Shimer and Werning analyse unemployment insurance policies in a setup with job arrivals, deterministic or stochastic job duration and individual savings under constant absolute risk aversion. CARA preferences allow them to derive closed-form solutions which, however, cannot be obtained for constant relative risk aversion as used here. Lise (2007) derives a deterministic Keynes-Ramsey rule (i.e. for periods between labour market transitions) similar to the one here for employed workers in a model with on-the-job search and firm heterogeneity. He abstracts from general equilibrium considerations and does not provide formal existence proofs.

The structure of the paper is as follows. Section 2 presents the model. Section 3 derives implications of optimal behaviour. Section 4 presents the phase diagram analysis to understand consumption-wealth patterns over time and across labour mar-

[2] The structure of our model differs slightly as we allow individuals to borrow.

[3] The latter would eventually provide a background for structural estimation as is typical for the empirical search and matching literature (van den Berg, 1990; Postel-Vinay and Robin, 2002; Flinn, 2006; see also Launov and Wälde, 2010).

[4] Work completed before the present paper includes an unpublished PhD dissertation by Sennewald (2006) supervised by one of the authors which contains the Keynes-Ramsey rules. Toche (2005) considers the saving problem of an individual where job-loss is permanent and unemployment benefits are zero. Lise (2006) developed a Keynes-Ramsey rule for times between jumps as well.

ket states. The existence result for optimal consumption rules is stated. Section 5 describes the joint distribution of the labour market status and wealth of one individual. The corresponding Fokker-Planck equations are introduced, boundary conditions are discussed and uniqueness and convergence results are stated. Section 6 shows how to obtain the aggregate distribution of wealth and how to formulate appropriate initial distributions. This allows to link macro to micro features of the model and to define general equilibrium. The final section concludes. To save on space, all proofs are in the companion papers by Bayer and Wälde (2010a,b).

2 The model

We consider a model where all aggregate variables are in a steady state. At the micro level, individuals face idiosyncratic uninsurable risk and variables evolve in a dynamic and stochastic way.

2.1 Technologies

The production of output requires capital K and labour L. Both the capital stock and employment are endogenous but constant. The technology is given by $Y = Y(K, L)$ and $Y(.)$ has the usual neoclassical properties.

As is common for Mortensen-Pissarides type search and matching models, the employment status $z(t)$ of any individual jumps between the state of employment, w, and unemployment, b, with corresponding labour income w – the net wage – and unemployment benefits b. As an individual cannot lose her job when she does not have one and as finding a job makes (in the absence of on-the-job search) no sense for someone who has a job, both the job arrival rate $\mu(z(t))$ and the separation rate $s(z(t))$ are state dependent. As an example, when an individual is employed, $\mu(w) = 0$, when she is unemployed, $s(b) = 0$.

$z(t)$	w	b
$\mu(z(t))$	0	$\mu > 0$
$s(z(t))$	$s > 0$	0

Table 1 *State dependent arrival rates*

The process $z(t)$ can be viewed as a continuous-time Markov chain with state space $\{w, b\}$. This Markov-chain view will be used further below for the derivation of the Fokker-Planck equations describing the distributional properties of wealth and employment status. For our analysis of the saving problem of the household, it is most convenient to describe the related labour income process $z(t)$ by a stochastic differential equation,

$$dz(t) = \Delta dq_\mu - \Delta dq_s, \qquad \Delta \equiv w - b. \tag{1}$$

The Poisson process q_s counts how often our individual moves from employment into unemployment. The arrival rate of this process is given by $s(z(t))$. The Poisson process related to job finding is denoted by q_μ with an arrival rate $\mu(z(t))$. It counts how often the individual finds a job.

When the individual is employed, $z(t) = w$, the employment equation (1) simplifies to $dw = -(w-b)\,dq_s$. Whenever the process q_s jumps, i.e. when the individual loses her job and $dq_s = 1$, the change in labour income is given by $-w+b$ and, given that the individual earns w before losing the job, earns $w - w + b = b$ afterwards. Similarly, when unemployed, the employment status follows $db = (w-b)\,dq_\mu$ and finding a job, i.e. $dq_\mu = 1$, means that labour income increases from b to w.

2.2 Households and government

Each individual can save in an asset a (which is capital used by firms). Her budget constraint reads

$$da(t) = \{ra(t) + z(t) - c(t)\}\,dt. \tag{2}$$

Per unit of time dt wealth $a(t)$ increases (or decreases) if capital income $ra(t)$ plus labour income $z(t)$ is larger (or smaller) than consumption $c(t)$. Following (1), labour income $z(t)$ is given either by w or b. Dividing the budget constraint by dt and using $\dot{a}(t) \equiv da(t)/dt$ would yield a more standard expression, $\dot{a}(t) = ra(t) + z(t) - c(t)$. As $a(t)$ is not differentiable with respect to time at moments where individuals jump between employment and unemployment (or vice versa), we prefer the above representation. The latter is also more consistent with (1).

The objective function of the individual is a standard intertemporal utility function,

$$U(t) = E_t \int_t^\infty e^{-\rho[\tau - t]} u(c(\tau))\,d\tau, \tag{3}$$

where expectations need to be formed due to the uncertainty of labour income which in turn makes consumption $c(\tau)$ uncertain.[5] The expectations operator is E_t and conditions on the current state in t. The planning horizon starts in t and is infinite. The time preference rate ρ is positive.

Even though most of our results should hold for general instantaneous utility functions with positive but decreasing first derivatives, we will work with a CRRA specification,

$$u(c(\tau)) = \left\{ \begin{array}{c} \frac{c(\tau)^{1-\sigma}-1}{1-\sigma} \\ \ln c(\tau) \end{array} \right\} \text{ for } \left\{ \begin{array}{l} \sigma \neq 1 \text{ and } \sigma > 0, \\ \sigma = 1. \end{array} \right. \tag{4}$$

All proofs will use the specification for a positive $\sigma \neq 1$. The log case is not proven.

There is a government who can tax the gross wage $w/(1-\xi)$ using a proportional tax ξ. Tax income from employed workers is used to finance unemployment benefits b. The tax adjusts such that a static government budget constraint

$$\xi \frac{w}{1-\xi} L = b[N - L] \tag{5}$$

[5]The notation $c(\tau)$ is not entirely correct. When solving the maximization problem below, consumption will be considered to be a function of current state variables.

is fulfilled at each point in time. The path of benefits b is determined by some political process which is exogenous to this model. This process makes sure that benefits are smaller than the net wage, $b < w$.

2.3 Endowment

The workforce of this economy has an exogenous and invariant size N. Individuals are initially endowed with wealth $a_i(t)$. This can be a fixed number or random (see sect. 5). The capital stock is defined as the sum over individual wealth holdings,

$$K \equiv \Sigma_{i=1}^{N} a_i(t). \tag{6}$$

Given our steady state setup, the aggregate capital stock K is endogenous but constant. Loosely speaking, there is a very large number of agents i such that all dynamics at the individual level wash out at the aggregate level. See our definition of an equilibrium below – especially (31) – for a precise formulation.

Given the job separation and matching setup, it is well-known that in a steady state, aggregate employment is an increasing function of the matching and a decreasing function of the separation rate,

$$L = \frac{\mu}{\mu + s} N. \tag{7}$$

3 Optimality conditions

3.1 Keynes-Ramsey rules

For our understanding of optimal consumption behaviour, it is useful to derive a Keynes-Ramsey rule. We extend the approach suggested by Wälde (1999, 2008) for the case of an uncertain interest rate to our case of uncertain labour income. We suppress the time argument for readability. Consumption $c(a_w, w)$ of an employed individual with current wealth a_w follows (see app. A.1)

$$\begin{aligned} -\frac{u''(c(a_w,w))}{u'(c(a_w,w))} dc(a_w,w) = {} & \left\{ r - \rho + s \left[\frac{u'(c(a_w,b))}{u'(c(a_w,w))} - 1 \right] \right\} dt \\ & - \frac{u''(c(a_w,w))}{u'(c(a_w,w))} \left[c(a_w,b) - c(a_w,w) \right] dq_s \end{aligned} \tag{8}$$

while her wealth evolves according to (2) with $z = w$, i.e.

$$da_w = \left[r a_w + w - c(a_w,w) \right] dt. \tag{9}$$

Analogously, solving for the optimal consumption of an unemployed individual with current wealth a_b yields

$$\begin{aligned} -\frac{u''(c(a_b,b))}{u'(c(a_b,b))} dc(a_b,b) = {} & \left\{ r - \rho - \mu \left[1 - \frac{u'(c(a_b,w))}{u'(c(a_b,b))} \right] \right\} dt \\ & - \frac{u''(c(a_b,b))}{u'(c(a_b,b))} \left[c(a_b,w) - c(a_b,b) \right] dq_\mu \end{aligned} \tag{10}$$

and her wealth follows

$$da_b = [ra_b + b - c(a_b, b)]dt. \tag{11}$$

Without uncertainty about future labor income, i.e. $s = \mu = dq_s = dq_\mu = 0$, the above Keynes-Ramsey rules reduce to the classical deterministic consumption rule, $-\frac{u''(c)}{u'(c)}\dot{c} = r - \rho$. The additional $s[.]$ term in (8) shows that consumption growth is faster under the risk of a job loss. Note that the expression $[u'(c(a_w, b))/u'(c(a_w, w)) - 1]$ is positive as consumption $c(a_w, b)$ of an unemployed worker is smaller than consumption of an employed worker $c(a_w, w)$ (this is proven in Bayer and Wälde, 2010a, lem. 8) and marginal utility is decreasing, $u'' < 0$. Similarly, the $\mu[.]$ term in (10) shows that consumption growth for unemployed workers is smaller.

As the additional term in (8) contains the ratio of marginal utility from consumption when unemployed relative to marginal utility when employed, this suggests that it stands for precautionary savings (Leland, 1968, Aiyagari, 1994, Huggett and Ospina, 2001). When marginal utility from consumption under unemployment is much higher than marginal utility from employment, individuals experience a high drop in consumption when becoming unemployed. If relative consumption shrinks as wealth rises, i.e. if $\frac{d}{da}\frac{c(a,w)}{c(a,b)} < 0$, reducing this gap and smoothing consumption is best achieved by fast capital accumulation. This fast capital accumulation would go hand in hand with fast consumption growth as visible in (8).

In the case of unemployment, the $\mu[.]$ term in (10) suggests that the possibility to find a new job induces unemployed individuals to increase their current consumption level. Relative to a situation in which unemployment is an absorbing state (once unemployed, always unemployed, i.e. $\mu = 0$), the prospect of a higher labor income in the future reduces the willingness to give up today's consumption. With higher consumption levels, wealth accumulation is lower and consumption growth is reduced.

The stochastic dq-terms in (8) and (10) (tautologically) represent the discrete jumps in the level of consumption whenever the employment status changes. We will understand more about these jumps after the phase-diagram analysis below.

3.2 Factor rewards

Workers find markets ("islands") with an infinite supply of jobs with arrival rates μ. Once a market is found, there is perfect competition and factor rewards are given by marginal productivities as in Lucas and Prescott (1974). Firms rent capital on a spot market and choose an amount such that marginal productivity equals the rental rate. At the aggregate level, this fixes capital returns r and the gross wage $w/(1-\xi)$ at

$$r = \frac{\partial Y(K, L)}{\partial K}, \quad \frac{w}{1-\xi} = \frac{\partial Y(K, L)}{\partial L}. \tag{12}$$

4 Consumption and wealth dynamics

This section characterizes consumption and wealth properties under optimal behaviour, taking factor rewards as given. We will return to endogenous factor rewards and thereby to general equilibrium in sect. 6.

4.1 Consumption growth and the interest rate

We first focus on individuals in periods between jumps. The evolution of consumption is then given by the deterministic part, i.e. the dt-part, in (8) and (10). We then easily understand

Lemma 1 *Individual consumption rises if and only if current consumption relative to consumption in the other state is sufficiently high.*

For the employed worker, consumption rises if and only if $c(a_w, w)$ relative to $c(a_w, b)$ is sufficiently high,

$$\frac{dc(a_w, w)}{dt} \geq 0 \Leftrightarrow \frac{u'(c(a_w, b))}{u'(c(a_w, w))} \geq 1 - \frac{r-\rho}{s} \Leftrightarrow \frac{c(a_w, w)}{c(a_w, b)} \geq 1/\psi, \tag{13}$$

where

$$\psi \equiv \left(1 - \frac{r-\rho}{s}\right)^{-1/\sigma}. \tag{14}$$

For the unemployed worker, consumption rises if and only if $c(a_b, b)$ relative to $c(a_b, w)$ is sufficiently high,

$$\frac{dc(a_b, b)}{dt} \geq 0 \Leftrightarrow \frac{u'(c(a_b, w))}{u'(c(a_b, b))} \geq 1 - \frac{r-\rho}{\mu} \Leftrightarrow \frac{c(a_b, b)}{c(a_b, w)} \geq \left(1 - \frac{r-\rho}{\mu}\right)^{1/\sigma}. \tag{15}$$

Proof. Rearranging (8) and (10) for $dq_s = dq_\mu = 0$ and taking (4) into account gives the results (see app. A.2). Note that in what follows ψ will be used only for r sufficiently small making sure that ψ is a real number. ■

We can now establish our first main findings. As the conditions in lem. 1 show, consumption and wealth dynamics crucially depend on how high the interest rate is. We therefore subdivide our discussion into three parts with r lying in the three ranges given by $(0, \rho]$, $(\rho, \rho + \mu)$, $[\rho + \mu, \infty)$. For the proofs of all of our results, we rely on one very weak

Assumption 1 *Relative consumption $c(a, w)/c(a, b)$ is continuously differentiable in wealth a. The number of sign changes of its first derivative with respect to wealth in any interval of finite length is finite.*[6]

Starting with the third range $[\rho + \mu, \infty)$, we obtain

Proposition 1 *For a high interest rate, i.e. if $r \geq \rho + \mu$, consumption of employed and unemployed workers always increases.*

[6] The second sentence of this assumption is required to rule out "pathological cases". One can construct continuously differentiable functions that change sign infinitly often in a finite neighborhood (think of $x \sin(1/x)$ in a neighborhood of zero). None of these functions would be economically plausible in any way. This second sentence is used in the proof of prop. 4.

Proof. Consumption of the employed worker increases as can be directly seen from the first expression in (13). As long as $r > \rho$ and $c(a, w) > c(a, b)$, the latter is proven in Bayer and Wälde (2010a, lem. 8), condition (13) is fulfilled: The right-hand side (RHS) is smaller than one and the left-hand side is larger than one as long as $u'' < 0$ which holds for (4). The case of the unemployed worker can also most easily be seen from the first expression in (15). For $r = \rho + \mu + \varepsilon$ with $\varepsilon \geq 0$, the RHS is given by $1 - \frac{r-\rho}{\mu} = -\frac{\varepsilon}{\mu} \leq 0$. As $\frac{u'(c(a_b,w))}{u'(c(a_b,b))} \geq 0$, (15) holds for $r \geq \rho + \mu$. ■

The high interest rate case reminds of the standard optimal saving result in deterministic setups. If the interest rate is only high enough, consumption and wealth increase over time. This is true here as well. The only difference consists in the fact that the interest rate must be higher than the time preference rate *plus* the job arrival rate.

While we leave a quantitative analysis to ongoing numerical work, it is interesting already at this stage to note that the difference for the interest rate as compared to deterministic models is quite substantial. In deterministic models, the interest rate must be larger than the time preference rate. As the job arrival rate is around four times higher than the time preference rate, the interest rate must be much higher here to guarantee wealth growth in all employment states.

As in other setups with growing consumption, we need to make sure that consumption does not grow too fast. If it does, utility grows too fast and the expected value of the integral in the objective function (3) is not finite. Optimization would then be more involved, which we would like to avoid. We therefore have to impose a boundedness condition which implies an upper limit on the interest rate. This condition can easily be derived for the limiting case where a is very large, i.e. where the difference between w and b can be neglected. The boundedness condition then reads $(1 - \sigma) r < \rho$.[7]

The second result is summarized in

Proposition 2 *If the interest rate is at an intermediate level, i.e.* $\rho < r < \rho + \mu$,

(i) consumption of employed workers always increases.

(ii) consumption of an unemployed worker increases only if she is sufficiently wealthy, i.e. if her wealth a *exceeds the threshold level* a_b^**, where the threshold level is implicitly given by*

$$\frac{u'(c(a_b^*, w))}{u'(c(a_b^*, b))} \equiv 1 - \frac{r - \rho}{\mu}. \tag{16}$$

Consumption decreases for $a < a_b^*$.

(iii) At the threshold level a_b^**, consumption of employed workers exceeds consumption of unemployed workers.*

Proof. The proof is in complete analogy to the proof of the following prop. 3 for the low interest rate. As prop. 3 is more important for our purposes, we will prove prop. 3 but not this one. ■

[7] An interest rate r can satisfy both this boundedness condition and the condition $r \geq \rho + \mu$ for the high-interest-rate case if $\mu < \frac{\sigma}{1-\sigma}\rho$. This condition on μ needs to be taken into account in any quantitative analysis.

This proposition points to the central new insight for optimal consumption. For the employed worker, the result from deterministic worlds survives: If the interest rate is higher than the time preference rate, consumption and wealth rise. For the unemployed worker, however, this is not true. Consumption and wealth rise only if the unemployed worker is sufficiently rich. In a way, this is a "dramatic" result. If a worker loses a job, consumption continues to rise only if the worker is sufficiently rich at the moment of the job loss. If, by contrast, a worker losing a job is below the threshold level a_b^*, consumption and wealth is reduced.[8]

Finally, we have

Proposition 3 *Consider a low interest rate, i.e.* $0 < r \leq \rho$. *Define a threshold level* a_w^* *by*

$$\frac{u'\left(c\left(a_w^*, b\right)\right)}{u'\left(c\left(a_w^*, w\right)\right)} \equiv 1 - \frac{r - \rho}{s}. \tag{17}$$

For our instantaneous utility function (4), this definition reads

$$c\left(a_w^*, b\right) = \psi c\left(a_w^*, w\right) \tag{18}$$

where ψ *is from (14).*

(i) Consumption of employed workers increases if the worker owns a sufficiently low wealth level, $a < a_w^*$. *Employed workers with* $a > a_w^*$ *choose falling consumption paths.*

(ii) Consumption of unemployed workers always decreases.

(iii) Consumption of employed workers exceeds consumption of unemployed workers at the threshold a_w^*, *i.e.* $\psi \leq 1$ *in (18) for* $r \leq \rho$.

Proof. see Bayer and Wälde (2010a). ■

We are now in a position to intuitively understand all three propositions. In deterministic setups, an interest rate exceeding the time preference rate is enough to imply positive consumption growth. In a world with precautionary saving, only employed workers will experience rising consumption for sure when $r > \rho$. Unemployed workers experience rising consumption only for a high interest rate $r > \rho + \mu$ or for r close to but larger than ρ only if they are sufficiently rich. The reason for these results is the "optimism" of unemployed workers that they will find a job in the future. Anticipating higher future income, they choose a higher consumption level than in a situation where the state of unemployment is permanent. Due to this higher consumption level, consumption and wealth growth is reduced. Only if the interest rate exceeds $\rho + \mu$ or if an unemployed worker is sufficiently rich, this higher consumption does still allow for consumption growth.

Similarly for employed workers: In deterministic worlds, an interest rate below the time preference rate implies falling consumption and wealth levels. Here, as there is precautionary saving of the employed worker, a situation of $r < \rho$ still implies growing consumption and wealth.

[8] See an earlier version of this paper (Bayer and Wälde, 2009) for a phase diagram illustration of this proposition.

These propositions also clearly show that if we are interested in a general equilibrium result with stationary properties, the interest rate cannot be larger than the time preference rate. If the interest rate exceeded the time preference rate, consumption would grow without bound – at least for some employment states and levels of wealth. Only for $r \leq \rho$ there are consumption dynamics which indicate that a stationary distribution of consumption can exist.

4.2 The reduced form

Before we can derive further properties of optimal behaviour, we need a "reduced form" for optimal behaviour of individuals. A reduced form is a system of equations with as few equations as possible which determines an identical number of endogenous variables and which allow us to derive all other endogenous variables subsequently. When searching for such a reduced form, we can exploit the fact that Poisson uncertainty allows to divide the analysis of a system into what happens between jumps and what happens at jumps. Between jumps, the system evolves in a deterministic way – but does of course take the possibility of a jump into account as is clearly visible in the precautionary savings terms in the Keynes-Ramsey rules (8) and (10).[9]

We obtain such a reduced form by focusing on the evolution between jumps and by eliminating time as exogenous variable. Computing the derivatives of consumption with respect to wealth in both states and considering wealth as the exogenous variable, we obtain a two-dimensional system of non-autonomous ordinary differential equations (ODE). As wealth is now the argument for these two differential equations, there is no longer a need to distinguish between wealth of employed and unemployed workers (i.e. between a_w and a_b). We simply ask how wealth changes in one or the other state given a certain wealth level a. Between jumps, the reduced form therefore reads

$$-\frac{u''(c(a,w))}{u'(c(a,w))}\frac{dc(a,w)}{da} = \frac{r-\rho+s\left[\frac{u'(c(a,b))}{u'(c(a,w))}-1\right]}{ra+w-c(a,w)}, \tag{19a}$$

$$-\frac{u''(c(a,b))}{u'(c(a,b))}\frac{dc(a,b)}{da} = \frac{r-\rho-\mu\left[1-\frac{u'(c(a,w))}{u'(c(a,b))}\right]}{ra+b-c(a,b)}. \tag{19b}$$

With two boundary conditions, this system provides a unique solution for $c(a,w)$ and $c(a,b)$. Once solved, the effect of a jump is then simply the effect of a jump of consumption from, say, $c(a,w)$ to $c(a,b)$.

[9] One could be tempted to think of the deterministic parts of the two Keynes-Ramsey rules (8) and (10), jointly with the budget constraints (9) and (11) to provide such a reduced form. With an initial condition for wealth and the consumption levels in the different states, one could think of the evolution between jumps as being described by four ordinary differential equations. When solving these equations (conceptionally or numerically), the solution in t for consumption of, say, the unemployed, $c(a_b,b)$ from (10) would not correspond to consumption $c(a_w,b)$ as required in the precautionary savings part in (8) for the employed as wealth levels are accumulated at different speed, i.e. $a_b(t)$ generally differs from $a_w(t)$. Equations (8) to (11) do therefore not constitute a system of ODEs and cannot be used as a reduced form.

4.3 Phase diagram and policy functions

Given the findings on consumption in the above propositions and our reduced form in (19), we can now describe the link between optimal consumption and wealth of unemployed and employed workers. We will focus on the case of an interest rate below the time preference rate as this is the endogenous property of a stationary general equilibrium solution.[10]

- Natural borrowing limit

The subsequent analysis will be facilitated by noting that there is an endogenous "natural" borrowing limit. The idea is similar to Aiyagari's (1994) borrowing limit resulting from non-negative consumption. This limit is derived in the following

Proposition 4 *Any individual with initial wealth $a \geq -b/r$ will never be able to or willing to borrow more than $-b/r$. Consumption of an unemployed worker at $a = -b/r$ is zero, $c(-b/r, b) = 0$.*

Proof. "willing to": An employed individual with $a \geq -b/r$ will increase wealth for any wealth levels below a_w^* from (17). If a_w^* is larger than $-b/r$ – which we can safely assume – employed workers with wealth below a_w^* increase wealth and are not willing to borrow more than $-b/r$.

"able to": Imagine an unemployed worker had wealth lower than $-b/r$. Even if consumption is equal to zero, wealth would further fall, given that $\dot{a} = ra + b < 0 \Leftrightarrow a < -b/r$. If an individual could commit to zero consumption when employed and if the separation rate was zero, the maximum debt an individual could pay back is $-w/r$. Imagine an unemployed worker succeeded in convincing someone to lend her "money" even though current wealth is below $-b/r$. Then, with a strictly positive probability, wealth will fall below $-w/r$ within a finite period of time. Hence, anyone lending to an unemployed worker with wealth below $-b/r$ knows that not all of this loan will be paid back with positive probability. This cannot be the case in our setup with one riskless asset. Hence, the maximum debt level is b/r and consumption is zero at $a = -b/r$ for an unemployed worker. ■

- Laws of motion and policy functions

The following fig. 1 plots wealth on the horizontal and consumption $c(a, z)$ on the vertical axis. It plots dashed zero-motion lines for a_w and $c(a, w)$ and a solid zero-motion line for a_b following from (9), (17) and (11), respectively. We assume for this figure that the threshold level a_w^* is positive.[11] The intersection point of the zero-motion lines for $c(a, w)$ and a_w is the temporary steady state (TSS),

$$\Theta \equiv (a_w^*, c(a_w^*, w)). \tag{20}$$

[10] See Bayer and Wälde (2009) for more on the intermediate and high interest rate case.

[11] This is of course a quantitative issue. In ongoing numerical work, the threshold is positive for reasonable parameter values. It approaches infinity for r approaching ρ.

We call this point *temporary* steady state for two reasons. On the one hand, employed workers experience no change in wealth, consumption or any other variable when at this point (as in a standard steady state of a deterministic system). On the other hand, the expected spell in employment is finite and a random transition into unemployment will eventually occur. Hence, the state in Θ is steady only temporarily.

As we know from prop. 3 that consumption for the unemployed always falls, both consumption and wealth fall above the zero-motion line for a_b. The arrow-pairs for the employed workers are also added. They show that one can draw a saddle-path through the TSS. To the left of the TSS, wealth and consumption of employed workers rise, to the right, they fall.

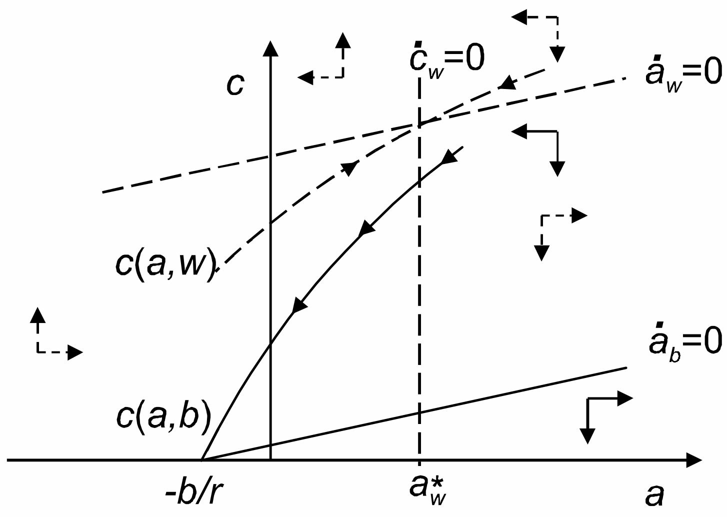

Figure 1 *Policy functions for employed and unemployed workers (low interest rate)*

Relative consumption when the employed worker is in the TSS is given by (18). A trajectory going through $(a_w^*, c(a_w^*, b))$ and hitting the zero-motion line of a_b at $-b/r$ is in accordance with laws of motions for the unemployed worker.

- Properties of optimal behaviour

The case of a low interest rate is particularly useful as the range of wealth a worker can hold is bounded. Whatever the initial wealth level, there is a positive probability that the wealth level will be in the range $[-b/r, a_w^*]$ after some finite length of time. For an illustration, consider the policy functions in fig. 1: Wealth decreases both for employed and unemployed workers for $a > a_w^*$. The transition into the range $[-b/r, a_w^*]$ will take place only in the state of unemployment which, however, occurs with positive probability.

When wealth of an individual is within the range $[-b/r, a_w^*]$, consumption and wealth will rise while employed and fall while unemployed. While employed, precautionary saving motives drive the worker to accumulate wealth. While unemployed, the worker runs down current wealth as higher income for the future is anticipated – "postcautionary dissaving" takes place. When a worker loses a job at a wealth level of, say, $a_w^*/2$, his consumption level will drop from $c(a_w^*/2, w)$ to $c(a_w^*/2, b)$. Conversely, if an unemployed worker finds a job at, say, $a = 0$, her consumption increases from $c(0, b)$ to $c(0, w)$. A worker will therefore be in a permanent consumption and wealth cycle. Given these dynamics, one can easily imagine a distribution of wealth over the range $[-b/r, a_w^*]$.

4.4 Existence of equilibrium

All steps undertaken so far were explorative. We now turn to a proof for the existence of a path $c(a, z)$ as depicted in fig. 1.

In fig. 1, we implicitly considered solutions of our system in the set $Q = \{a \geq -b/r\} \cap \{c(a, w) \leq ra + w\} \cap \{c(a, b) \geq ra + b\} \cap \{c(a, b) \geq 0\} \cap \{c(a, w) \geq c(a, b)\}$. In words, wealth is at least as large as the maximum debt level b/r, consumption of the employed worker is below the zero-motion line for her wealth, consumption of the unemployed worker is above her zero-motion line for wealth, consumption of the unemployed worker is non-negative and consumption of employed workers always exceeds consumption of unemployed workers (the latter is proven in Bayer and Wälde, 2010a, lem. 8). For the existence proof we restrict this set to

$$\begin{aligned} R_{v,\Psi} = \big\{ & (a, c(a, w), c(a, b)) \in \mathbb{R}^3 | \, (a, c(a, w), c(a, b)) \in Q, \\ & c(a, w) \leq \Psi < \infty, \; a \leq (c(a, w) - w + v)/r \big\}, \end{aligned} \tag{21}$$

where Ψ is a finite large constant. There are two differences to Q : First, the set $R_{v,\Psi}$ is bounded. This is a purely technical necessity. Second, the set $R_{v,\Psi}$ excludes the zero-motion line for wealth a_w by subtracting a small positive number v. We need to do this as the fraction on the right-hand side of our differential equation (19a) is not defined for the TSS.[12]

We now introduce an auxiliary TSS (aTSS) in order to capture v. In analogy to the TSS Θ from (20), this point is defined by

$$\Theta_v \equiv (a_w^*, c_v(a_w^*, w)),$$

i.e. the wealth level a_w^* is unchanged but the consumption level is "a bit lower" than in the TSS. In the TSS, the consumption level is on the zero-motion line, i.e. $c(a_w^*, w) = ra_w^* + w$. In the aTSS, the consumption level is on the line $ra + w - v$ and therefore given by $c_v(a_w^*, w) = ra_w^* + w - v$. Let us now consider the following

Definition 1 *(Optimal consumption path) A* consumption path *is a solution* $(a, c(a, w), c(a, b))$ *of the ODE-system (19) for the range* $-b/r \leq a \leq a_w^*$ *in* $R_{v,\Psi}$ *with terminal condition* $(a_w^*, c_v(a_w^*, w), c_v(a_w^*, b))$. *In analogy to the aTSS and to (18), the terminal condition satisfies* $c_v(a_w^*, w) = ra_w^* + w - v$ *and* $c_v(a_w^*, b) = \psi c_v(a_w^*, w)$ *for an arbitrary* $a_w^* > -b/r$. *An* optimal consumption path *is a consumption path which in addition satisfies* $c(-b/r, b) = 0$.

[12] While this is a standard property of many steady states, the standard solutions (e.g. linearization around the steady state) do not work in our case. This is in part due to the fact that the original stochastic differential equation system (8) to (11) - even when stripped of its stochastic part - is not an ordinary differential equation system.

Bayer and Wälde (2010a) then prove

Theorem 1 *There is an optimal consumption path.*

This establishes that we can continue in our analysis by taking the existence of a path $c(a,z)$ as given. Intuitively speaking, i.e. looking at v as very small constant close to zero, we know that there are paths $c(a,w)$ and $c(a,b)$ as drawn in fig. 1. The approximation implied by the auxiliary TSS is very small compared to any measurement error in the data. Values of $v = 10^{-3}$ worked perfectly in numerical solutions.[13]

5 The distribution of labour income and wealth

Let us now describe distributional properties of $z(t)$ and $a(t)$. This is of importance per se from a micro perspective – but it will also allow us to close the model and obtain general equilibrium results.

5.1 Labour market probabilities

Consider first the distribution of the labour market state. Given that the transition rates between w and b are constant, the conditional probabilities of being in state $z(\tau)$ follow e.g. from solving Kolmogorov's backward equations as presented e.g. in Ross (1993, ch. 6). As an example, the probability of being employed in $\tau \geq t$ conditional on being in state $z \in \{w, b\}$ in t are

$$P(z(\tau) = w | z(t) = w) \equiv p_{ww}(\tau) = \frac{\mu}{\mu+s} + \frac{s}{\mu+s} e^{-(\mu+s)(\tau-t)}, \tag{22}$$

$$P(z(\tau) = w | z(t) = b) \equiv p_{bw}(\tau) = \frac{\mu}{\mu+s} - \frac{\mu}{\mu+s} e^{-(\mu+s)(\tau-t)}. \tag{23}$$

The complementary probabilities are $p_{wb}(\tau) = 1 - p_{ww}(\tau)$ and $p_{bb}(\tau) = 1 - p_{bw}(\tau)$. Letting $p_w(t)$ denote the probability of $z(t) = w$, i.e. letting it describe the initial distribution of $z(t)$, the unconditional probability of being in state z in τ is

$$p_z(\tau) = p_w(t) p_{wz}(\tau) + (1 - p_w(t)) p_{bz}(\tau). \tag{24}$$

5.2 Fokker-Planck equations for wealth

Now consider one individual with a level of wealth of $a(t)$ and an employment status $z(t)$. This individual faces an uncertain future labour income stream $z(\tau)$. What is the joint distribution of $a(\tau)$ and $z(\tau)$ for $\tau > t$? Using methods well-established in stochastics, we can compute the Fokker-Planck equations. They describe the evolution of the (joint) density of $(a(\tau), z(\tau))$, i.e. of the labour market status and wealth

[13] Working on the existence proof made clear that one cannot easily prove uniqueness. Ongoing numerical work indicates that the maximized Bellman equation can be satisfied by two different value functions which imply large quantitative differences.

for $\tau \geq t$. This density is denoted by $p(a, z, \tau)$ and obviously driven by a discrete and a continuous random variable. We can therefore split it into two "subdensities" $p(a, w, \tau)$ and $p(a, b, \tau)$ which can be understood as the product of a conditional probability times the probability of being in a certain employment state,

$$p(a, z, \tau) \equiv p(a, \tau|z)\, p_z(\tau)\,. \tag{25}$$

The probability $p_z(\tau)$ of an individual to be in a state z in τ is given by (24). The conditional density of $a(\tau)$ given $z(\tau)$ is denoted by $p(a, \tau|z)$.

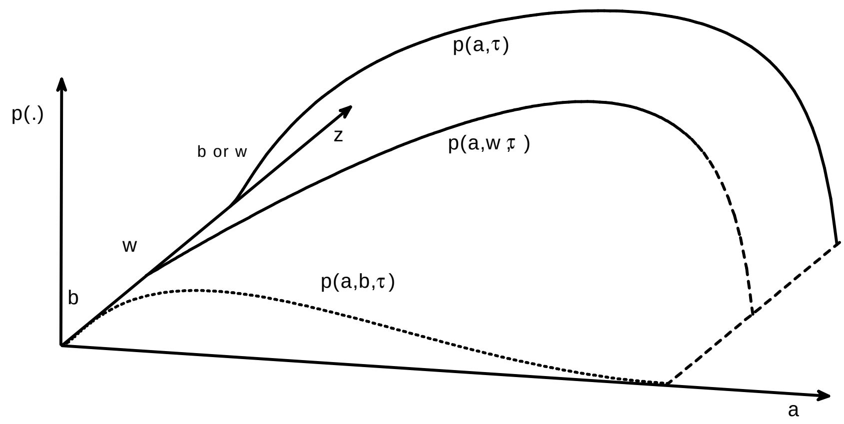

Figure 2 *The subdensities $p(a, b, \tau)$ and $p(a, w, \tau)$ and the density $p(a, \tau)$*

Note that the distribution of $(a(\tau), z(\tau))$ certainly depends on the initial condition $(a(t), z(t))$, which needs to be specified in order to calculate $p(a, z, \tau)$. In the notation we do not distinguish between the following two possibilities. Firstly, $(a(t), z(t))$ can be deterministic numbers, in which case $p(a, z, t)$ is a Dirac-distribution centered in $(a(t), z(t))$ (more precisely, the mapping $a \rightarrow p(z(t), a, t)$ is a Dirac-distribution). Secondly, $(a(t), z(t))$ can itself be random, either because we regard them as outcomes of the employment-wealth-process started at an even earlier time, or because there is some intrinsic uncertainty in measuring $a(t)$ (see below in sect. 5.4).

As is clear from (25), $p(a, z, \tau)$ are not conditional densities – they rather integrate to the probability of $z(\tau) = z$. Looking at an individual who is in state z in τ, we get

$$\int p(a, z, \tau)\, da = \int p(a, \tau|z)\, p_z(\tau)\, da = p_z(\tau) \int p(a, \tau|z)\, da = p_z(\tau)\,. \tag{26}$$

The density of a at some point in time τ for of an individual with initial condition $(a(t), z(t))$ is then simply

$$p(a, \tau) = p(a, w, \tau) + p(a, b, \tau)\,. \tag{27}$$

Fig. 2 illustrates these (sub-) densities and how they qualitatively look like at an arbitrary point in time $\tau \geq t$.[14]

The derivation of the Fokker-Planck equations is in Bayer and Wälde (2010a). The result is a system of two one-dimensional linear partial differential equations in $p(a,w,\tau)$ and $p(a,b,\tau)$,

$$\frac{\partial}{\partial \tau} p(a,w,\tau) + \{ra + w - c(a,w)\} \frac{\partial}{\partial a} p(a,w,\tau) + \left\{ r - \frac{\partial}{\partial a} c(a,w) + s \right\} p(a,w,\tau) - \mu p(a,b,\tau) = 0, \tag{28a}$$

$$\frac{\partial}{\partial \tau} p(a,b,\tau) + \{ra + b - c(a,b)\} \frac{\partial}{\partial a} p(a,b,\tau) + \left\{ r - \frac{\partial}{\partial a} c(a,b) + \mu \right\} p(a,b,\tau) - sp(a,w,\tau) = 0. \tag{28b}$$

The differential equations are linear with derivatives in τ and a. They are non-autonomous as coefficients of the densities and their derivatives are functions of a. As we can see, the density is linked to optimal behaviour through the consumption levels $c(a,w)$ and $c(a,b)$ (and their partial derivatives, i.e. the marginal propensities to consume out of wealth) obtained from the solution of the individual optimization problem. The solution of (28) gives the density $p(a,z,\tau)$ for any a, z and τ, i.e. the adjustment process of the density is captured as well.

Compared to closed-form solutions for transition densities which are used in finance (see e.g. Aït-Sahalia, 2004), our differential equations are of course less informative. The closed form solutions build on linear stochastic differential equations, however. The absence of a closed-form solution here is therefore simply the result of the non-linearity of our optimal consumption functions $c(a,z)$.

5.3 Existence and uniqueness of and convergence to a limiting distribution

Let us first note that the limiting distribution for the employment status $z(\tau)$ can easily be seen from (22) and (23) and their complementary probabilities. For any fixed initial condition $z(t) \in \{w,b\}$, the limiting distribution for $\tau \to \infty$ is given by $p(w) = \mu/(\mu+s)$ and $p(b) = s/(\mu+s)$. The same limiting distribution results if the initial condition itself is a distribution.

The question of existence of and convergence to a limiting distribution for the joint density $p(a,z,\tau)$ is far more involved. This question is of importance not only for the model under consideration, it is related to many other stochastic growth models in economics. We therefore state the central result here but refer the reader to a companion paper (Bayer and Wälde, 2010b) which treats this issue in much more detail.

[14] We should stress that even though we use the term density, $p(a,z,\tau)$ and $p(a,\tau)$ can contain one or more mass-points. Our formal derivation covers these cases as well.

For our purposes here, let us introduce the notion of an *invariant distribution.* A distribution for (a, z) is called invariant, if $(a(\tau), z(\tau))$ follows the distribution for any time $\tau > t$ provided that $(a(t), z(t))$ does. Obviously, any limiting distribution must be invariant. Building on the general ergodicity-theory for Markov-processes by Down et al. (1995), we obtain

Theorem 2 *(Bayer and Wälde, 2010b) Let $r < \rho$ and assume there is a temporary steady state (TSS), such that optimal consumption $c(a, z)$ is continuously differentiable in wealth a. Assume further that the initial distribution of wealth is supported in $[-b/r, a_w^*]$. Then,*

(i) the distribution of wealth at any subsequent time $\tau \geq t$ is supported in $[-b/r, a_w^*]$,

(ii) there is a unique invariant wealth distribution, and

(iii) for any such initial distribution, the distribution of wealth converges to the invariant distribution.

Note that this theorem can be proven for the set R_Ψ only, i.e. the parameter v from the set $R_{v,\Psi}$ in (21) used for the proof of theo. 1 must be zero. This proof therefore requires the existence of a TSS and not of an auxiliary TSS. Note also that if $r > \rho$, no invariant probability distribution and, hence, no limiting distribution exists.

5.4 Some background for numerical solutions

- Initial conditions

Obtaining a unique solution for ODEs generally requires certain differentiability conditions and as many initial conditions as differential equations. Conditions for obtaining a unique solution for PDEs differ in various respects, of which the most important one from an intuitive perspective is the fact that instead of initial conditions (i.e. an initial value or vector), initial functions are required. This can easily be understood for our case: Let us assume two initial functions for a, one for each labour market state $z \in \{w, b\}$. The obvious interpretation for these initial functions are densities, just as illustrated in fig. 2. Initial functions would therefore be given by $p(a, b, t) = p^{ini}(a, b)$ and $p(a, w, t) = p^{ini}(a, w)$. Clearly, they take positive values on the range $[-b/r, a_w^*]$ only and need to jointly integrate to unity. Given these initial functions, one can then compute the partial derivatives with respect to a in (28). This gives an ODE system which allows us to compute the density for the "next" τ. Repeating this gives us the densities for all z, a and τ we are interested in.

An initial function for wealth in each labour market state sounds unusual when thinking of one individual who, say, in t has wealth of $a(t)$ and is currently employed, $z(t) = w$. One can express these two deterministic numbers such that we obtain initial functions, however. First, $p^{ini}(a, b) = 0$: as the probability for an employed individual to be unemployed is zero and the probability of being unemployed is given by $\int_{-b/r}^{a_w^*} p^{ini}(a, b)\, da$ (compare the example in (26)), $p^{ini}(a, b)$ must be zero. Second, there are two possibilities for $p^{ini}(a, w)$. Either one considers $p^{ini}(a, w)$ as a Dirac-distribution, i.e. there is a degenerate density with mass-point at $a = a(t)$. Or, maybe most convenient both for numerical purposes and for intuition, one considers the current wealth level $a(t)$ to be observed with some imprecision. Pricing various types of assets (cars or other durable consumption goods like a house) might not be straightforward and one can easily imagine an initial function which is zero to the left of $a_{\min}$ and to the right of $a_{\max}$ and condenses all probability between these values (which can of course be arbitrarily close to $a(t)$).

- The long-run distribution of individual wealth

When we are interested in the long-run distribution of wealth and income only, the time derivatives of the densities would be zero and the long-run densities would be described by two linear ordinary differential equations,

$$\{ra + w - c(a, w)\} \frac{\partial}{\partial a} p(a, w) + \left\{r - \frac{\partial}{\partial a} c(a, w) + s\right\} p(a, w) - \mu p(a, b) = 0, \tag{29a}$$

$$\{ra + b - c(a, b)\} \frac{\partial}{\partial a} p(a, b) + \left\{r - \frac{\partial}{\partial a} c(a, b) + \mu\right\} p(a, b) - sp(a, w) = 0. \tag{29b}$$

The advantage of these two differential equation systems is clear: if numerical procedures can be found to easily solve them, short-run and long-run distributions can be obtained without having to simulate a system. These equations also open up new avenues for structural estimation. Parameters can easily be estimated such that an observed distribution is optimally fitted by these predicted distributions.

- Boundary conditions for the long-run distribution

For the long-run distribution in (29), we conjecture that two boundary conditions can be described as follows. Let the bounds of the range of a be given by $-b/r$ and a_w^* as illustrated in fig. 1. Boundary conditions are then provided by

$$p(a_w^*, w) = 0, \quad p(a_w^*, b) = 0. \tag{30}$$

We do not provide a formal proof here but rather give the following intuitive reasoning. We leave further investigation of both (29) and (28) to our ongoing work on the effect of labour market policies on wealth distributions. The intuition for $p(a_w^*, w) = 0$ comes from the saddle-path nature of the TSS Θ in (20): There is one path going into Θ from the left and one going into Θ from the right and two (not drawn) starting from Θ and going North and South. In saddle-points of ODE systems, one can prove by linearization around the fix point that local solutions of the ODE approach the saddle point asymptotically. Linearization here is more involved given the special structure of our system (see fn. 12). Assuming that the qualitative properties of local behaviour are not affected by this structure, we would observe asymptotic behaviour here as well and the TSS Θ would actually never be reached: $p(a_w^*, w) = 0$ would follow. The second boundary condition is then an immediate consequence. As the state (a_w^*, b) can occur only through a transition from (a_w^*, w) but the density at (a_w^*, w) is zero, $p(a_w^*, b) = 0$ as well.

6 Aggregate distributions and equilibrium

6.1 The aggregate distribution of wealth and employment

Using all the results we collected so far on individual behaviour, we are now in an easy position to describe the aggregate distribution of wealth and employment. One statistic one generally would like to understand is the share of the population which has a wealth below a certain level. The population consists of N individuals. Wealth and labour market status of an individual i is described by the density $p_i(a, z, \tau)$ given an initial condition $(a_i(t), z_i(t))$. The density of each single individual is described by the PDEs in (28). The density of individual wealth (without taking the labour market status into account) is $p_i(a, \tau)$ from (27).

Now define the share of individuals in the entire population with wealth below a certain level a at some point in time $\tau > t$ as $H(a, \tau) \equiv \Sigma_{i=1}^{N} I(a_i(\tau))/N$ where $I(a_i(\tau))$ is the indicator function taking a value of 1 if $a_i(\tau) < a$ and 0 otherwise. As the $a_i(\tau)$ are identically and independently distributed, the strong law of large numbers holds and we obtain $\lim_{N\to\infty} H(a, \tau) = \int_{-b/r}^{a} p(x, \tau)\, dx$. In words, the share of individuals in our population with wealth below a is given by the probability that an individual has wealth below a.[15] Computing the derivative of the distribution function gives the density of wealth for the population as a whole, $h(a, \tau) \equiv \frac{d}{da} \int_{-b/r}^{a} p(x, \tau)\, dx = p(a, \tau)$.

When we are interested in wealth distributions for each labour market status individually, we can define $H(a, z, \tau) \equiv \Sigma_{i=1}^{N} I(a_i(\tau), z(\tau))/N$ where the indicator function takes the value of one if $a_i(\tau) < a$ and $z(\tau) = z$. The density is then given by $h(a, z, \tau) = p(a, z, \tau)$.

As has been stressed in the discussion after (25), the initial condition $(a(t), z(t))$ can itself be random. This means that a solution of (28) with an initial distribution for a and z capturing some real world distribution of wealth and employment status provides a prediction how this aggregate distribution evolves over time. We describe our initial conditions by two subdensities, one for employed individuals and one for unemployed individuals, similar to the subdensities in (27),

$$h(a, w, t) = h^{ini}(a, w), \quad h(a, b, t) = h^{ini}(a, b).$$

Empirical information needed to find plausible initial functions (or to estimate them) is the distribution of wealth for employed and unemployed workers. If the share of unemployed workers is x%, the density $h^{ini}(a, w)$ must integrate to x/100, given the property of the subdensity $p(a, w, \tau)$ as shown in (26). If one is primarily interested in understanding the prediction for the aggregate distribution of wealth, any reasonable functions with range $[-b/r, a_w^*]$ and satisfying (26) will do.

[15] It is hard to imagine an economy with an infinite number of agents N. The alternative to this discrete law of large numbers is to work with a continuum of agents of mass N. The concept of infinity is then available by construction and laws of large numbers do not encounter the problem of having to imagine what an infinite number of individuals mean. On the downside, one runs into many well-known technical problems and, maybe more importantly, it might be just as difficult to imagine a continuum of individuals as an economy with an infinite number of inhabitants.

6.2 Equilibrium

We are now finally able to define general equilibrium. There is a deterministic macro level where all variables are constant. All uncertainty and all dynamics take place at the micro level. The average capital stock (for N approaching infinity) is given by the mean of the wealth distribution, given a density $p(a)$ of wealth,

$$\frac{K}{N} = \int ap(a)\,da. \tag{31}$$

This provides the link between the micro and macro level. We can now formulate

Definition 2 *A competitive stationary equilibrium is described by a constant aggregate capital stock K and employment level L, factor rewards w, r and the tax rate ξ, two functions $c(a,w)$ and $c(a,b)$ and a wealth density $p(a)$ such that*

1. *K satisfies (31) and L is given by (7),*

2. *given exogenous benefits b, the government budget constraint (5) and the first-order condition for labour in (12) jointly fix the tax rate ξ and wage rate w, the interest rate r satisfies the first-order condition for capital in (12),*

3. *the consumption functions $c(a,z)$ satisfy the reduced form (19) plus two boundary conditions of def. 1,*

4. *the density $p(a)$ is given by (27) for $\tau \to \infty$ where $p(a,w)$ and $p(a,b)$ are the solution to (29) with boundary conditions (30).*

In addition to this macro equilibrium, each individual's wealth distribution $p(a,z,\tau)$ is described by the solution to (28).

7 Conclusion

The objective of this paper was to extend the standard labour market matching model by allowing individuals to save. Allowing for savings in standard matching and search models is highly recommended given that individuals do self-insure in the presence of uninsurable risk. Due to the continuous-time setup chosen here, various results could be illustrated in an intuitive way.

The Keynes-Ramsey rule for this setup reveals that precautionary savings are at work here. Consumption grows faster while employed (as compared to a situation without labour income fluctuations) and grows slower while unemployed. Different qualitative behaviour concerning consumption and wealth growth was discovered for situations of a low, intermediate and high interest rate.

A phase-diagram analysis for the low-interest-rate case revealed consumption and wealth cycles within an endogenous wealth range. The lower bound of the distribution of wealth is given by minus the present value of permanent unemployment benefits, $-b/r$, the maximum wealth level is given by some a_w^* which depends on fundamental parameters of the model. In times of employment, consumption grows and wealth is built up. In times of unemployment consumption and wealth falls. A theorem on the existence of an optimal consumption path was stated.

The paper then derived the Fokker-Planck equations which describe the evolution of the distribution of wealth and labour market status for each individual worker starting from some initial condition. A result has been stated that there exists a unique long-run distribution and that initial distributions converge to this long-run distribution. The analysis of distributions of labour market status and wealth in an economy with many agents has also been undertaken. Using a standard law of large numbers, aggregate shares in the population can be linked to individual probabilities. This allows to close the model and obtain general equilibrium results. The advantage of using Fokker-Planck equations is very fast computation of densities. This should make this approach very suitable for structural estimation.

A problem often encountered in structural estimation with micro data is the lack of model guidance on how to control for aggregate time-series effects. Future work can address this issues by first allowing for explicit transitional dynamics. This would require time varying factor rewards and thereby a generalization of the Keynes-Ramsey rules and of the derivation of the Fokker-Planck equations. Eventually, one should allow for aggregate stochastic disturbances. This would yield exciting and highly promising new results opening up new avenues for estimation.

8 Appendix

Proofs are in Bayer and Wälde (2010a,b). Appendices numbered A.1 and A.2 are available upon request.

References

Acemoglu, D., and R. Shimer (1999): "Efficient Unemployment Insurance," Journal of Political Economy, 107, 893–928.

Aït-Sahalia, Y. (2004): "Disentangling diffusion from jumps," Journal of Financial Economics, 74, 487–528.

Aiyagari, S. R. (1994): "Uninsured Idiosyncratic Risk and Aggregate Saving," Quarterly Journal of Economics, 109, 659–84.

Andolfatto, D. (1996): "Business Cycles and Labor-Market Search," American Economic Review, 86, 112–32.

Bayer, C., and K. Wälde (2009): "Matching and Saving in Continuous Time : Theory (long version)," mimeo - available at www.waelde.com.

——— (2010a): "Matching and Saving in Continuous Time: Proofs," CESifo Working Paper 3027.

——— (2010b): "Matching and Saving in Continuous Time: Stability," mimeo - available at www.waelde.com.

Bertola, G., L. Guiso, and L. Pistaferri (2005): "Uncertainty and Consumer Durables Adjustment," Review of Economic Studies, 72, 973 – 1007.

Bils, M., Y. Chang, and S.-B. Kim (2007): "Comparative Advantage in Cyclical Unemployment," NBER Working Paper 13231.

——— (2009): "Heterogeneity and Cyclical Unemployment," NBER Working Paper 15166.

Burdett, K., and D. T. Mortensen (1998): "Wage Differentials, Employer Size, and Unemployment," International Economic Review, 39, 257–273.

Down, D., S. P. Meyn, and R. L. Tweedie (1995): "Exponential and Uniform Ergodicity of Markov Processes," Annals of Probability, 23, 1671 – 1691.

Flinn, C. (2006): "Minimum Wage Effects on Labor Market Outcomes under Search, Matching, and Endogenous Contact Rates," Econometrica, 74, 1013–1062.

Gautier, P., C. Teulings, and A. van Vuuren (2010): "On-the-Job Search, Mismatch and Efficiency," Review of Economic Studies, 77, 245–272.

Guo, X., J. Miao, and E. Morelle (2005): "Irreversible investment with regime shifts," Journal of Economic Theory, 122, 37–59.

Hassler, J., and J. V. R. Mora (1999): "Employment turnover and the public allocation of unemployment insurance," Journal of Public Economics, 73, 55 – 83.

——— (2008): "Unemployment insurance design: Inducing moving and retraining," European Economic Review, 52, 757 – 791.

Hassler, J., J. V. R. Mora, K. Storesletten, and F. Zilibotti (2005): "A Positive Theory of Geographic Mobility and Social Insurance," International Economic Review, 46(1), 263–303.

Huggett, M. (1993): "The risk-free rate in heterogeneous-agent incomplete-insurance economies," Journal of Economic Dynamics and Control, 17, 953–969.

Huggett, M., and S. Ospina (2001): "Aggregate precautionary savings: when is the third derivative irrelevant?," Journal of Monetary Economics, 48, 373–396.

Krusell, P., T. Mukoyama, R. Rogerson, and A. Sahin (2008): "Aggregate implications of indivisible labor, incomplete markets, and labor market frictions," Journal of Monetary Economics, 55, 961–979.

Krusell, P., T. Mukoyama, and A. Sahin (2010): "Labor-Market Matching with Precautionary Savings and Aggregate Fluctuations," Review of Economic Studies, forthcoming.

Launov, A., and K. Wälde (2010): "Estimating Incentive and Welfare Effects of Non-Stationary Unemployment Benefits," mimeo - available at www.waelde.com.

Lentz, R. (2009): "Optimal unemployment insurance in an estimated job search model with savings," Review of Economic Dynamics, 12, 37–57.

Lentz, R., and T. Tranaes (2005): "Job Search and Savings: Wealth Effects and Duration Dependence," Journal of Labor Economics, 23, 467–489.

Lise, J. (2006): "On-the-Job Search and Precautionary Savings: Theory and Empirics of Earnings and Wealth Inequality," mimeo Queen's University.

——— (2007): "On-the-Job Search and Precautionary Savings: Theory and Empirics of Earnings and Wealth Inequality," mimeo UCL.

Lucas, R. E., and E. C. Prescott (1974): "Equilibrium Search and Unemployment," Journal of Economic Theory, 7, 188 – 209.

Marcet, A., F. Obiols-Homs, and P. Weil (2007): "Incomplete markets, labor supply and capital accumulation," Journal of Monetary Economics, 54(8), 2621–2635.

Merton, R. C. (1969): "Lifetime Portfolio Selection Under Uncertainty: The Continuous-Time Case," Review of Economics and Statistics, 51, 247–257.

Merz, M. (1995): "Search in the labor market and the real business cycle," Journal of Monetary Economics, 36, 269–300.

Meyn, S. P., and R. L. Tweedie (1993): "Stability of Markovian Processes II: Continuous-Time Processes and Sampled Chains," Advances in Applied Probability, 25, 487–517.

Mortensen, D., and C. Pissarides (1994): "Job Creation and Job Destruction in the Theory of Unemployment," Review of Economic Studies, 61, 397–415.

Moscarini, G. (2005): "Job Matching and the Wage Distribution," Econometrica, 73(2), 481–516.

Nakajima, M. (2008): "Business Cycle in the Equilibrium Model of Labor Market Search and Self-Insurance," mimeo FED Philadelphia.

Pissarides, C. A. (1985): "Short-run Equilibrium Dynamics of Unemployment Vacancies, and Real Wages," American Economic Review, 75, 676–90.

Postel-Vinay, F., and J.-M. Robin (2002): "Equilibrium Wage Dispersion with Worker and Employer Heterogeneity," Econometrica, 70, 2295–2350.

Ross, S. M. (1993): Introduction to Probability Models, 5th edition. Academic Press, San Diego.

Sennewald, K. (2006): "Stochastic Control, Optimal Saving, and Job Search in Continuous Time," Dissertation submitted to the University of Dresden.

Shimer, R. (2005): "The Cyclical Behavior of Equilibrium Unemployment and Vacancies," American Economic Review, 95, 25–49.

——— (2007): "Mismatch," American Economic Review, 97(4), 1074–1101.

Shimer, R., and I. Werning (2007): "Reservation Wages and Unemployment Insurance," Quarterly Journal of Economics, 122(3), 1145–1185.

——— (2008): "Liquidity and Insurance for the Unemployed," American Economic Review, 98(3), 1922 – 1942.

Toche, P. (2005): "A tractable model of precautionary saving in continuous time," Economics Letters, 87, 267–272.

Turnovsky, S. J., and W. T. Smith (2006): "Equilibrium consumption and precautionary savings in a stochastically growing economy," Journal of Economic Dynamics and Control, 30(2), 243–278.

van den Berg, G. J. (1990): "Nonstationarity in Job Search Theory," Review of Economic Studies, 57, 255–277.

Wälde, K. (1999): "Optimal Saving under Poisson Uncertainty," Journal of Economic Theory, 87, 194–217.

——— (2005): "Endogenous Growth Cycles," International Economic Review, 46, 867–894.

——— (2008): "Corrigendum to 'Optimal Saving under Poisson Uncertainty'," Journal of Economic Theory, 140, e360 – e364.

Institut de Recherches Économiques et Sociales
Université catholique de Louvain

Place Montesquieu, 3
1348 Louvain-la-Neuve, Belgique

ISSN 1379-244X D/2010/3082/013