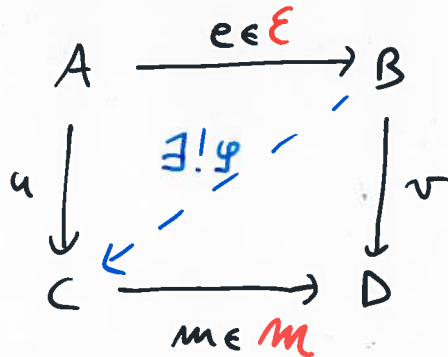


SEMI-LOCALISATIONS AND FACTORISATION SYSTEMS

A factorisation system $(\mathcal{E}, \mathcal{M})$ in a category \mathcal{C} is a pair of classes of arrows in \mathcal{C} s.t.

- 1) \mathcal{E}, \mathcal{M} contain all identities
- 2) \mathcal{E}, \mathcal{M} are stable by composition with ISOS
- 3) \mathcal{E} and \mathcal{M} are ORTHOGONAL: for any commutative



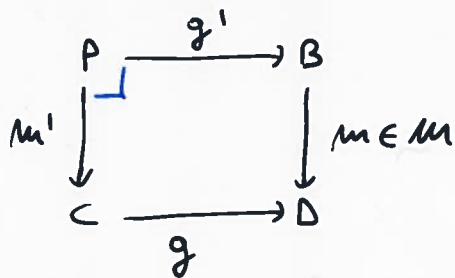
such that $m \cdot \varphi = v$, $\varphi \cdot e = u$

4) any arrow $A \xrightarrow{f} B$ has a FACTORISATION
 $e \in \mathcal{E} \rightarrow C \xrightarrow{m \in \mathcal{M}}$

$f = m \cdot e$, with $e \in \mathcal{E}$ and $m \in \mathcal{M}$

REMARK The factorisation in 4) is unique.

The class \mathcal{M} is always stable under pullbacks:



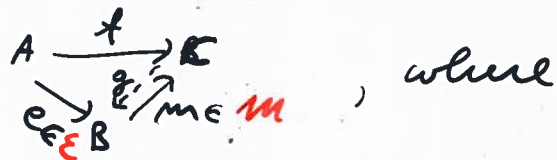
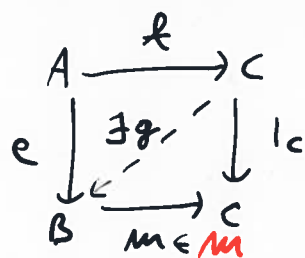
LEMMA (15)

Let $(\mathcal{E}, \mathcal{M})$ be a factorisation system in \mathcal{E} . Then:

- 1) $f \in \mathcal{E}$ (\Rightarrow) f is ORTHOGONAL to all $m \in \mathcal{M}$
- 2) $f \in \mathcal{M}$ (\Rightarrow) f is ORTHOGONAL to all $e \in \mathcal{E}$

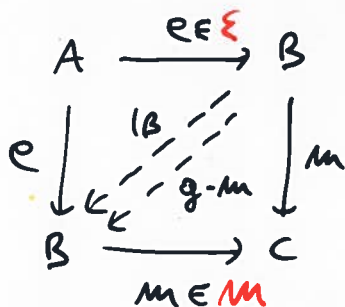
PROOF

- 1) Assume that f is ORTHOGONAL to all $m \in \mathcal{M}$, and consider its factorisation



$$g \cdot f = e, \quad m \cdot g = l_c$$

By considering the diagram



we see that $g \cdot m = l_B$

Since \mathcal{E} is stable under composition with isos, we conclude that $f = m \cdot e \in \mathcal{E}$

□

EXAMPLES

1) Ab

Any torsion theory (τ, \mathcal{F}) in **Ab** induces
a HOMOLOGY-LIGHT FACTORISATION SYSTEM.

Indeed, (N) holds since every monomorphism is NORMAL.

One can show that \mathcal{F} contains all free groups (whenever $(\tau, \mathcal{F}) \neq (\text{Ab}, 0)$).

Indeed, if there is an $F_1 \notin \mathcal{F}$ one has a FREE GROUP in τ :

$$0 \longrightarrow T(F_1) \longrightarrow F_1 \longrightarrow F(F_1) \longrightarrow 0$$

$T(F_1) \in \mathcal{F}$, since $F_1 \in \mathcal{F}$!

But τ is stable in **Ab** under COPRODUCTS

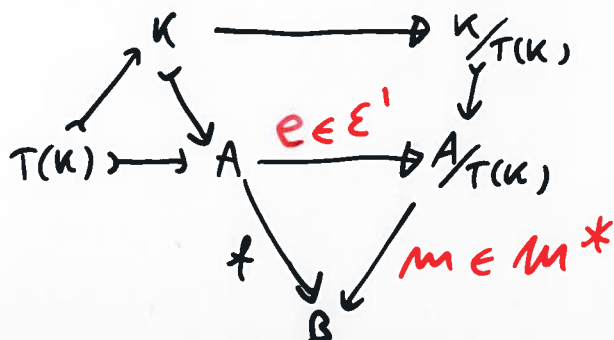
$\Rightarrow \tau = \text{Ab}$ (since it would contain ALL FREE GROUPS)

& contradiction! For any $A \in \text{Ab}$, $\exists F \xrightarrow{p} A$, with $F \in \mathcal{F}$.

CONCLUSION: $\bar{\mathcal{E}} = \{ f: A \rightarrow B \text{ SURJ. HOM.} \mid K(f) \in \tau \} = \mathcal{E}'$

$$\mathcal{M}^* = \{ f: A \rightarrow B \mid K(f) \in \mathcal{F} \}$$

$(\mathcal{E}', \mathcal{M}^*)$ IS A HOMOLOGY-LIGHT FACTORISATION SYSTEM.



2) GRP

Any torsion theory (τ, \mathcal{A}) in GRP induces a MONOTONE-LIGHT FACTORISATION SYSTEM.

(N) holds because any RADICAL

$$T(G) \twoheadrightarrow G$$

in GRP is "fully invariant", so that for any

NORMAL MONO $K \xrightarrow{k} A$, the composite

$$\begin{array}{ccccc}
 T(K) & \xrightarrow{\epsilon_K} & K & \xrightarrow{k} & A \\
 & & & & \nearrow \\
 & & & & k \cdot \epsilon_K
 \end{array}$$

is a NORMAL SUBGROUP in A.

Since any subgroup of a FREE GROUP is FREE (NIELSEN - SCHREIER THEOREM), the same argument as for Ab shows that any torsion-free subcategory \mathcal{A} contains ALL FREE GROUPS.

$$\forall A \in \text{GRP}, \exists F \xrightarrow{p} A \text{ WITH } F \in \mathcal{A}$$

$$\mathcal{E}' = \{ f: A \twoheadrightarrow B \text{ SURJ. HOM.} \mid K(f) \in \text{PERF} \}$$

$$\mathcal{M}^* = \{ f: A \rightarrow B \mid K(f) \in \text{HYP} \text{ Ab} \}$$

IS A MONOTONE-LIGHT FACTORISATION SYSTEM IN GRP.

3) The torsion theory $(\text{GRP}(\text{COM}), \text{GRP}(\text{TOT DIS}))$ in the category $\text{GRP}(\text{HAUS})$ of Hausdorff groups satisfies property (N).

Moreover any Hausdorff group is a QUOTIENT of a TOTALLY DISCONNECTED GROUP:

$$\forall H \in \text{GRP}(\text{HAUS}), \exists D \in \text{GRP}(\text{TOT DIS})$$

$$\text{WITH } D \xrightarrow{P} H \text{ A REGULAR EPI}$$

(ARKHANGEL'SKII, 1981).

One then gets the factorisation

$$\begin{array}{ccccc}
 & & K & \xrightarrow{\quad} & K/\Gamma_0(K) \in \text{GRP}(\text{TOT DIS}) \\
 & \nearrow & \downarrow & & \downarrow \\
 & & A & \xrightarrow{e \in \mathcal{E}'} & A/\Gamma_0(K) \\
 \Gamma_0(K) \nearrow & & \downarrow & & \downarrow \\
 & & B & & \\
 & & \downarrow & \swarrow & \\
 & & & m \in \mathcal{M}^* &
 \end{array}$$

$$\mathcal{E}' = \{ f: A \rightarrow B \text{ OPEN SURJ. HOM} \mid K[f] \in \text{GRP}(\text{COM}) \}$$

$$\mathcal{M}^* = \{ f: A \rightarrow B \mid K[f] \in \text{GRP}(\text{TOT DIS}) \}$$

This is a MONOTONE-LIGHT factorisation system!

4) CRNG

The torsion theory $(\text{Nil CRNG}, \text{Red CRNG})$
 in the category CRNG of COMMUTATIVE RINGS
 is HEREDITARY: for any MONO

$$R \xrightarrow{m} N$$

where $N \in \text{Nil CRNG}$, then $R \in \text{Nil CRNG}$.

This implies that CONDITION (N) holds.

Since FREE COMMUTATIVE RINGS are REDUCED,

$\forall R \in \text{CRNG}$ there is $F \xrightarrow{p} R$,
 with $F \in \text{Red CRNG}$ and p a SURJECTIVE HOMOMORPH.

It follows that

$$\mathcal{E}' = \{ f: A \rightarrow B \text{ SURJ. HOM.} \mid \ker(f) \in \text{Nil CRNG} \}$$

$$\mathcal{M}^* = \{ f: A \rightarrow B \mid \ker(f) \in \text{Red CRNG} \}$$

is a MONOTONE-LIGHT factorisation system
 in CRNG .

REMARK

The category $\mathcal{M}^* = \{ \text{light maps} \}$

is itself reflective in $\text{Arr}(\mathcal{E})$:

$$\mathcal{M}^* \begin{array}{c} \xleftarrow{F_1} \\ \xrightarrow{\perp} \\ \xleftarrow{U_1} \end{array} \text{Arr}(\mathcal{E})$$

the reflection of an arrow $f: A \rightarrow B$ is given by

$$\begin{array}{ccc} A & \xrightarrow{e' \in \mathcal{E}'} & I \\ & \searrow f & \swarrow m^* \in \mathcal{M}^* \\ & B & \end{array}$$

This yields a **NEW TORSION THEORY** in $\text{Arr}(\mathcal{E})$

$$(\tau_1, \mathcal{F}_1)$$

where $\tau_1 = \{ T \rightarrow 0 \mid T \in \tau \}$

$$\mathcal{F}_1 = \{ A \xrightarrow{f} B \mid \kappa(f) \in \mathcal{F} \}$$

This torsion theory satisfies again the assumption of the THEOREM, yielding a chain of MONOTONE-LIGHT FACTORISATION SYSTEMS in the categories $\text{Arr}^n(\mathcal{E})$, $\forall n \geq 1$.

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