On the Optimality of PAYG Pension Systems in an Endogenous Fertility Setting

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Abstract

In order to help in designing an accurate pension reform, we determine the optimal resource allocation in an endogenous fertility model generating a demographic transition. Extending Samuelson's (1975) work in such a setting, we analyze the problem of the interiority of the optimal solution and discuss the serendipity theorem. We then characterize the decentralization of the first best, showing that a pension policy linking pension benefits to the number of children constitutes an optimal social security program able to restore both the optimal capital stock and the optimal rate of population growth as a unique instrument. We also show that neither a Beveridgean pension scheme nor a Bismarckian one can decentralize the first best.

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1 Introduction

It is generally agreed upon that the ageing process experienced in most of the OECD countries requires a reform of the public pension scheme. This is necessary in order to insure their financial solvency threatened by the increase in the dependency ratio. Different policies have been proposed to face this demographic change, but all of them seem to imply a cost in terms of welfare for some generations.

A way to shed some light in this debate is to provide a benchmark by determining the optimal social security program, namely the social security program able to lead to the social optimum. Indeed, as shown by Atkinson and Sandmo (1980), a transfer system that redistributes wealth among generations, such as a pension system, constitutes a potential instrument to achieve the first best allocation. In Diamond's overlapping generations model, Samuelson (1975a) determines the social optimum, the so-called *goldenest golden-rule state*: it is defined as the maximum of the steady-state lifetime well-being of the representative agent, which is attained when the population grows at the optimum population growth rate. He shows that, in general, the competitive solution fails in achieving the goldenest golden-rule state. However, he proves that in the particular case where the population growth rate is optimal, private savings exactly lead to the golden-rule capital level. This result is known as 'Samuelson's *Serendipity Theorem'*. Samuelson (1975b) then shows that, when the population does not grow at its optimal rate, there exists a social security program that converts the laissez-faire equilibrium into a golden-rule state. However, such a program does not allow to reach the goldenest golden rule since population growth is exogenously fixed. It therefore does not constitute an optimal social security system.

While in such a model fertility is exogenous, the *endogeneity* of such a decision seems largely recognized and may have important implications in this debate. First, if fertility is endogenous, it may be possible to design an instrument allowing to achieve the first best by insuring both the optimal capital stock and the optimal population growth rate. Second, if fertility is an individual decision, it may be directly affected by the presence and the design of the pension system. Our purpose is therefore to extend the analysis of Samuelson to an *endogenous* fertility setting and investigate the consequences for Samuelson's results as well as for policy recommendations. This analysis indeed highlights the effects of pension policies on fertility, which is often ignored in the literature, whereas the drop in fertility is well at the root of the current problems of pension schemes.

In order to be consistent with the observed fertility evolution, we endogenize fertility using a model that is able to explain and reproduce the demographic transition at the origin of the financial problems of pay-as-you-go (PAYG) pension systems[1]. Apart from the social security hypothesis[2], the various models that generate an endogenous demographic transition are usually based on one of these two arguments. One deals with the substitution of quality for quantity of children due to economic development. Becker, Murphy and Tamura (1990) and Galor and Weil (2000) rely on this argument in different endogenous growth settings. The other main argument put forward is the increase in the opportunity cost of having children experienced by women in a developed economy. This argument is attractive because, apart from explaining the demographic transition, it is also able to replicate the observed increase in female labor participation. We therefore focus on this last explanation, following Galor and Weil (1996).

In addition, our analysis deals also with the issue of the existence of an interior solution. As noticed by Deardorff (1976) and then analyzed by Michel and Pestieau (1993), the maximization problem of the planner in Samuelson's model is generally not concave and does not have an interior solution for a wide variety of production and utility functions. We examine the shape of the utility function in our endogenous fertility setting.

The paper is organized as follows. In section 2 we describe the model with endogenous fertility. Section 3 examines the planner's problem as well as the existence of an interior optimal solution, comparing it with Samuelson's case. We find that, contrary to Deardorff's case, with a log-linear utility function and a Cobb-Douglas production function an interior optimal population growth rate exists if parameters satisfy a particular condition and children are costly enough. Hence the shape of the indirect utility function changes notoriously when fertility is endogenous, allowing for the existence of an interior global maximum. Section 4 analyzes the steady state solution of the laissez-faire economy and discusses the serendipity theorem in the case of endogenous fertility. In section 5 we show that a policy that links pension benefits to the number of children is able to decentralize the social optimum as a *unique* instrument. This pension policy restores the optimal incentives for individuals to have children, so that the optimal capital and the optimal population growth rate are achieved simultaneously when the payroll tax is properly chosen. Such a pension policy has also the interesting property to solve the pension crisis –by restoring the financial equilibrium of the system. We also show that other types of PAYG pension system, such as a Beveridgean or a Bismarckian system, are not useful to restore the first best.

[1]Some theoretical models with endogenous fertility have studied the capacity of a PAYG social security system to solve the pension crisis, but none of these models are able to replicate the drop in fertility observed since the sixties. For example, Eckstein and Wolpin (1985) introduce a voluntary social security program that gives a return equal to the population growth rate, although this system does not insure the financial equilibrium of the system out of the steady state. Bental (1989) introduces a PAYG system similar to the one we propose, in an economy where children support their parents according to an exogenous social norm.

[2]See for example Cigno (1993) and Wigger (1999).

2 The Model

We consider a two-period overlapping generations model. The economy is constituted of couples, each one formed by one man and one woman. Men and women differ in their ability in the production process. It is assumed that both men and women are endowed with one unit of gender-specific labor. The production function includes three inputs, physical capital and two types of labor input:

$$Y_t = F(K_t, L_t^f, L_t^m)$$

with K_t denoting the capital stock, L_t^f the stock of female labor and L_t^m the amount of male labor. For simplicity, capital is assumed to totally depreciate in the production process.

The crucial assumption allowing for an *endogenous* demographic transition is that the production function $F(\cdot)$ is such that capital is more complementary to female labor than it is to male labor. This assumption can be justified, as in Galor and Weil (1996), by the fact that men have a comparative advantage in physical labor, which is less complementary to capital than mental labor. They therefore provide more physical labor than women, which have a comparative advantage in mental labor. This insures that, as the economy develops and capital increases, the female wage –which, as will be seen further, constitutes the opportunity cost of having children– increases proportionately more than total household income –the sum of the two wages. Hence the substitution effect of an increase in wages dominates the income effect and households decide to have less children. This, in turn, further increases the stock of capital, producing a demographic transition.

Men supply inelastically their unit of labor in the market, while women divide their unit of time between working in the market and raising children[3]. As the total amount of male labor, L_t^m, is equal to the number of working-age couples in the economy, assuming that the production function exhibits constant returns to scale we can express it in per-couple terms as:

$$y_t = f(k_t, l_t^f) \tag{1}$$

with its derivatives $f_k(k_t, l_t^f) > 0$ and $f_l(k_t, l_t^f) > 0$, and where y_t, k_t, l_t^f are respectively per-couple units of output, capital[4] and female labor.

To endogenize fertility, we introduce a taste for children and a cost of children. First, we suppose individuals derive utility from having descendants. Individual preferences can be represented by the following utility function:

$$U(n_t, c_t, d_{t+1}) = \gamma u(n_t) + (1-\gamma)\left[u(c_t) + \beta u(d_{t+1})\right] \tag{2}$$

[3]We could think that women earn lower wages than men, hence have a lower opportunity cost of raising children; or, alternatively, we could think that women have a comparative advantage in childcare due to some natural reason. In any case, this assumption will permit to explain the observed drop in fertility together with the observed increase in female participation in the labor market.

[4]In the following we will use the term capital to refer to the per couple –or per male labor– capital stock.

where $u(\cdot)$ is increasing and concave in its argument, $\gamma \in [0,1]$ is a parameter reflecting the taste for children, $\beta \in [0,1]$ is the subjective discount factor, $n_t = \frac{N_{t+1}}{N_t}$ represents the number of children (expressed in terms of number of couples) that each couple has, while c_t and d_{t+1} are respectively the *couple's* consumption in the first and second period of life. The first derivatives of the utility function with respect to each argument can be written as $\gamma u'(n_t) > 0$, $(1-\gamma)u'(c_t) > 0$, $(1-\gamma)\beta u'(d_{t+1}) > 0$.

Second, we assume children are costly in terms of time. Each couple of children consumes a fraction z of the woman's endowment of time. The inclusion of a time cost of children implies the endogeneity of female labor supply, as it introduces a trade-off between working and having children.

3 The Planner's Problem and the Existence of an Interior Optimal Solution

Samuelson (1975a) determines the optimum growth rate for population in the simple two-period overlapping generations model à la Diamond (1965), in which population grows at an *exogenous* constant rate. In his work, Samuelson assumes the existence of an interior optimal solution. However, a year after its publication, Deardorff (1976) states that, for a wide range of utility and production functions, Samuelson's problem does not have an interior global maximum. In this section, we solve the planner's problem in our model with endogenous fertility, and describe how the optimal rate of population growth is determined. We then turn to the problem of the interiority of the optimal solution, and show that, contrary to Samuelson's case, with endogenous fertility there exists the possibility of having an interior global maximum with Cobb-Douglas utility and production functions.

3.1 The Planner's Problem

Following Samuelson (1975a), we assume that the planner maximizes the utility of the representative agent at the steady state.

Definition 1 *An optimal allocation at the steady state is a set of positive quantities (c,d,n,k,l^f) that solve the planner's problem:*

$$\max_{c,d,n,k} \gamma u(n) + (1-\gamma)\left[u(c) + \beta u(d)\right]$$

subject to the resource constraint of the economy:

$$f\left(k, l^f\right) = c + \frac{d}{n} + nk$$

where $l^f = 1 - zn$.

If it exists[5], an interior optimal solution is characterized by the following optimality conditions:

$$\frac{u'(c)}{u'(d)} = \beta n \tag{3}$$

$$f_k\left(k, l^f\right) = n \tag{4}$$

$$\frac{\gamma}{1-\gamma}\frac{u'(n)}{u'(c)} + \frac{d}{n^2} = z f_l\left(k, l^f\right) + k \tag{5}$$

$$f\left(k, l^f\right) = c + \frac{d}{n} + nk \tag{6}$$

$$l^f = 1 - zn \tag{7}$$

The first equation replicates the individual allocation of consumption across time. The second condition is the so-called golden rule, which determines the optimal stock of capital. Equation (5) is the first order condition determining the optimal number of children or population growth rate by equalizing the marginal benefit of children to their marginal cost. The former is given by the marginal utility provided by children to their parents –in terms of consumption– plus the so-called *intergenerational transfer effect* ($\frac{d}{n^2}$). This effect captures the fact that, when population grows, there are more working individuals to support each retired person, which reduces the relative cost of consumption of the old. The marginal cost of children for the planner is the loss in production due to the time cost of children plus the so-called *capital dilution effect* (k), according to which, the higher the population growth, the higher the investment requirement to keep a constant capital stock. In Samuelson's model, the optimal population growth rate was given by the equalization of the intergenerational transfer effect to the capital dilution effect. The other two terms were not present as in his model there is no taste nor cost of children.

3.2 The Interiority of the Optimal Solution

Deardorff (1976) shows that with standard preferences[6] the planner's problem in Samuelson's model does not have an interior solution whenever the production function is unbounded, i.e. when "it places no upper bound to the per capita output as the capital-labor ratio becomes very large". This includes the typical case of a Cobb-Douglas production function and also a CES production function

[5] The problem of the planner is indeed not necessarily concave. The next subsection details the conditions insuring the existence of an interior global maximum.

[6] Utility increasing monotonically in both of its arguments.

with an elasticity of substitution greater than 1. For instance, with a Cobb-Douglas production function and a log-linear utility function, there exists no interior solution to the planner's problem when capital depreciation is total, while there may only exist an interior *minimum* if capital depreciation is lower than 1. Michel and Pestieau (1993), using CES utility and production functions and assuming total depreciation of capital, analyze the conditions guaranteeing the existence of an interior solution. They conclude that, in order to have an *interior global maximum*, there must exist complementarity between labor and capital in production, as in Deardorff's analysis. Alternatively, if the production function is of the Cobb-Douglas type, complementarity between first and second period consumption in preferences is required. In all other cases, the optimal population growth rate is a corner solution.

In the following we analyze whether it is possible to have an *interior global maximum* in the planner's problem when the framework has been changed to include children as a decision variable for the agents. The existence of an interior solution depends on the shape of the utility and production functions. After giving necessary and sufficient conditions for the existence of such a solution for general preferences and technology, we focus on the case of a log-linear utility function and a Cobb-Douglas production function. This allows us to contrast our results in an endogenous fertility setting with the exogenous fertility setting considered by Samuelson where the optimal solution of the planner was in this case a corner solution.

The general case

A necessary condition to have an interior global maximum is that there is a solution to the planner's set of first order conditions. In addition, the Hessian matrix corresponding to the planner's problem evaluated at such a critical point must be negative semidefinite; this guarantees that this point is a local maximum. A sufficient condition that guarantees that the maximum is *unique* and *global* is that the planner's objective is strictly concave, i.e. that the Hessian matrix is negative definite at all points. In appendix A we determine necessary and sufficient conditions to have this result.

The case of a log-linear utility function and a Cobb-Douglas production function

In such a case, the problem is drastically simplified since the indirect utility function can be expressed as a function of n only. Preferences and technology are assumed to satisfy:

$$U_t(n_t, c_t, d_{t+1}) = \gamma \log(n_t) + (1-\gamma)\left[\log(c_t) + \beta \log(d_{t+1})\right] \tag{8}$$

$$f(k_t, l_t) = A k_t^{\alpha} (l_t^f)^{1-\alpha} + B \tag{9}$$

with $A > 0$ and $B > 0$. Observe that this production function allows a demographic transition to be generated[7] but represents a strong assumption as it implies that male labor is not complementary to capital. It however strongly eases the analytical resolution.

Using the first order conditions of the planner's problem, the indirect utility function can be expressed as:

$$V(n) = \gamma \log(n) + (1-\gamma)\left[\log(c(n)) + \beta \log(d(n))\right]$$

with:

$$k = k(n) = (1 - zn)\left(\frac{A\alpha}{n}\right)^{\frac{1}{1-\alpha}}$$

$$c = c(n) = \frac{1}{1+\beta}\left[\frac{1-\alpha}{\alpha}(A\alpha)^{\frac{1}{1-\alpha}}(1-zn)\, n^{\frac{-\alpha}{1-\alpha}} + B\right]$$

$$d = d(n) = \beta n \frac{1}{1+\beta}\left[\frac{1-\alpha}{\alpha}(A\alpha)^{\frac{1}{1-\alpha}}(1-zn)\, n^{\frac{-\alpha}{1-\alpha}} + B\right]$$

Proposition 2 *With a log-linear utility function and a Cobb-Douglas production function satisfying (8) and (9), there exists a* ***unique interior global maximum*** *iff:*

$$\alpha \leq \frac{\gamma + \beta(1-\gamma)}{1 + 2\beta(1-\gamma)} \equiv \widetilde{\alpha}_1$$

and

$$z > z_{\min}$$

If $\widetilde{\alpha}_1 < \alpha < 1/2$, the global maximum is reached when $n \to 0$.

The proof is given in appendix B[8]. The intuition for these conditions can be easily seen from the planner's first order condition with respect to n, equation (5). In order for the planner's objective to be hump-shaped and to achieve an interior maximum, we need that the marginal benefit of children –which corresponds to the left-hand-side of equation (5)– dominates the marginal cost –i.e. the right-hand-side of (5)– for low values of n, and that the opposite happens for high enough values of n. The first condition in proposition 2 insures that, for sufficiently low values of n, the marginal benefit of children is higher than their marginal cost. Observe that it requires that labor is sufficiently

[7] Recall that the only condition required is that female labor is more complementary to capital than male labor is.

[8] It can also be shown that, in a model with a monetary cost (instead of a time cost) of children and with exogenous labor supply, the result of the existence of an interior optimum for a sufficiently low value of α would still hold. See Abío (2002).

important in the production process (α sufficiently low, so that the capital dilution effect is not very important), that the taste for children γ is sufficiently high (so that the marginal utility term is important enough), and that future consumption is not discounted too much by individuals (β sufficiently high, so that the intergenerational transfer effect is sufficiently valued). For larger values of n, due to the different proportions in which the terms in equation (5) depend on n, the marginal cost dominates the marginal benefit and utility is decreasing in n. The second condition requires however that the cost per child is high enough so that utility is maximized for a feasible value of the fertility rate, that is for $zn < 1$.

4 The Laissez-Faire Solution and the Serendipity Theorem

4.1 The Laissez-Faire Economy

In the first period of their life, couples raise their children, supply labor in the labor market, consume and save. In the second period, they consume the products of their savings. The budget constraints of the couple are then given by:

$$c_t + s_t = w_t^m + w_t^f(1 - zn_t) \tag{10}$$

$$d_{t+1} = R_{t+1}s_t \tag{11}$$

with w_t^f and w_t^m being respectively the wages for female and male labor, R_{t+1} the gross interest rate in period $t+1$ and s_t denoting the savings made in period t.

Maximizing the utility function of the couple (2) subject to these two budget constraints gives the two following first order conditions:

$$\frac{u'(c_t)}{u'(d_{t+1})} = \beta R_{t+1} \tag{12}$$

$$\frac{\gamma}{1-\gamma}\frac{u'(n_t)}{u'(c_t)} = zw_t^f \tag{13}$$

Equation (12) determines the allocation of consumption across time and therefore the amount of savings s_t. The second first order condition (13) determines the total amount of time devoted to child-raising by equalizing the marginal utility of children to their opportunity cost, both in terms of consumption. As noted above, the higher the female wage, the higher is the opportunity cost of children.

The competitive behavior of the representative firm leads to the equalization of factor prices to their marginal productivity:

$$R_t = f_k(k_t, l_t^f) \tag{14}$$

$$w_t^f = f_l(k_t, l_t^f) \tag{15}$$

$$w_t^m = f(k_t, l_t^f) - k_t f_k(k_t, l_t^f) - l_t^f f_l(k_t, l_t^f) \tag{16}$$

Capital comes from savings in the previous period. The capital market equilibrium condition is therefore given by:

$$k_{t+1} n_t = s_t \tag{17}$$

Finally, the labor market equilibrium condition is:

$$l_t^f = 1 - z n_t \tag{18}$$

Definition 3 *A steady state in the laissez-faire economy is a stationary path of variables* $(c, d, n, s, k, l^f, w^f, w^m, R)$ *with positive quantities verifying the following conditions:*

$$\frac{u'(c)}{u'(d)} = \beta R \tag{19}$$

$$\frac{\gamma}{1-\gamma} \frac{u'(n)}{u'(c)} = z w^f \tag{20}$$

$$c + s = w^m + w^f (1 - zn) \tag{21}$$

$$d = Rs \tag{22}$$

$$l^f = 1 - zn \tag{23}$$

$$s = kn \tag{24}$$

$$R = f_k(k, l^f) \tag{25}$$

$$w^f = f_l(k, l^f) \tag{26}$$

$$w^m = f(k, l^f) - k f_k(k, l^f) - l^f f_l(k, l^f) \tag{27}$$

Appendix C proves the existence of a unique steady state in the case of log-linear utility and Cobb-Douglas production specified in equations (8) and (9).

4.2 The Serendipity Theorem

Comparing the planner's solution with the laissez-faire solution at the steady state, Samuelson (1975a) shows that if population grows –by chance– at its optimal rate, the laissez-faire equilibrium just reaches what he called the *goldenest golden-rule state*. Hence, at the optimal rate of population growth, laissez-faire private savings exactly lead to the golden-rule capital stock: when $n = n^*$, $k = k^*$ also. This result is known as 'Samuelson's Serendipity Theorem'.

By comparing the steady state solutions for the planner and for the laissez-faire economy, it can be shown (see appendix D) that these two solutions can be simplified to the following two-equation sets:

$$\frac{\gamma}{1-\gamma}\frac{u'(n)}{u'(c)} + \frac{d}{nf_k\left(k,l^f\right)} = zf_l\left(k,l^f\right) + k \tag{28}$$

$$f_k\left(k,l^f\right) = n \tag{29}$$

for the planner, and:

$$\frac{\gamma}{1-\gamma}\frac{u'(n)}{u'(c)} + \frac{d}{nf_k\left(k,l^f\right)} = zf_l\left(k,l^f\right) + k \tag{30}$$

$$k = \frac{d}{nf_k(k,l^f)} \tag{31}$$

for the laissez-faire. Note that, as in the case with exogenous population growth analyzed by Samuelson, one of the equations that characterize the laissez-faire steady state solution can be expressed as the equation determining the optimal population growth rate or fertility rate[9]. The other equation is different. While the planner chooses the optimal capital stock from the golden-rule equation, k in the laissez-faire economy is determined by equalizing the capital dilution to the intergenerational transfer effect. Hence, the laissez-faire and optimal solutions do in general not coincide.

They would only coincide if by chance the capital dilution and intergenerational transfer effects cancelled each other in the optimal solution[10]. In this situation, equation (31) would be satisfied in the planner's problem, so the two solutions would be identical and the laissez-faire would reach the social optimum. However, in the general case where these two effects do not cancel each other, it can in principle happen that in the laissez-faire solution the optimal population growth rate is reached. Hence, we may have a situation where the golden rule would not be attained in the laissez-faire solution (the capital would not be optimal), but where the population growth would be optimal. In other

[9] With exogenous fertility, that equation corresponds to the capital market equilibrium condition.

[10] Note that this might not be possible, i.e. if the set of equations (28), (29) and (31) has no solution.

words, we cannot rule out that a steady state with the optimal n but a non-optimal k could be a solution of the laissez-faire economy.

Hence we do not know whether the serendipity theorem holds in our framework[11]. In any case, with endogenous fertility, this theorem might not be relevant any more. In this case, fertility is an individual decision, and may therefore be potentially affected by policy variables. Hence, in place of trying to determine whether the achievement of the optimal n by chance in the laissez-faire necessarily leads to the optimal k, it seems more interesting to focus on the more general case in which population does not grow at its optimal rate and to determine the policy able to decentralize this optimum. The next section investigates the role of pension policy in that case.

5 The Optimal Pension Policy

As shown above, and as it typically happens in OLG models, the laissez-faire solution is in general not optimal . With endogenous fertility, contrary to Samuelson's case, the rate of population growth can be affected by policy instruments. This implies that there may exist a decentralization policy able to achieve the first best, namely the optimal capital stock and the optimal population growth rate. In this section, we study how different pension policies affect the decisions of the couples in our model, and in particular, whether they can be used to decentralize the first best. Samuelson (1975b) shows that, in the case of overaccumulation of capital in the laissez-faire economy, the introduction of a Beveridgean PAYG pension system can lead to the golden rule. In this section we show that in the model with endogenous fertility this policy also allows the golden rule to be attained, although it is not able to decentralize the first best –i.e. the potentially achieved golden rule is not the goldenest one, as in Samuelson's case. We then analyze the effects of introducing other PAYG pension policies. We show that a Bismarckian pension system similar to those present in most OECD countries cannot be used to decentralize the social optimum either. By contrast, a system that links pensions to the number of children, by introducing the social security system's budget constraint into the couple's constraints, is shown to provide couples with the optimal incentives to choose both the optimal capital stock and the optimal fertility rate[12].

5.1 A Beveridgean PAYG Pension System

Suppose that we introduce an instrument in the competitive economy in order to induce individuals to choose the golden-rule level of capital. Following Samuelson (1975b), we use a pension system and we assume this system is a pure PAYG scheme[13]. A proportional taxation rate, τ_t, is levied on male labor

[11] i.e. when $n = n^*$, it is not possible to prove that $k = k^*$ nor that $k \neq k^*$.

[12] Throughout the section we assume that the planner's solution is interior.

[13] Samuelson (1975b) introduces a social security system that has both a PAYG component and a funded component. The use of a pure PAYG transfer system implies that, in case of

income[14] and each retired person at t receives a pension benefit p_t coming from contributions in the same period. For convenience, we will refer to this type of pension system as *pure Beveridgean PAYG pension system.*

Definition 4 *A steady state competitive equilibrium under a pure Beveridgean PAYG pension system is a transfer system* (p, τ) *satisfying:*

$$p = n\tau w^m \tag{32}$$

and a vector of variables $(c, d, n, s, l^f, k, w, R)$ *satisfying (19), (20), (23), (24), (25), (26), (27) and:*

$$c + s = w^m(1 - \tau) + w^f(1 - zn) \tag{33}$$

$$d = sR + p \tag{34}$$

Observe that couples perceive their pension as a fixed amount of money, not being aware of the social security budget constraint. Suppose that the government fixes the value of the payroll tax τ and pension benefits are determined endogenously from equation (32). Then the government can choose the value of the contribution rate such that the golden rule is attained. However, as stated in the following proposition, this instrument cannot decentralize the *goldenest* golden rule state, as it is not able to insure simultaneously the optimal capital stock and the optimal rate of population growth.

Proposition 5 *A Beveridgean PAYG pension system cannot be used to decentralize the social optimum.*

The proof is developed in appendix E. By comparing the two sets of equations, we show that this policy is not enough to restore the social optimum. If the policy instrument is used to attain the golden rule, the optimal values of the capital stock and the population growth rate are not achieved. Alternatively, if τ is chosen so as to reach the optimal k, neither the golden rule nor the optimal n would be attained. And something similar would happen if we chose the payroll tax so as to reach the optimal population growth rate. Hence, another instrument is required to decentralize the first best.

5.2 A Bismarckian PAYG Pension System

Although the previous analysis is interesting as a starting point, one might point out that in reality individuals do not perceive their pension as fixed. In fact, in most OECD countries, pension benefits are defined as a replacement rate on past

underaccumulation, the payroll tax will be negative and the system will no longer be a pension system.

[14]This does not produce any distortion in the choice of labor, since male labor supply is inelastic. Note also that it can be shown that proposition 5 would still hold if both types of labor were taxed.

wages. This replacement rate often depends on labor participation, as benefits are a function of the number of years of contribution to the system. Moreover, in many countries, recent reforms of the pension system tend to increase the proportionality between contribution years and pension benefits.

In the following we analyze a pension system where benefits are defined according to the following pension formula:

$$p_{t+1} = \theta_{t+1} \left[w_t^m + w_t^f (1 - zn_t) \right] \tag{35}$$

where θ_{t+1} is the gross replacement rate on the couple's labor earnings, so that, from the *couple's point of view*, the pension is positively related to female labor force participation and hence negatively related to fertility. We refer to this PAYG pension system as *Bismarckian pension system.*

The social security system's budget constraint, which must be balanced every period, can be expressed as:

$$\theta_{t+1} \left[w_t^m + w_t^f (1 - zn_t) \right] = n_t \tau_{t+1} \left[w_{t+1}^m + w_{t+1}^f (1 - zn_{t+1}) \right] \tag{36}$$

Under such a system, which is assumed to tax all wages[15], the first order conditions of the couple's maximization problem are:

$$\frac{u'(c_t)}{u'(d_{t+1})} = \beta R_{t+1} \tag{37}$$

$$\frac{\gamma}{1-\gamma} \frac{u'(n_t)}{u'(c_t)} = z w_t^f \left[(1 - \tau_t) + \frac{\theta_{t+1}}{R_{t+1}} \right] \tag{38}$$

Factor prices are still given by (14), (15), (16), and the equilibrium conditions for the labor market and the capital market by (17) and (18).

Definition 6 *A steady state competitive equilibrium under a Bismarckian PAYG pension system is a transfer system (p, θ, τ) satisfying:*

$$p = \theta \left[w^m + w^f (1 - zn) \right] \tag{39}$$

$$\theta = n\tau \tag{40}$$

and a vector of variables $(c, d, n, s, l^f, k, w^f, w^m, R)$ satisfying (19), (23), (24), (25), (26), (27) and:

$$\frac{\gamma}{1-\gamma} \frac{u'(n)}{u'(c)} = z w^f \left[(1 - \tau) + \frac{\theta}{R} \right] \tag{41}$$

$$c + s = \left[w^m + w^f (1 - zn) \right] (1 - \tau) \tag{42}$$

$$d = sR + \theta \left[w^m + w^f (1 - zn) \right] \tag{43}$$

[15]As it corresponds to reality, i.e. female labor is also taxed by the social security administration. However, the main result that the system cannot decentralize the social optimum does not change if only male labor is taxed.

As can be seen from equation (41), the existence of a Bismarckian pension system affects the fertility decision in two ways. On the one hand, the payroll tax *reduces* the opportunity cost of children in terms of net wage, having a positive effect on fertility. On the other hand, the social security system increases such a cost by reducing second period consumption (through lower pension benefits), having a negative effect on the choice of the number of children. We could then expect that these two opposite effects on the fertility decision would allow to internalize the capital dilution and the intergenerational transfer effects. However, it is not the case, as explained in the following proposition.

Proposition 7 *The Bismarckian pension system cannot be used to decentralize the planner's optimum.*

Proof. Suppose that the government chooses the value of the replacement rate, and then determines the payroll tax according to equation (40) and pension benefits according to (39)[16]. Using (26) and (40), equation (41) can be written as:

$$\frac{\gamma}{1-\gamma}\frac{u'(n)}{u'(c)} = zf_l(k,l^f)\left[1-\frac{\theta}{n}+\frac{\theta}{R}\right]$$

Therefore, if we choose the payroll tax so that the golden rule is satisfied, i.e. $R=n$, the last two terms of the previous expression cancel out and the first order condition with respect to n becomes:

$$\frac{\gamma}{1-\gamma}\frac{u'(n)}{u'(c)} = zf_l(k,l^f)$$

which always differs from the optimal fertility decision, unless by chance the capital dilution and intergenerational transfer effects cancel each other in the planner's solution and thus disappear. However, in such a case the optimal pension policy would be to have no transfer system, as we have seen in the previous section. Hence, this pension policy cannot be used to decentralize the first best. ■

We can also remark that, in addition to the two direct effects on the couple's fertility decision, a Bismarckian pension system produces several *general equilibrium effects* by affecting capital accumulation and therefore factor prices. Taking into account all these effects, the sign of the impact of such a Bismarckian pension system on fertility at the steady state is *ambiguous* and crucially depends on the benefit formula as well as on preferences and technology. It can however be proven that with the Cobb-Douglas production function and the log-linear utility function (8) and (9), an increase in the size of the Bismarckian pension system *increases* the steady state fertility level, while it decreases the steady state capital stock[17].

[16]Results would not change if the government fixed the contribution rate instead and the replacement rate was adjusted.

[17]As the utility is log-linear, the substitution and income effects cancel each other and the pension system only affects the fertility decision through general equilibrium effects. The payroll tax crowds out capital and pushes the economy back to an earlier stage of the demographic transition. See Abío (2002).

5.3 A PAYG System Linking Pensions to the Number of Children

In the following we analyze whether there is a transfer system under which the achievement of the golden rule restores the social optimum, so that *both* the optimal capital stock and the optimal population growth rate are achieved *simultaneously.* The answer is positive: a PAYG pension system that links pension benefits to fertility and to future wages[18] –i.e. to the children's contribution to the pension scheme– can decentralize the first best.

Suppose we introduce a pension system in which pension benefits are distributed proportionally to fertility behavior as well as to the level of future wages. As before, to avoid any distortions on the labor supply choice, we assume this system is financed by proportional taxation on male labor income only[19]. We refer to this type of pension system as *PAYG pension system with fertility link.* With such a pension policy, the budget constraints of the couple become:

$$c_t + s_t = w_t^m(1-\tau_t) + w_t^f(1-zn_t) \tag{44}$$

$$d_{t+1} = s_t R_{t+1} + w_{t+1}^m \tau_{t+1} n_t \tag{45}$$

where τ_t is the social security contribution rate, fixed by the government. As can be seen in (45), the pension formula exactly replicates the budget constraint of the PAYG pension system, given by:

$$p_{t+1} = n_t w_{t+1}^m \tau_{t+1}$$

Hence, this pension policy *de facto* insures the financial balance of the system and provides couples with the information that their pensions depend on the productivity growth rate as well as on the population growth rate.

The first order conditions of the maximization program of the couples expressed at the steady state become:

$$\frac{u'(c)}{u'(d)} = \beta R \tag{46}$$

$$\frac{\gamma}{1-\gamma}\frac{u'(n)}{u'(c)} + \frac{w^m\tau}{R} = zw^f \tag{47}$$

In contrast with the pension schemes considered before, such a PAYG pension scheme constitutes an *optimal* social security system, as stated in the following proposition:

[18] As the pension scheme is still PAYG financed.

[19] It is therefore equivalent to using a lump-sum tax, which should be used if male labor was also elastically supplied. It can indeed be shown that a system taxing both types of labor would not restore the social optimum.

Proposition 8 *In a model with endogenous fertility, if an interior optimal allocation exists, a PAYG social security system with fertility link decentralizes the first best if the payroll tax satisfies:*

$$\tau^* = \frac{\frac{d}{n} - kn}{f(k, l^f) - kn - l^f f_l(k, l^f)} \tag{48}$$

The proof is shown in appendix F. The reason for such a policy to constitute a *unique* instrument to reach the first best comes from the fact that such a pension system introduces the links that are missing in the couple's fertility decision. As can be seen in (45), this policy introduces a specific link between n and d and between n and k. It introduces a positive link between consumption of the old and the fertility decision (internalizing the intergenerational transfer effect). Moreover, it affects the capital accumulation because it introduces another way of saving through having children. This allows the capital dilution effect to be internalized. Hence, for any value of the payroll tax, the proposed policy corrects the divergences from the optimal fertility decision. The value of the payroll tax can then be chosen so as to restore the optimal capital stock, accomplishing the golden rule. Since the individual allocation rule of consumption over the life-cycle is the same as the optimal one, there is no need for another instrument to allocate consumption optimally. Thus, for any interior optimal allocation there exists a transfer system τ^* such that this allocation is a steady state intertemporal equilibrium with perfect foresight. This policy is the *only* instrument required to reach the optimal fertility rate and capital.

Note that, as it is typically the case when using such a decentralization instrument, the sign of the transfer is not necessarily positive. Since one effect is positive –intergenerational transfer effect– and the other one is negative –capital dilution effect–, the sign of the transfer depends on the balance of the two. The weight of each effect depends on the parameters of the production and utility functions as well as on the size of n. A negative value for τ^* implies a transfer mechanism from the old to the young that would no longer be a pension system.

Finally remark that in addition to restore the social optimum, such a transfer system is attractive as it suggests a pension policy that could solve the financial crisis of the pension system. Indeed, the financial crisis of PAYG pension systems experienced by many developed countries is mainly due to the difference between the macroeconomic and the microeconomic return of the system: the pension formula is not related to the population growth rate, whereas this rate really matters at the macroeconomic level. Hence, an interesting way of solving the crisis suggested by this analysis could be to redefine the system in a way that links pension benefits to the fertility behavior of individuals. Alternatively, a compensatory family allowance –which may be more politically feasible– equivalent to the present value of the future contribution of their children to the pension scheme could be introduced[20].

[20] See Loupias and Wigniolle (2000).

6 Conclusion

In order to shed some light on the current debate over the pension reform required by the drop in fertility, this paper extends the analysis of Samuelson to a framework of endogenous fertility. To this effect, we introduce a taste for children and a time cost of raising them, so that female labor supply becomes an endogenous joint decision. Compared with the Samuelson's case, the following results are obtained. First, we show that in the case of endogenous fertility the existence of a global maximum in the planner problem is more likely. In particular, the introduction of these elements eliminates the problem of the non-existence of an interior optimal solution in the case of a Cobb-Douglas production function and a log-linear utility function. Second, while in Samuelson's model no policy instruments could be used to restore the social optimum –which could only be reached if the exogenous population growth rate happened to be the optimal (the so-called 'Serendipity Theorem')– we find that a policy linking pension benefits to the social security contribution of children can be used as the *unique* instrument to restore the social optimum. The crucial point is that this policy introduces the links that are missing in the individual's fertility decision, so that the instrument –the contribution rate– can be chosen so as to attain the golden rule. In addition, from a positive point of view, the policy that links pension benefits to fertility and wages has also the interesting characteristic of restoring the financial equilibrium of the PAYG pension system, isolating it from demographic shocks. By contrast, we also show that both a pure Beveridgean system with constant pensions and a Bismarckian system where benefits are proportional to labor force participation fail to lead the economy to the first best.

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A Concavity of the Objective Function of the Planner

In this appendix we determine sufficient conditions guaranteeing that the objective function of the planner is strictly concave.

The assumptions imposed on the utility and production functions are:

- $U_i(\cdot) > 0, U_{ii}(\cdot) < 0, U_{ij}(\cdot) = 0, \qquad \forall i, j = n, c, d, \qquad i \neq j.$
- $f_k(\cdot) > 0, f_l(\cdot) > 0, f_{kk}(\cdot) < 0, f_{ll}(\cdot) < 0, f_{kl}(\cdot) = f_{lk}(\cdot) > 0.$

We first reduce two dimensions of the problem by eliminating variables d and l^f, using the resource constraint. Hence the planner's objective is expressed as $U(c, d(k), n)$ and the production function as $f(k, l^f(n))$. The Hessian matrix can be written as:

$$H = \begin{bmatrix} \frac{\partial U^2}{\partial^2 c} & \frac{\partial U^2}{\partial c \partial k} & \frac{\partial U^2}{\partial c \partial n} \\ \frac{\partial U^2}{\partial c \partial k} & \frac{\partial U^2}{\partial^2 k} & \frac{\partial U^2}{\partial n \partial k} \\ \frac{\partial U^2}{\partial c \partial n} & \frac{\partial U^2}{\partial n \partial k} & \frac{\partial U^2}{\partial^2 n} \end{bmatrix}$$

Using the principal minors' method, we know the Hessian is negative definite if the following conditions are satisfied:

1. $\frac{\partial U^2}{\partial^2 c} < 0$
2. $\frac{\partial U^2}{\partial^2 k} < 0$
3. $\frac{\partial U^2}{\partial^2 n} < 0$
4. $\frac{\partial U^2}{\partial^2 c}\frac{\partial U^2}{\partial^2 k} - \left(\frac{\partial U^2}{\partial c \partial k}\right)^2 > 0$
5. $\frac{\partial U^2}{\partial^2 c}\frac{\partial U^2}{\partial^2 n} - \left(\frac{\partial U^2}{\partial c \partial n}\right)^2 > 0$
6. $\frac{\partial U^2}{\partial^2 k}\frac{\partial U^2}{\partial^2 n} - \left(\frac{\partial U^2}{\partial n \partial k}\right)^2 > 0$
7. $\frac{\partial U^2}{\partial^2 c}\frac{\partial U^2}{\partial^2 k}\frac{\partial U^2}{\partial^2 n} + 2\frac{\partial U^2}{\partial c \partial k}\frac{\partial U^2}{\partial c \partial n}\frac{\partial U^2}{\partial n \partial k} - \left(\frac{\partial U^2}{\partial c \partial n}\right)^2 \frac{\partial U^2}{\partial^2 k} - \left(\frac{\partial U^2}{\partial n \partial k}\right)^2 \frac{\partial U^2}{\partial^2 c} - \left(\frac{\partial U^2}{\partial c \partial k}\right)^2 \frac{\partial U^2}{\partial^2 n} < 0$

Conditions (1)-(4), given the assumptions made on preferences, are automatically satisfied:

$$\frac{\partial U^2}{\partial^2 c} = \underbrace{U_{cc}}_{<0} + n^2 \underbrace{U_{dd}}_{<0} < 0$$

$$\frac{\partial U^2}{\partial^2 k} = \underbrace{U_{dd}}_{<0} n^2 \underbrace{(f_k - n)^2}_{>0} + \underbrace{U_d}_{>0} n \underbrace{f_{kk}}_{<0} < 0$$

$$\frac{\partial U^2}{\partial^2 n} = \underbrace{U_{dd}}_{<0} \underbrace{\left[f_n n + \frac{d}{n} - kn\right]^2}_{>0} + \underbrace{U_d}_{>0} \underbrace{\left[\underset{<0}{f_{nn}} n + 2\underset{<0}{f_n} - 2k\right]}_{<0} + \underbrace{U_{nn}}_{<0} < 0$$

$$\frac{\partial U^2}{\partial^2 c}\frac{\partial U^2}{\partial^2 k} - \left(\frac{\partial U^2}{\partial c \partial k}\right)^2 = \underbrace{\underbrace{\left[U_{cc} + n^2 U_{dd}\right]}_{<0} \underbrace{\left[U_{dd} n^2 (f_k - n)^2 + U_d n f_{kk}\right]}_{<0}}_{>0} -$$

$$-\underbrace{\left[n^2 U_{dd}(f_k - n)\right]^2}_{>0} = \underbrace{\underbrace{U_{cc}}_{<0} \underbrace{\left[U_{dd} n^2 (f_k - n)^2 + U_d n f_{kk}\right]}_{<0}}_{>0} + \underbrace{\underbrace{U_{dd}}_{<0} \underbrace{U_d}_{>0} n^3 \underbrace{f_{kk}}_{<0}}_{>0} > 0$$

Regarding condition (5),

$$\frac{\partial U^2}{\partial^2 c}\frac{\partial U^2}{\partial^2 n} - \left(\frac{\partial U^2}{\partial c \partial n}\right)^2 = \underbrace{\left[U_{cc} + n^2 U_{dd}\right]}_{<0} \left[\underbrace{U_{dd}\left[f_n n + \frac{d}{n} - kn\right]^2}_{>0} + \right.$$

$$\left. + \underbrace{\left[U_d \left[f_{nn} n + 2f_n - 2k\right] + U_{nn}\right]}_{>0}\right] - \underbrace{\left[U_{dd} n \left[f_n n + \frac{d}{n} - kn\right] + U_d\right]^2}_{>0} =$$

$$= \underbrace{U_{cc} U_{dd}\left[f_n n + \frac{d}{n} - kn\right]^2}_{>0} + \underbrace{\left[U_{cc} + n^2 U_{dd}\right]\left[U_d \left[f_{nn} n + 2f_n - 2k\right] + U_{nn}\right]}_{>0} -$$

$$\underbrace{-2 U_d U_{dd} n \left[f_n n + \frac{d}{n} - kn\right]}_{?} - \underbrace{(U_d)^2}_{>0}$$

which is positive if the following condition is satisfied:

$$\underbrace{U_{cc} U_{dd}\left[f_n n + \frac{d}{n} - kn\right]^2}_{>0} + \underbrace{U_{cc} U_d \left[f_{nn} n + 2f_n - 2k\right]}_{>0} + \underbrace{\left[U_{cc} + n^2 U_{dd}\right] U_{nn}}_{>0} +$$

$$\underbrace{+ U_d U_{dd} n \left[f_{nn} n^2 - 2\frac{d}{n}\right]}_{>0} > \underbrace{(U_d)^2}_{>0}$$

A sufficient condition on the utility function is hence that:

$$-2U_d U_{dd} d - (U_d)^2 > 0 \tag{49}$$

which is always satisfied in the case of log-linear utility.

Condition (6), on the other hand, is also ambiguous:

$$\underbrace{\left[U_{dd} n^2 (f_k - n)^2 + U_d n f_{kk}\right]}_{<0} \underbrace{\left[U_{dd} \left[f_n n + \frac{d}{n} - kn\right]^2 + U_d (f_{nn} n + 2f_n - 2k) + U_{nn}\right]}_{<0} -$$
$$-\underbrace{\left[U_{dd} \left(f_n n + \frac{d}{n} - kn\right) n (f_k - n) + U_d (f_{kn} n + f_k - 2n)\right]^2}_{>0}$$

and will be positive if the following condition is satisfied:

$$\left[U_{dd} n^2 (f_k - n)^2 + U_d n f_{kk}\right] \left[U_d (f_{nn} n + 2f_n - 2k) + U_{nn}\right] -$$
$$-2U_{dd} U_d n \underbrace{(f_k - n)}_{?} \underbrace{\left(f_n n + \frac{d}{n} - kn\right)}_{?} \underbrace{(f_{kn} n + f_k - 2n)}_{?} +$$
$$+U_d n f_{kk} U_{dd} \left[f_n n + \frac{d}{n} - kn\right]^2 - U_d^2 (f_{kn} n + f_k - 2n)^2 > 0$$

Therefore, the following sufficient conditions on the production function insure that the sixth principal minor is positive:

$$0 \leq f_k - n < n \tag{50}$$

$$\underbrace{\underbrace{2(\underbrace{f_n}_{<0} - k)n}_{<0} \underbrace{f_{kk}}_{<0}}_{>0} > \underbrace{(f_k - 2n)(2f_{kn} n + f_k - 2n)}_{>0 \text{ if condition (50) above holds}} \tag{51}$$

The seventh principal minor corresponds to the whole Hessian matrix and is more complex. The following is a necessary condition for its determinant to be negative, as required by condition (7):

$$\underbrace{U_d n f_{kk}\left[U_{nn}\left(U_{cc}+n^2U_{dd}\right)+U_{dd}U_{cc}\left(f_n n+\frac{d}{n}-kn\right)^2+\underbrace{\left(-2U_dU_{dd}d-(U_d)^2\right)}_{>0 \text{ if condition (49) holds}}\right]}_{<0 \text{ if condition (49) above holds}}+$$

$$\underbrace{+U_{cc}\left(U_d\right)^2\underbrace{\left[2f_{kk}n(f_n-k)-(f_k-2n)\left(2f_{kn}n+f_k-2n\right)\right]}_{>0 \text{ if condition (51) above holds}}}_{<0 \text{ if conditions (50) and (51) hold}}+U_{dd}U_{cc}n\left(f_k-n\right)\cdot$$

$$\cdot\left[\left(f_k-n\right)\left(U_{nn}n+U_d\left[f_{nn}n^2-2\frac{d}{n}\right]\right)-2U_dn\underbrace{\left(f_nn+\frac{d}{n}-kn\right)}_{?}\left(f_{kn}-1\right)\right]+$$

$$+\underbrace{\left[-U_{dd}\left(U_d\right)^2n^4\left(1-2f_{kn}\right)\right]}_{>0}<0 \tag{52}$$

Hence a sufficient condition that guarantees that the objective of the planner is concave is that conditions (49), (50), (51) and (52) are satisfied.

B Existence of an Interior Solution in the Double Cobb-Douglas Case

In this appendix we analyze how the objective function of the planner behaves in the particular case of log-linear preferences and a Cobb-Douglas production, as specified in equations (8) and (9). For these specific functions, using the first order conditions of the planner's problem, we can express c, d and k as a function of n only:

$$k=k\left(n\right)=\left(1-zn\right)\left(\frac{A\alpha}{n}\right)^{\frac{1}{1-\alpha}}$$

$$c=c\left(n\right)=\frac{1}{1+\beta}\left[\frac{1-\alpha}{\alpha}\left(A\alpha\right)^{\frac{1}{1-\alpha}}\left(1-zn\right)n^{\frac{-\alpha}{1-\alpha}}+B\right]$$

$$d=d\left(n\right)=\beta n\frac{1}{1+\beta}\left[\frac{1-\alpha}{\alpha}\left(A\alpha\right)^{\frac{1}{1-\alpha}}\left(1-zn\right)n^{\frac{-\alpha}{1-\alpha}}+B\right]$$

Hence we can write the following indirect utility function:

$$V\left(n\right)=\gamma\log\left(n\right)+\left(1-\gamma\right)\left[\log\left(c\left(n\right)\right)+\beta\log\left(d\left(n\right)\right)\right]$$

which has the following limit values:

$$\lim_{n\to 0} V(n) = \begin{cases} -\infty & \text{if } \alpha < \widetilde{\alpha}_1 \\ +\infty & \text{if } \alpha > \widetilde{\alpha}_1 \\ I & \text{if } \alpha = \widetilde{\alpha}_1 \end{cases}$$

with

$$\widetilde{\alpha}_1 \equiv \frac{\gamma + \beta(1-\gamma)}{1+2\beta(1-\gamma)}$$

and

$$I \equiv \log\left(\beta^{\beta(1-\gamma)} A^{1+2\beta(1-\gamma)} \frac{(1-\gamma)^{(1+\beta)(1-\gamma)}\left[\gamma+\beta(1-\gamma)\right]^{\gamma+\beta(1-\gamma)}}{\left[1+2\beta(1-\gamma)\right]^{1+2\beta(1-\gamma)}}\right)$$

while

$$\lim_{n\to 1/z} V(n) = \left[\gamma+\beta(1-\gamma)\right]\log(\frac{1}{z}) + (1-\gamma)(1+\beta)\log(\frac{\beta}{1+\beta}) + \beta(1-\gamma)\log\beta$$

is a constant.

Second, taking the first derivative of $V(n)$ and grouping terms together we get:

$$V'(n) = \frac{\left[\gamma+\beta(1-\gamma)\right]}{n} - (1-\gamma)(1+\beta)\left(\frac{A\alpha}{n}\right)^{\frac{1}{1-\alpha}} \frac{1+zn\left(\frac{1-2\alpha}{\alpha}\right)}{\left[\frac{1-\alpha}{\alpha}(A\alpha)^{\frac{1}{1-\alpha}}(1-zn)\,n^{\frac{-\alpha}{1-\alpha}}+B\right]}$$

which can also be written as:

$$V'(n) = \underbrace{\frac{B\left[\gamma+\beta(1-\gamma)\right]}{n^{\frac{1}{1-\alpha}}\left[\frac{1-\alpha}{\alpha}(A\alpha)^{\frac{1}{1-\alpha}}(1-zn)\,n^{\frac{-\alpha}{1-\alpha}}+B\right]}}_{>0} g(n)$$

where

$$g(n) \equiv n^{\frac{\alpha}{1-\alpha}} - \Gamma n + \Theta$$

with

$$\Gamma \equiv \frac{z(A\alpha)^{\frac{1}{1-\alpha}}\left[1+2\beta(1-\gamma)-\alpha\left[1+2\beta(1-\gamma)+(1-\gamma)(1+\beta)\right]\right]}{\alpha B\left[\gamma+\beta(1-\gamma)\right]}$$

and

$$\Theta \equiv \frac{(A\alpha)^{\frac{1}{1-\alpha}}\left[\gamma+\beta(1-\gamma)-\alpha\left[1+2\beta(1-\gamma)\right]\right]}{\alpha B\left[\gamma+\beta(1-\gamma)\right]}$$

It can be shown that $\Gamma > 0$ iff $\alpha < \widetilde{\alpha}_2 \equiv \frac{1+2\beta(1-\gamma)}{1+2\beta(1-\gamma)+(1-\gamma)(1+\beta)} > 1/2$ and $\Theta > 0$ iff $\alpha < \widetilde{\alpha}_1 < \widetilde{\alpha}_2$.

Note also that:

$$\lim_{n\to 0} V'(n) = \{.+\infty \quad \text{if } \alpha \leq \widetilde{\alpha}_1 - \infty \quad \text{if } \alpha > \widetilde{\alpha}_1$$

As $V'(n)$ crucially depends on $g(n)$, the function $g(n)$ can be used to analyze the critical points of $V(n)$. Assuming $\alpha < 1/2$, Γ is always positive and $g(n)$ has one critical point given by:

$$\widehat{n} := g'(n) = 0$$

so

$$\widehat{n} = \left[\frac{1-\alpha}{\alpha}\Gamma\right]^{-\frac{1-\alpha}{1-2\alpha}}$$

Since $g''(n) < 0 \;\; \forall n > 0$, this critical point is a maximum.

On the other hand, the function $g(n)$ has the following limit values:

$$\lim_{n\to 0} g(n) = \Theta$$

and

$$\lim_{n\to 1/z} g(n) = \left(\frac{1}{z}\right)^{\frac{\alpha}{1-\alpha}} - \frac{(A\alpha)^{\frac{1}{1-\alpha}}}{B\left[\gamma+\beta(1-\gamma)\right]}\left[\gamma+\beta(1-\gamma)+\frac{1-2\gamma}{\alpha}\right]$$

which will be negative under the sufficient condition that:

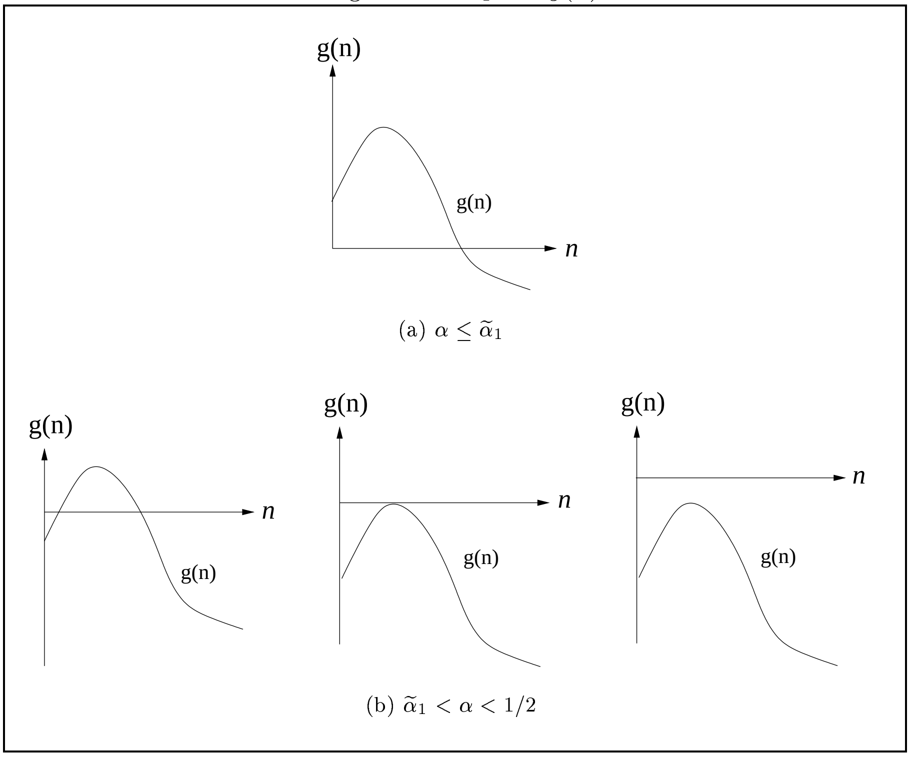
$$z > z_{\min} \equiv \frac{1}{\alpha A^{1/\alpha}}\left(\frac{B\left[\gamma+\beta(1-\gamma)\right]}{(1+\beta)(1-\gamma)(1-\alpha)}\right)^{\frac{1-\alpha}{\alpha}}$$

This condition guarantees that $zn^* < 1$.

Hence, two cases can be distinguished:

- If $\alpha \leq \widetilde{\alpha}_1$ and $z > z_{\min}$, $\Theta \geq 0$, $\Gamma > 0$ and $g(n)$ has one root for $n \in [0, 1/z]$. This is depicted in panel (a) of figure 1. Hence, $V(n)$ has one critical point. The utility function first increases, reaches a maximum and then decreases, as represented graphically in panel (a) of figure 2.
- If $\widetilde{\alpha}_1 < \alpha < 1/2$, $\Theta < 0$ but $\Gamma > 0$, and $g(n)$ can have one, two or no roots depending on the values of the parameters, as shown in panel (b) of figure 1. The utility function starts decreasing, as can be seen in panels (b) and (c) of figure 2. Depending on parameter values, utility may always decrease (in case $g(n)$ has no root) with the possibility of having an inflexion point (in case $g(n)$ has one root) or it may have first a local minimum and then a local maximum (in case $g(n)$ has two roots).

Figure 1: Shape of $g(n)$



Therefore, with a log-linear utility function and a Cobb-Douglas production function, there exists a unique interior global maximum if and only if the following conditions are satisfied:

$$\alpha \leq \frac{\gamma + \beta (1-\gamma)}{1 + 2\beta (1-\gamma)} \equiv \widetilde{\alpha}_1$$

and

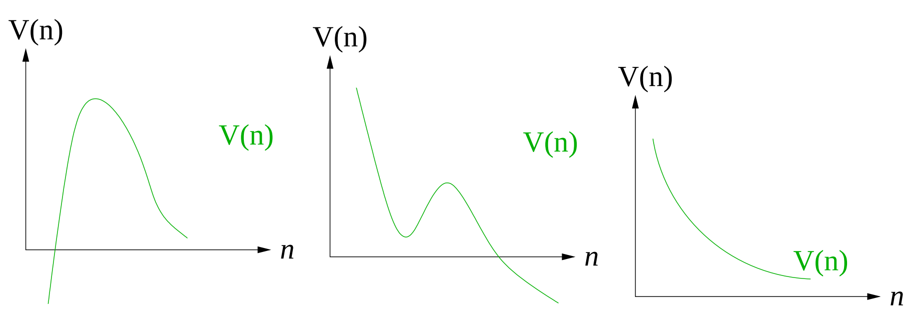
$$z > z_{\min}$$

If $\widetilde{\alpha}_1 < \alpha < 1/2$, the global maximum is reached when $n \to 0$.

C Dynamics and Steady State of the Competitive Economy

This appendix analytically studies the dynamics and the steady state equilibrium of the laissez-faire economy. We focus on the log-linear utility and Cobb-Douglas production case as specified in equations (8) and (9).

Figure 2: Shape of $V(n)$



(a) $\alpha \le \tilde{\alpha}_1$ (b) $\tilde{\alpha}_1 < \alpha < 1/2$ (c) $\tilde{\alpha}_1 < \alpha < 1/2$

Definition 9 *Assume an initial capital stock k_0. A perfect foresight, intertemporal laissez-faire equilibrium is a vector $(c_t, d_t, n_t, s_t, k_t, l_t^f, w_t^f, w_t^m, R_t)$ starting at k_0 and satisfying the following conditions:*

$$c_t + s_t = w_t^m + w_t^f(1 - zn_t) \tag{53}$$

$$d_{t+1} = R_{t+1}s_t \tag{54}$$

$$\frac{u'(c_t)}{u'(d_{t+1})} = \beta R_{t+1} \tag{55}$$

$$\frac{\gamma}{1-\gamma}\frac{u'(n_t)}{u'(c_t)} = zw_t^f \tag{56}$$

$$w_t^f = f_l(k_t, l_t^f) \tag{57}$$

$$R_t = f_k(k_t, l_t^f) \tag{58}$$

$$w_t^m = f(k_t, l_t^f) - k_t f_k(k_t, l_t^f) - l_t^f f_l(k_t, l_t^f) \tag{59}$$

$$k_{t+1}n_t = s_t \tag{60}$$

$$l_t^f = 1 - zn_t \tag{61}$$

With the production function in (9), factor prices are given by:

$$1 + r_t = R_t = A\alpha k_t^{\alpha-1}(l_t^f)^{1-\alpha} \tag{62}$$

$$w_t^f = A(1-\alpha)k_t^{\alpha}(l_t^f)^{-\alpha} \tag{63}$$

$$w_t^m = B \tag{64}$$

With the utility function in (8), the first order condition of the maximization program with respect to consumption (55), together with the budget constraints (53) and (54), allow to define the savings function as follows:

$$s_t = \frac{\beta}{1+\beta}\left[w_t^m + w_t^f(1 - zn_t)\right] \tag{65}$$

The first order condition relative to fertility (56) determines the time spent by women raising children:

$$zn_t = \min\left[1, \frac{\gamma}{1+\beta(1-\gamma)}\left(1 + \frac{w_t^m}{w_t^f}\right)\right] \tag{66}$$

The log-linear utility function insures the positivity of zn_t, and the fertility choice is restricted by the maximum available time of women (1 unit). After substituting (61), (63) and (64) in (66), the time devoted to raising children can be defined by the following implicit function:

$$H(zn_t, k_t) \equiv zn_t - \frac{\gamma}{1+\beta(1-\gamma)}\left(1 + \frac{B(1-zn_t)^{\alpha}}{A(1-\alpha)k_t^{\alpha}}\right) = 0 \tag{67}$$

with

$$\frac{\partial H(zn_t, k_t)}{\partial zn_t} = 1 - \frac{\gamma}{1+\beta(1-\gamma)}\frac{B}{A(1-\alpha)k_t^{\alpha}}\alpha(1-zn_t)^{\alpha-1}(-1) > 0$$

as any zn_t that satisfies the above equation will always be inferior to 1 for $k_t > 0$. Indeed, we can isolate k_t from (67) as:

$$k_t = (1 - zn_t)\left(\frac{\gamma}{1+\beta(1-\gamma)}\frac{B}{A(1-\alpha)}\frac{1}{zn_t - \frac{\gamma}{1+\beta(1-\gamma)}}\right)^{1/\alpha}$$

Then, since $\gamma/(1+\beta(1-\gamma)) < 1$, $zn_t > 1$ implies that $k_t < 0$. Thus for positive values of the capital stock, $zn_t < 1$. Women will always supply some labor in the market, implying that the time devoted to raising children is strictly smaller than 1.

Since $\frac{\partial H(zn_t,k_t)}{\partial zn_t} > 0$, there exists a function $\Phi(k_t)$ such that:

$$zn_t = \Phi(k_t)$$

Hence, we can rewrite (67) as:

$$\Phi(k_t) - \frac{\gamma}{1+\beta(1-\gamma)}\left(1 + \frac{B\left[1-\Phi(k_t)\right]^{\alpha}}{A(1-\alpha)k_t^{\alpha}}\right) = 0$$

Differentiating this expression, we obtain:

$$\Phi'(k_t) - \frac{\gamma}{1+\beta(1-\gamma)}\frac{B}{A(1-\alpha)}\left[-\frac{\alpha\Phi'(k_t)}{\left[1-\Phi(k_t)\right]^{1-\alpha}k_t^{\alpha}} - \frac{\alpha\left[1-\Phi(k_t)\right]^{\alpha}}{k_t^{\alpha+1}}\right] = 0$$

and therefore:

$$\Phi'(k_t) = -\frac{\left[1-\Phi(k_t)\right]}{k_t\left(1+\frac{1+\beta(1-\gamma)}{\gamma}\frac{A}{B}\frac{1-\alpha}{\alpha}\left[1-\Phi(k_t)\right]^{1-\alpha}k_t^{\alpha}\right)} < 0 \tag{68}$$

As capital accumulates, the female wage increases and fertility decreases.

Let's now turn to the accumulation of capital. Using (65) and (66), equation (60) can be written as:

$$k_{t+1} = \frac{1-\gamma}{\gamma}z\beta w_t^f \tag{69}$$

Substituting factor prices,

$$k_{t+1} = \frac{1-\gamma}{\gamma}z\beta A\left(1-\alpha\right)\left(\frac{k_t}{1-zn_t}\right)^{\alpha} \tag{70}$$

Hence the evolution of the capital stock is defined by:

$$k_{t+1} = \frac{1-\gamma}{\gamma}z\beta A\left(1-\alpha\right)\left(\frac{k_t}{1-\Phi(k_t)}\right)^{\alpha}$$

Differentiating this expression:

$$\frac{dk_{t+1}}{dk_t} = \frac{1-\gamma}{\gamma}z\beta A\left(1-\alpha\right)\alpha\left(\frac{k_t}{1-\Phi(k_t)}\right)^{\alpha-1}\frac{1-\Phi(k_t)+k_t\Phi'(k_t)}{\left[1-\Phi(k_t)\right]^2}$$

Using (68), it is easy to see that:

$$\frac{1-\Phi(k_t)+k_t\Phi'(k_t)}{\left[1-\Phi(k_t)\right]^2} = \frac{\left[1-\Phi(k_t)\right]^{2-\alpha}\frac{1-\alpha}{\alpha}\frac{A}{B}\frac{1+\beta(1-\gamma)}{\gamma}k_t^{\alpha}}{1+\frac{1-\alpha}{\alpha}\frac{A}{B}\frac{1+\beta(1-\gamma)}{\gamma}k_t^{\alpha}\left[1-\Phi(k_t)\right]^{1-\alpha}} > 0 \tag{71}$$

and therefore, the capital accumulation is increasing over time. In addition:

$$\lim_{k_t\to 0} k_{t+1} = 0$$

Note also that capital stops increasing in the long run:

$$\lim_{k_t \to \infty} \frac{dk_{t+1}}{dk_t} = 0$$

Hence, we can conclude the **existence of at least one non-trivial steady state**.

We will now prove the existence of a *unique* non-trivial steady state. The fertility equation (67) at the steady state can be written as:

$$zn = \frac{\gamma}{1+\beta(1-\gamma)} \left[1 + \frac{B(1-zn)^{\alpha}}{A(1-\alpha)k^{\alpha}} \right] \tag{72}$$

whereas the capital accumulation equation (70) at the steady state becomes:

$$k = \frac{1-\gamma}{\gamma} A\beta z(1-\alpha) \left(\frac{k}{1-zn} \right)^{\alpha} \tag{73}$$

These two equations fully characterize the steady state, determining n and k. In fact, we can isolate capital per couple in (72) as follows:

$$k = \left(\frac{CB}{A(1-\alpha)(zn-C)} \right)^{\frac{1}{\alpha}} (1-zn)$$

where $C \equiv \frac{\gamma}{1+\beta(1-\gamma)}$. Substituting this into (73), we obtain only one equation in one variable, i.e. the number of children n:

$$\left(\frac{CB}{A(1-\alpha)(zn-C)} \right)^{\frac{1}{\alpha}} (1-zn) = \frac{1-\gamma}{\gamma} \frac{\beta zCB}{zn-C}$$

which we rewrite as:

$$\left(\frac{CB}{A(1-\alpha)} \right)^{\frac{1}{\alpha}} \left[(zn-C)^{\frac{\alpha-1}{\alpha}} (1-zn) \right] - \frac{1-\gamma}{\gamma} \beta zCB = 0 \tag{74}$$

Taking the first derivative with respect to zn[21]:

$$\left(\frac{CB}{A(1-\alpha)} \right)^{\frac{1}{\alpha}} \left[\underbrace{\frac{\alpha-1}{\alpha}(zn-C)^{\frac{\alpha-1}{\alpha}-1}(1-zn) + (zn-C)^{\frac{\alpha-1}{\alpha}}(-1)}_{<0} \right]$$

which means that this function is decreasing in zn. Hence, (74) will have, at the most, one root. Since we have proven the existence of at least one non-trivial steady state, there will be a **unique steady state solution**.

[21] Observe that we know that the term $(zn - C)$ is positive from equation (72).

D A Comparison of the Planner and Laissez-Faire Solutions

The optimal values of c, d, k, n and l^f are given by the set of equations (3)-(7). Substituting (4) into (3) and (5), these two equations can be rewritten as:

$$\frac{u'(c)}{u'(d)} = \beta f_k\left(k, l^f\right) \tag{75}$$

$$\frac{\gamma}{1-\gamma}\frac{u'(n)}{u'(c)} + \frac{d}{n f_k\left(k, l^f\right)} = z f_l\left(k, l^f\right) + k \tag{76}$$

On the other hand, the laissez-faire values of $c, d, k, n, l^f, s, R, w^f$ and w^m are given by equations (19)-(27). Using (22), equation (21) can be rewritten as the intertemporal budget constraint:

$$c + \frac{d}{R} = w^m + w^f(1 - zn) \tag{77}$$

Using equations (23) to (27), equations (19), (20) and (77) can be rewritten as:

$$\frac{u'(c)}{u'(d)} = \beta f_k(k, l^f) \tag{78}$$

$$\frac{\gamma}{1-\gamma}\frac{u'(n)}{u'(c)} = z f_l(k, l^f) \tag{79}$$

$$f(k, l^f) = c + \frac{d}{n} + kn \tag{80}$$

Substituting (24) into (22), this latter equation can be expressed as:

$$k = \frac{d}{n f_k(k, l^f)} \tag{81}$$

Finally, we can sum equations (79) and (81) and express them as:

$$\frac{\gamma}{1-\gamma}\frac{u'(n)}{u'(c)} + \frac{d}{n f_k\left(k, l^f\right)} = z f_l\left(k, l^f\right) + k \tag{82}$$

$$k = \frac{d}{n f_k(k, l^f)} \tag{83}$$

Written in this way, equations (78), (80), (82), (83) and (23) jointly determine c, d, k, n and l^f –the same variables as the planner– while s is given by

(24) and factor prices by (25)-(27). Therefore, we can focus on the equations that determine the same variables as the planner and compare them.

When doing so, we see that (78), (80), (23) and (82) are identical to (75), (6), (7) and (76). Thus there is only one equation that is different, i.e. (83) in the laissez-faire and (4) in the planner. Female labor supply is given by (23) or (7). From (78) or (75) we can usually express future consumption as a function of first period consumption, and then from (80) or (6) we can obtain c as a function of k and n only. Hence we end up with two equations determining n and k, (82) and (83) in the laissez-faire, and (76) and (4) in the planner.

E Proof of Proposition 5

As we have seen in the previous appendix, using equations (25) and (26), equations (19) and (20) can respectively be rewritten as:

$$\frac{u'(c)}{u'(d)} = \beta f_k(k, l) \tag{84}$$

$$\frac{\gamma}{1-\gamma}\frac{u'(n)}{u'(c)} = z f_l(k, l^f) \tag{85}$$

Using (23) to (27) and (34), (33) can be expressed as:

$$f(k, l^f) = c + \frac{d}{n} + kn \tag{86}$$

Using (24), (25), (27) and (32), we can rewrite (34) as:

$$\frac{d}{f_k(k,l)} - \tau\frac{n}{f_k(k,l)}[f(k,l^f) - kf_k(k,l^f) - l^f f_l(k,l^f)] = kn \tag{87}$$

Finally, we have also equation (23):

$$l^f = 1 - zn \tag{88}$$

Suppose the government sets the tax rate so as to induce the level of capital satisfying the golden rule (given by equation (87) with $f_k(k, l) = n$). Then the intertemporal allocation of consumption (84) is the same as the optimal one (3), and the planner's optimality conditions (4), (6) and (7) are also satisfied, as can be seen from the equations above. However, equation (85) determining the value of n is still different from the optimal one given by (5).

F Proof of proposition 8

A steady state competitive equilibrium under a PAYG pension system with fertility link is a transfer system (p, τ) and a vector of variables $(c, d, n, s, k, l^f, w^f,$

w^m, R) with positive quantities verifying the following conditions:

$$\frac{u'(c)}{u'(d)} = \beta R \tag{89}$$

$$\frac{\gamma}{1-\gamma}\frac{u'(n)}{u'(c)} + \frac{w^m \tau}{R} = z w^f \tag{90}$$

$$c + s = w^m(1-\tau) + w^f(1-zn) \tag{91}$$

$$d = sR + w^m \tau n \tag{92}$$

$$l^f = 1 - zn \tag{93}$$

$$s = kn \tag{94}$$

$$R = f_k(k, l^f) \tag{95}$$

$$w^f = f_l(k, l^f) \tag{96}$$

$$w^m = f(k, l^f) - k f_k(k, l^f) - l^f f_l(k, l^f) \tag{97}$$

$$p = w^m \tau n \tag{98}$$

We assume the government fixes the payroll tax and then determines pension benefits according to (98).

Isolating the term $w^m \tau$ from (92), (90) can be written as:

$$\frac{\gamma}{1-\gamma}\frac{u'(n)}{u'(c)} + \frac{d}{nR} = z w^f + \frac{s}{n}$$

Substituting (94)-(97) into this new expression and in (89), (91) and (92), we can rewrite (89)-(92) as:

$$\frac{u'(c)}{u'(d)} = \beta f_k(k, l^f) \tag{99}$$

$$\frac{\gamma}{1-\gamma}\frac{u'(n)}{u'(c)} + \frac{d}{n f_k(k, l^f)} = z f_l(k, l^f) + k \tag{100}$$

$$f(k,l^f) = c + \frac{d}{n} + kn \tag{101}$$

$$d = f_k(k,l^f)kn + \left[f(k,l^f) - kf_k(k,l^f) - l^f f_l(k,l^f)\right]\tau n \tag{102}$$

The first three equations are identical to the planner's first order conditions (3), (5) and (6), once we take into consideration (4), and (93) is the same as (7). Hence, this pension policy keeps the right incentives to choose the optimal fertility rate. The optimal value of τ can then be obtained from (102) once the golden rule has been introduced in that expression:

$$d = kn^2 + \left[f(k,l^f) - kn - l^f f_l(k,l^f)\right]\tau^* n$$

Isolating the payroll tax in the previous expression, equation (48) is obtained.