Semi-localisations and factorisation systems

A factorisation system $(\mathcal{E}, \mathcal{M})$ in a category $\mathcal{C}$ is a pair of classes of arrows in $\mathcal{C}$ s.t.

1) $\mathcal{E}, \mathcal{M}$ contain all identities
2.) $\mathcal{E}, \mathcal{M}$ are stable by composition with isos
3) $\mathcal{E}$ and $\mathcal{M}$ are ORTHOGONAL : for any commutative

$$\begin{tikzcd} A \arrow[r, "e \in \mathcal{E}"] \arrow[d, "u"'] & B \arrow[d, "v"] \arrow[dl, dashed, "\exists ! \varphi"'] \\ C \arrow[r, "m \in \mathcal{M}"'] & D \end{tikzcd}$$

such that $m \cdot \varphi = v$, $\varphi \cdot e = u$

4) any arrow $A \xrightarrow{f} B$ has a FACTORISATION

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "e \in \mathcal{E}"'] & & B \\ & C \arrow[ur, "m \in \mathcal{M}"'] & \end{tikzcd}$$

$f = m \cdot e$, with $e \in \mathcal{E}$ and $m \in \mathcal{M}$

<u>Remark</u> The factorisation in 4) is unique.

The class $\mathcal{M}$ is always stable under pullbacks:

$$\begin{tikzcd} P \arrow[r, "g'"] \arrow[d, "m'"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "m \in \mathcal{M}"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

LEMMA (15)

Let $(\mathcal{E}, \mathcal{M})$ be a factorisation system in $\mathcal{C}$. Then:

1) $f \in \mathcal{E} \iff f$ is ORTHOGONAL to all $m \in \mathcal{M}$

2) $f \in \mathcal{M} \iff f$ is ORTHOGONAL to all $e \in \mathcal{E}$

PROOF

1) Assume that f is ORTHOGONAL to all $m \in \mathcal{M}$, and consider its factorisation $\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "e \in \mathcal{E}"'] & C \arrow[dl, dashed, "g"'] \\ B \arrow[ur, "m \in \mathcal{M}"'] & \end{tikzcd}$, where

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "e"'] & C \arrow[d, "1_C"] \arrow[dl, dashed, "\exists g"'] \\ B \arrow[r, "m \in \mathcal{M}"'] & C \end{tikzcd}$$

$$g \cdot f = e \quad , \quad m \cdot g = 1_C$$

By considering the diagram

$$\begin{tikzcd} A \arrow[r, "e \in \mathcal{E}"] \arrow[d, "e"'] & B \arrow[d, "m"] \arrow[dl, dashed, shift right, "1_B"'] \arrow[dl, dashed, shift left, "g \cdot m"] \\ B \arrow[r, "m \in \mathcal{M}"'] & C \end{tikzcd}$$

we see that $g \cdot m = 1_B$

Since $\mathcal{E}$ is stable under composition with isos, we conclude that $f = m \cdot e \in \mathcal{E}$

$\square$

EXAMPLES

1) Ab

Any torsion theory $(\mathcal{T}, \mathcal{F})$ in Ab induces a MONOTONE-LIGHT FACTORISATION SYSTEM.

Indeed, (N) holds since any mono is NORMAL.

One can show that $\mathcal{F}$ contains all free groups (whenever $(\mathcal{T}, \mathcal{F}) \neq (Ab, 0)$).

Indeed, if there is an $F_1 \notin \mathcal{F}$ one has a FREE GROUP in $\mathcal{T}$:

$$0 \longrightarrow T(F_1) \rightarrowtail F_1 \longrightarrow F(F_1) \longrightarrow 0$$

$T(F_1) \in \mathcal{F}$, since $F_1 \in \mathcal{F}$!

But $\mathcal{T}$ is stable in Ab under COPRODUCTS

$\Rightarrow$ $\mathcal{T} = Ab$ (since it would contain ALL FREE GROUPS), a contradiction! For any $A \in Ab$, $\exists\, F \xrightarrow{p} A$, with $F \in \mathcal{F}$.

CONCLUSION: $\bar{\mathcal{E}} = \{ f : A \to B \text{ SURJ. HOM.} \mid K[f] \in \mathcal{T} \} = \mathcal{E}'$

$\mathcal{M}^* = \{ f : A \to B \mid K[f] \in \mathcal{F} \}$

$(\mathcal{E}', \mathcal{M}^*)$ IS A MONOTONE-LIGHT FACTORISATION SYSTEM.

$$\begin{tikzcd}
 & K \arrow[r] \arrow[d, tail] & K/T(K) \arrow[d, tail] \\
T(K) \arrow[ur, tail] \arrow[r, tail] & A \arrow[r, "e \in \mathcal{E}'"] \arrow[d, "f"'] & A/T(K) \arrow[dl, "m \in \mathcal{M}^*"] \\
 & B &
\end{tikzcd}$$

2) GRP

Any torsion theory $(\mathcal{T}, \mathcal{F})$ in GRP induces a MONOTONE-LIGHT FACTORISATION SYSTEM.

(N) holds because any RADICAL

$$T(G) \rightarrowtail G$$

in GRP is "fully invariant", so that for any NORMAL MONO $K \overset{k}{\rightarrowtail} A$, the composite

$$\begin{tikzcd} T(K) \arrow[r, tail, "\varepsilon_K"] \arrow[rr, bend right, dashed, "k \cdot \varepsilon_K"'] & K \arrow[r, tail, "k"] & A \end{tikzcd}$$

is a NORMAL SUBGROUP in A.

Since any subgroup of a FREE GROUP is FREE (NIELSEN - SCHREIER THEOREM), the same argument as for Ab shows that any torsion-free subcategory $\mathcal{F}$ contains ALL FREE GROUPS.

$$\forall A \in \mathrm{GRP}, \quad \exists \quad F \overset{p}{\twoheadrightarrow} A \quad \text{WITH} \quad F \in \mathcal{F}$$

$$\mathcal{E}' = \{ f : A \twoheadrightarrow B \ \text{SURJ. HOM.} \mid K[f] \in \mathrm{Perf} \}$$

$$\mathcal{M}^* = \{ f : A \to B \mid K[f] \in \mathrm{HypoAb} \}$$

IS A MONOTONE-LIGHT FACTORISATION SYSTEM IN GRP.

3) The torsion theory $(\mathrm{Grp(Conn)}, \mathrm{Grp(TotDis)})$ in the category $\mathrm{Grp(Haus)}$ of Hausdorff groups satisfies property (N).

Moreover any Hausdorff group is a QUOTIENT of a TOTALLY DISCONNECTED GROUP:

$$\forall H \in \mathrm{Grp(Haus)}, \ \exists D \in \mathrm{Grp(TotDis)}$$

WITH $D \xrightarrow{p} H$ A REGULAR EPI

(ARKHANGEL'SKII, 1981).

One then gets the factorisation

$$\begin{tikzcd}
 & K \arrow[r] \arrow[d] & K/\Gamma_0(K) \in \mathrm{Grp(TotDis)} \arrow[d] \\
\Gamma_0(K) \arrow[ur] \arrow[r, tail] & A \arrow[r, "e \in \mathcal{E}'"] \arrow[d, "f"'] & A/\Gamma_0(K) \arrow[dl, "m \in \mathcal{M}^*"] \\
 & B &
\end{tikzcd}$$

$\mathcal{E}' = \{ f : A \to B$ OPEN SURJ. HOM $\mid K[f] \in \mathrm{Grp(Conn)}$

$\mathcal{M}^* = \{ f : A \to B \mid K[f] \in \mathrm{Grp(TotDis)} \}$

This is a MONOTONE-LIGHT factorisation system!

4) CRNG

The torsion theory (NilCRNG, RedCRNG)
in the category CRNG of COMMUTATIVE RINGS
is HEREDITARY: for any MONO

$$R \rightarrowtail^{m} N$$

where $N \in$ NilCRNG, then $R \in$ NilCRNG.

This implies that CONDITION (N) holds.

Since FREE COMMUTATIVE RINGS are REDUCED,
$\forall R \in$ CRNG there is $F \xrightarrow{p} R$,
with $F \in$ RedCRNG and p a SURJECTIVE HOMOMORPH.

It follows that

$$\mathcal{E}' = \{ f: A \twoheadrightarrow B \text{ SURJ. HOM.} \mid K[f] \in \text{NilCRNG} \}$$

$$\mathcal{M}^* = \{ f: A \rightarrow B \mid K[f] \in \text{RedCRNG} \}$$

is a MONOTONE-LIGHT factorisation system
in CRNG.

REMARK

The category $\mathcal{M}^* = \{\text{light maps}\}$ is itself reflective in $\mathrm{ARR}(\mathcal{C})$:

$$\begin{tikzcd} \mathcal{M}^* \arrow[r, shift right, hook, "U_1"'] \arrow[r, phantom, "\perp"] & \mathrm{ARR}(\mathcal{C}) \arrow[l, shift right, bend right, "F_1"'] \end{tikzcd}$$

the reflection of an arrow $f : A \to B$ is given by

$$\begin{tikzcd} A \arrow[rr, "e' \in \mathcal{E}'"] \arrow[dr, "f"'] & & I \arrow[dl, "m^* \in \mathcal{M}^*"] \\ & B & \end{tikzcd}$$

This yields a NEW TORSION THEORY in $\mathrm{ARR}(\mathcal{C})$

$$(\mathcal{T}_1 , \mathcal{F}_1)$$

where
$$\mathcal{T}_1 = \{ T \longrightarrow 0 \mid T \in \mathcal{T} \}$$
$$\mathcal{F}_1 = \{ A \xrightarrow{f} B \mid K[f] \in \mathcal{F} \}$$

This torsion theory satisfies again the assumption of the THEOREM, yielding a chain of MONOTONE-LIGHT FACTORISATION SYSTEMS in the categories $\mathrm{ARR}^n(\mathcal{C})$, $\forall n \geq 1$.

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