

DEFINITION (JANELIDZE, MÁRKI, THOLEN, 2001)

A category  $\mathcal{C}$  is **SEMI-ABELIAN** if

- $\mathcal{C}$  is **HOMOLOGICAL**
- $\mathcal{C}$  is **EXACT**
- $\mathcal{C}$  has **BINARY COPRODUCTS**

REMARK

**ABELIAN  $\Rightarrow$  SEMI-ABELIAN**

The classical result by Tierney:

**ABELIAN  $\equiv$  EXACT + ADDITIVE**

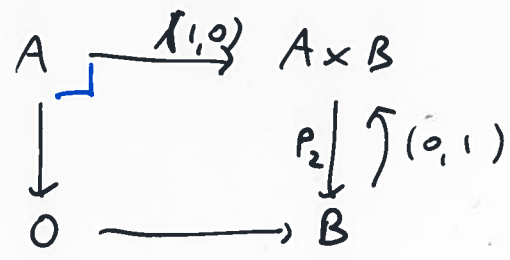
tells us that it suffices to check that the **SPLIT SHORT FIVE LEMMA** holds in any abelian category. This is clear, since any split short exact sequence is of the form

$$0 \longrightarrow K \longrightarrow K \oplus B \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} B \longrightarrow 0$$

LEMMA (7) If  $\mathcal{E}$  and  $\mathcal{E}^{op}$  are **HOMOLOGICAL**, then the canonical arrow  $\alpha: A+B \rightarrow A \times B$  is an isomorphism.

PROOF

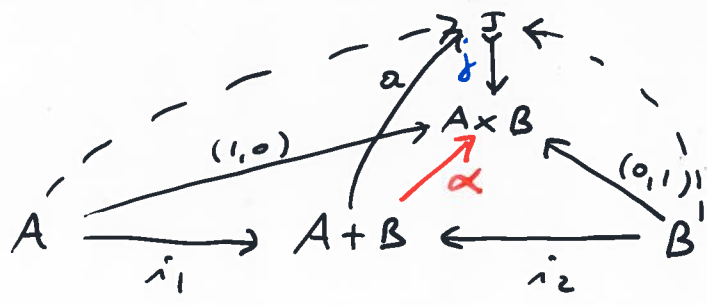
The pair  $(1, 0)$  and  $(0, 1)$  in the diagram



is **JOINTLY EXTREMAL EPIMORPHIC**. This implies

that  $\alpha: A+B \rightarrow A \times B$  is **EXTREMAL EPI**:

given a mono  $J \xrightarrow{j} A \times B$  such that  $\exists a$



with  $j \cdot a = \alpha$ , then  $j$  is an ISO.

The arrow  $\alpha: A+B \rightarrow A \times B$  is also an EPI in  $\mathcal{E}^{op}$  (since this latter is homological).

Accordingly,  $\alpha: A+B \rightarrow A \times B$  is a MONO + EXTREMAL EPI

$\Rightarrow \alpha$  is an ISOMORPHISM

□

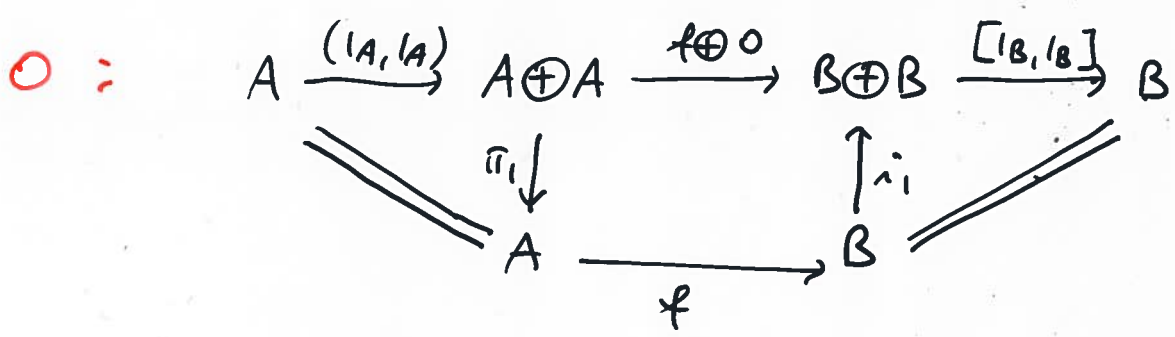
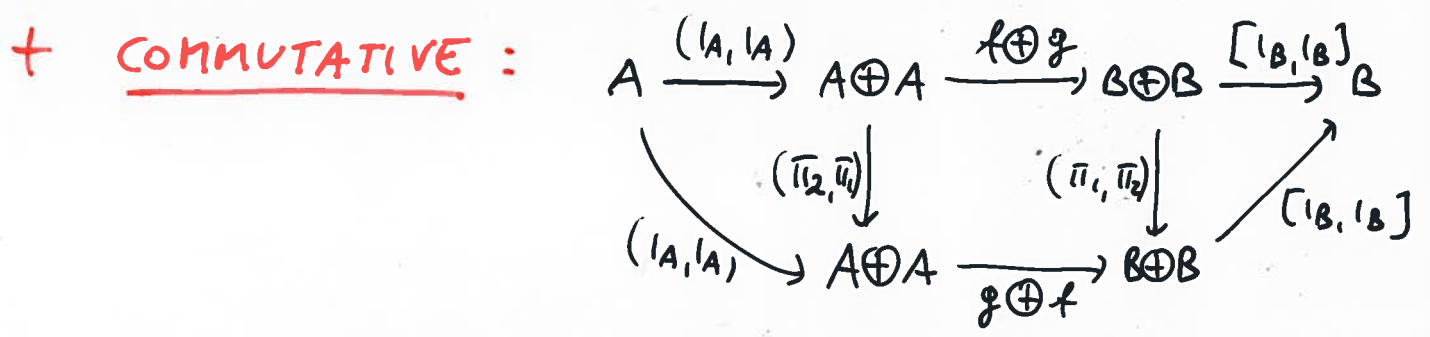
PROPOSITION (8)  $\mathcal{C}$  POINTED PROMONOIDAL + BINARY PRODUCTS ARE BIPRODUCTS. THEN  $\mathcal{C}$  IS **ADDITIVE**.

PROOF (SKETCH)

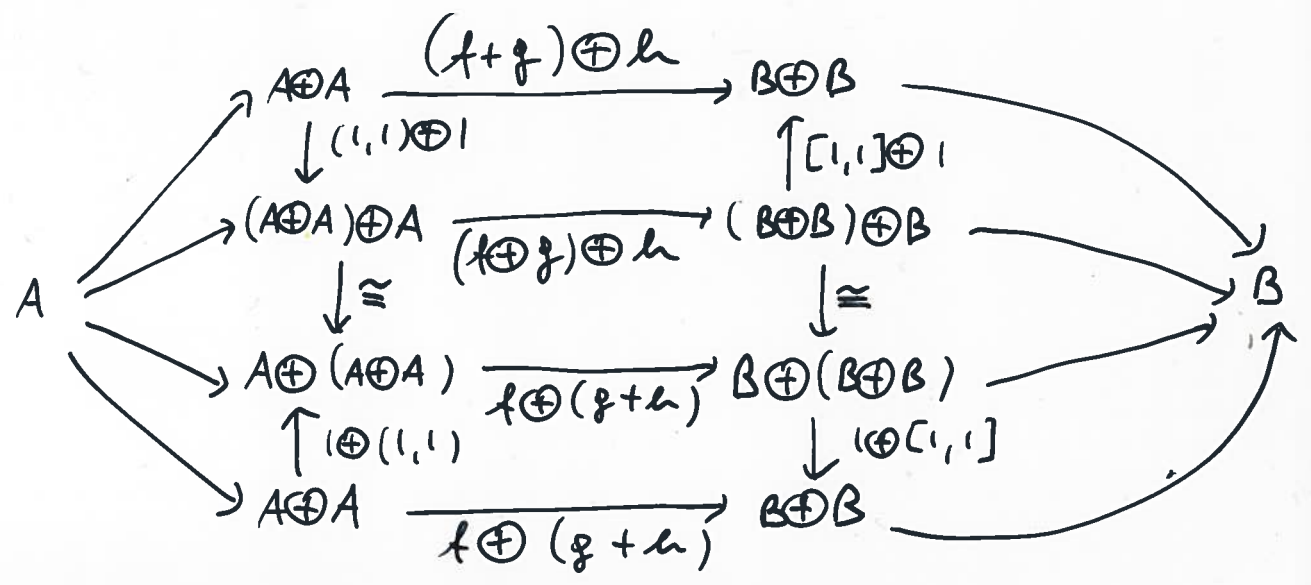
Given two arrows  $A \begin{matrix} \xrightarrow{f} \\ \xrightarrow{g} \end{matrix} B$  one defines

$$f + g = A \xrightarrow{(1_A, 1_A)} A \times A \xrightarrow{f \times g} B \times B \xrightarrow{[1_B, 1_B]} B$$

$$= [1_B, 1_B] \cdot f \oplus g \cdot (1_A, 1_A)$$



**+ ASSOCIATIVE !**



+ DISTRIBUTIVE WITH RESPECT TO COMPOSITION:

Given  $A \xrightarrow[f]{g} B$  and  $E \xrightarrow{e} A$ , the commutativity

of

$$\begin{array}{ccccc}
 A & \xrightarrow{(1,1)} & A \oplus A & \xrightarrow{f \oplus g} & B \oplus B & \xrightarrow{[1,1]} & B \\
 e \uparrow & & \uparrow e \oplus e & \nearrow (f \cdot e) \oplus (g \cdot e) & & & \\
 E & \xrightarrow{(1,1)} & E \oplus E & & & & 
 \end{array}$$

shows that  $(f + g) \cdot e = f \cdot e + g \cdot e$ .

Similarly one checks the other distributivity.

+ HAS AN "OPPOSITE"

One checks that any identity  $1_A$  has an opposite  $-1_A$  (using **PROTOMODULARITY**) and then  $f \cdot (-1_A) = -f$

□

COROLLARY (9)

$$\mathcal{C} \text{ ABELIAN} \iff \begin{array}{l} \mathcal{C} \text{ SEMI-ABELIAN} \\ + \\ \mathcal{C}^{op} \text{ SEMI-ABELIAN} \end{array}$$

PROOF

It follows immediately from Tierney's result **ABELIAN = EXACT + ADDITIVE** and **PROPOSITION (8)**

□

REMARK

Any SEMI-ABELIAN CATEGORY  $\mathcal{C}$  contains a full subcategory  $Ab(\mathcal{C})$  of ABELIAN OBJECTS, which is a full reflective subcategory of  $\mathcal{C}$ :

$$Ab(\mathcal{C}) \begin{matrix} \xleftarrow{ab} \\ \xrightarrow{\perp} \\ \xrightarrow{\nu} \end{matrix} \mathcal{C} \quad (*)$$

the left adjoint  $ab: \mathcal{C} \rightarrow Ab(\mathcal{C})$  sends an element  $x \in \mathcal{C}$  to the quotient by the largest commutator:

$$x \xrightarrow{\eta_x} D^+ / [x, x]$$

$Ab(\mathcal{C})$  is an ABELIAN CATEGORY, and the adjunction  $(*)$  has been studied in relationship with the categorical theory of CENTRAL EXTENSIONS (BOURN - GRAM, 2002, GRAM - VAN DER LINDEN, 2008)

# TORSION THEORIES

Originally defined by DICKSON (1966) in the realm of **ABELIAN CATEGORIES**, torsion theories have been recently investigated in a **NON-ADDITIVE CONTEXT**:

## DEFINITION

A pair of full (replete) subcategories  $(\mathcal{T}, \mathcal{F})$  of a **HOMOLOGICAL CATEGORY**  $\mathcal{C}$  is a **TORSION THEORY** if

1)  $\forall x \in \mathcal{C} \exists$  a short exact sequence

$$0 \longrightarrow T(x) \xrightarrow{t_x} x \xrightarrow{f_x} F(x) \longrightarrow 0$$

with  $T(x) \in \mathcal{T}$ ,  $F(x) \in \mathcal{F}$

2)  $\mathcal{C}(\mathcal{T}, \mathcal{F}) = \{ o_{\mathcal{T}, \mathcal{F}} \}$

$\forall T \in \mathcal{T}$

$\forall F \in \mathcal{F}$

$$\begin{array}{ccc} T & \xrightarrow{\quad} & F \\ & \searrow \text{---} & \nearrow \text{---} \\ & & 0 \end{array}$$

# EXAMPLES

- 1) Any torsion theory  $(\tau, \mathcal{F})$  in an **ABELIAN CATEGORY**. In particular,  $(\text{Ab}_{\epsilon.}, \text{Ab}_{\epsilon.\neq})$ , or  $(\text{Ab}_{\text{DIV}}, \text{Ab}_{\text{RED}})$  in  $\text{Ab}$ .
- 2) Let  $X\text{-MOD}$  be the category of **CROSSED MODULES**,  $\text{NORM MONO}$  its full subcategory of **NORMAL MONOS**,  $\text{Ab}$  the full subcategory whose objects are

$$A \in \text{Ab} \\ \downarrow \\ 0$$

Then, for any crossed module  $A \xrightarrow{\alpha} B$  one has an exact sequence

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \pi_1(\alpha) & \longrightarrow & A & \longrightarrow & \alpha(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \alpha & & \downarrow & & \\
 0 & \longrightarrow & 0 & \longrightarrow & B & \xlongequal{\quad} & B & \longrightarrow & 0 \\
 & & \uparrow \text{Ab} & & & & \in \text{NormMono} & & 
 \end{array}$$

Moreover, any arrow

$$\begin{array}{ccc}
 A & \xrightarrow{f_1} & N \\
 \downarrow & & \downarrow m \\
 0 & \xrightarrow{f_0} & B
 \end{array}$$

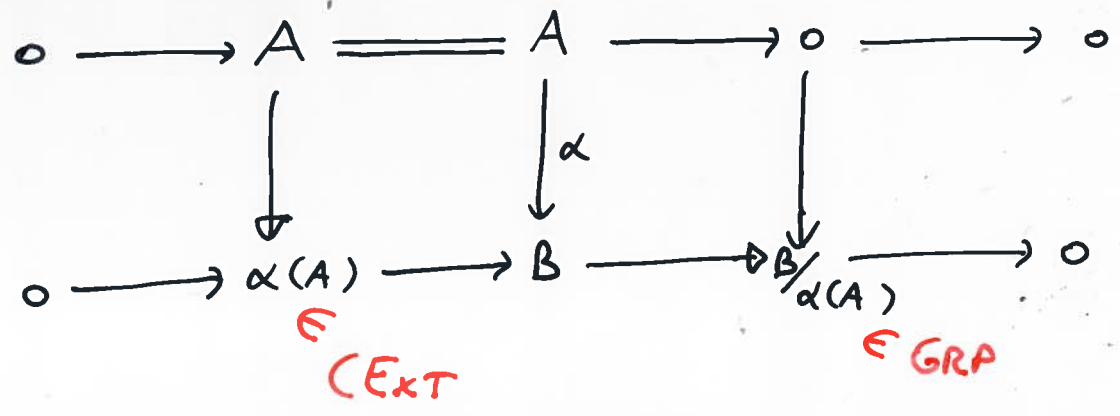
is the **0-ARROW**, since  $m: N \rightarrow B$  is **INJECTIVE**

3) In  $X\text{-Mod}$  consider the full subcategories

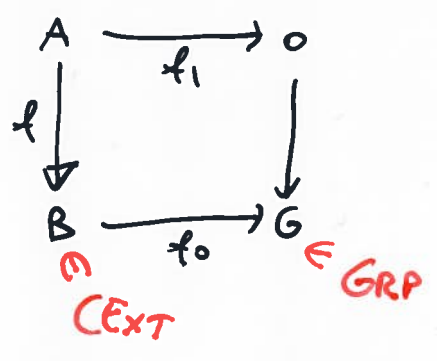
$\mathcal{T} = \text{CEXT}$  whose objects are **CENTRAL EXTENSIONS**

$\mathcal{F} = \text{GRP}$  whose objects are  $0 \rightarrow G$ , with  $G \in \text{GRP}$ .

Then any crossed module  $A \xrightarrow{\alpha} B$  yields an exact sequence



Furthermore, an arrow



is necessarily the **0-ARROW**, since  $f$  is SURJECTIVE

**MORE GENERALLY:**  $\mathcal{C}$  HOMOLOGICAL  $\Rightarrow \text{GRP}(\mathcal{C})$  HOMOL.

$(\text{COM} \text{GRP}(\mathcal{C}), \text{DIS}(\mathcal{C}))$  is a **TORSION THEORY** in  $\text{GRP}(\mathcal{C})$



4) Let  $\text{CRNG}$  be the category of COMMUTATIVE RINGS  
 $\text{NILCRNG}$  its subcategory of NILPOTENT RINGS  
 $\text{REDCRNG}$  " " " REDUCED RINGS

The pair  $(\text{NILCRNG}, \text{REDCRNG})$  is a TORSION THEORY in  $\text{CRNG}$ .

Indeed, given a ring  $A$ , let  $\text{Nil}(A) = \{a \in A \mid \exists m \in \mathbb{N}^+ a^m = 0\}$ .  
 It is an IDEAL of  $A$ , and

$$0 \longrightarrow \text{Nil}(A) \xrightarrow{\quad} A \xrightarrow{\quad} A/\text{Nil}(A) \longrightarrow 0$$

$\in \text{NILCRNG}$ 

 $\in \text{REDCRNG}$

One easily sees that, for any arrow

$$\begin{array}{ccc} N & \xrightarrow{f} & R \\ & \searrow & \nearrow \\ & 0 & \end{array}$$

from a nilpotent ring  $N$  to a reduced  $R$ ,

$$f(x) = 0 \quad \forall x \in N$$

5) Let  $\text{GRP}(\text{TOP})$  be the category of **TOPOLOGICAL GROUPS**,  
 $\text{GRP}(\text{IND})$  the category of **INDISCRETE GROUPS**,  
 $\text{GRP}(\text{HAUS})$  the category of **HAUSDORFF GROUPS**.

Then  $(\text{GRP}(\text{IND}), \text{GRP}(\text{HAUS}))$  is a torsion theory in  $\text{GRP}(\text{TOP})$   
indeed, if we denote by  $\overline{\{0\}}_G$  the closure of  $\{0\}$   
in a topological group  $G$ , one has an exact  
sequence in  $\text{GRP}(\text{TOP})$ :

$$0 \longrightarrow \overline{\{0\}}_G \longrightarrow G \longrightarrow \frac{G}{\overline{\{0\}}_G} \longrightarrow 0$$

Furthermore, if  $I \in \text{GRP}(\text{IND})$  and  $H \in \text{GRP}(\text{HAUS})$

$$\begin{array}{ccc} I & \xrightarrow{f} & H \\ & \searrow & \nearrow \\ & 0 & \end{array}$$

This follows from the fact that  $\overline{\{0\}}_H = \{0\}$   
and  $f^{-1}(\overline{\{0\}}_H) = \overline{\{0\}}_I = I$ .

6) In the category  $\text{GRP}(\text{COMP})$  of compact Hausdorff  
groups, the pair  $(\text{GRP}(\text{COMP}), \text{GRP}(\text{PROF}))$   
is a torsion theory. Here  $\text{GRP}(\text{COMP})$  is the  
category of connected compact groups,  $\text{GRP}(\text{PROF})$   
the category of profinite groups.

7) In the category  $GRP$  consider the full subcategories

$PERF$  of PERFECT GROUPS:  $G \in GRP$  SUCH THAT  $G = [G, G]$ .

$HypoAb$  of HYPOABELIAN GROUPS: these are the groups whose largest PERFECT SUBGROUP is TRIVIAL.

Then  $(PERF, HypoAb)$  is a torsion theory in  $GRP$ .

Another torsion theory in  $GRP$  is given by

$(GRP_T, GRP_{T.F.})$ , where  $GRP_T$  is the category of TORSION GROUPS (= generated by torsion elements) and  $GRP_{T.F.}$  the category of TORSION-FREE GROUPS.

THEOREM (10)

Let  $(\tau, \mathcal{F})$  be a **TORSION THEORY** in a HOMOLOGICAL CATEGORY  $\mathcal{C}$ . Then the **TORSION-FREE** subcategory  $\mathcal{F}$  is reflective in  $\mathcal{C}$ :

$$\mathcal{F} \begin{array}{c} \xleftarrow{F} \\ \xrightarrow{\perp} \\ \xrightarrow{U} \end{array} \mathcal{C} \quad (*)$$

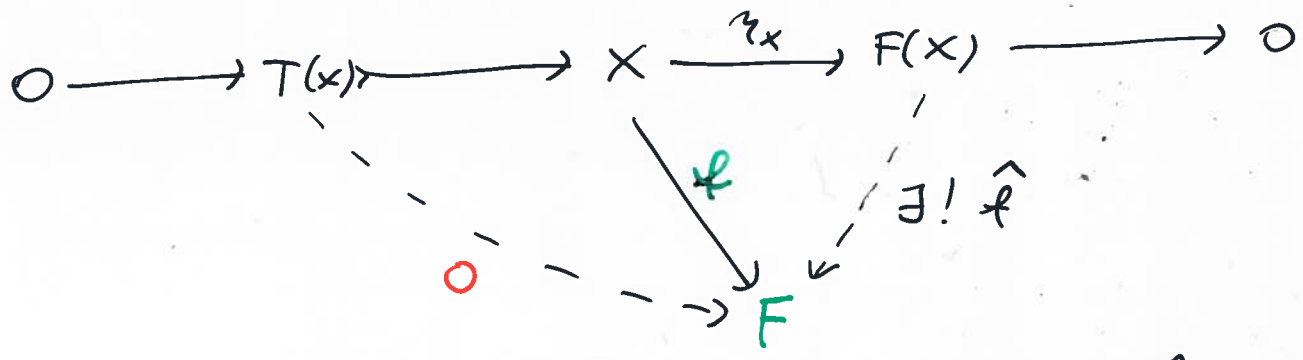
PROOF:

Let us show that the canonical quotient

$$X \xrightarrow{\eta_x} F(X) = X/T(X)$$

is the **UNIT** of the adjunction  $(*)$ .

Given an object  $F \in \mathcal{F}$  and an arrow  $X \xrightarrow{f} F$  one gets the diagram



and a **UNIQUE**  $\hat{f}$  such that  $\hat{f} \cdot \eta_x = f$ .

□

Any torsionfree subcategory  $\mathcal{A}$  of a HOMOLOGICAL CATEGORY  $\mathcal{C}$  is then a (NORMAL EPI) - REFLECTION :

$$\mathcal{A} \begin{array}{c} \xleftarrow{F} \\ \perp \\ \xrightarrow{U} \end{array} \mathcal{C}$$

FULL, REFLECTIVE,

$$\forall x \in \mathcal{C} \quad x \xrightarrow{\eta_x} U F(x)$$

IS A NORMAL EPI

We are going to prove that torsion theories correspond to a special kind of (NORMAL EPI) - REFLECTION, namely those ones where the REFLECTOR  $F: \mathcal{C} \rightarrow \mathcal{A}$  is SEMI-LEFT-EXACT (CASSIDY, HEBERT, KELLY, 1985)

DEFINITION

A full reflective subcategory  $\mathcal{A}$  of a finitely complete category  $\mathcal{C}$

$$\mathcal{A} \begin{array}{c} \xleftarrow{F} \\ \perp \\ \xrightarrow{U} \end{array} \mathcal{C}$$

has a SEMI-LEFT-EXACT reflector if  $F$  preserves the pullbacks of the form

$$\begin{array}{ccc} P & \xrightarrow{P_2} & Y \\ P_1 \downarrow \perp & & \downarrow \eta_Y \\ UX & \xrightarrow{U \eta} & UY \end{array}$$

where  $\eta_Y$  is the  $Y$ -component of the UNIT, and  $\eta$  lies in  $\mathcal{A}$ .

THEOREM (11) Let  $\mathcal{C}$  be a **HOMOLOGICAL CATEGORY**,

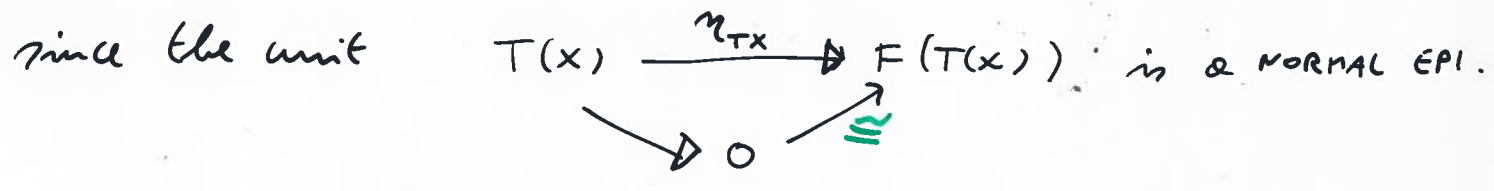
$\mathcal{A}$  a full replete subcategory of  $\mathcal{C}$ . TFCAE:

- 1)  $\mathcal{A}$  is a **TORSION-FREE** subcategory of  $\mathcal{C}$ ;
- 2)  $\mathcal{A}$  is **(NORMAL EPI)-REFLECTIVE** in  $\mathcal{C}$  and the induced radical  $T$  is IDEMPOTENT:  $\forall x \in \mathcal{C}, T(T(x)) \cong T(x)$ ;
- 3)  $\mathcal{A}$  is **(NORMAL EPI)-REFLECTIVE** in  $\mathcal{C}$  and the reflector  $F: \mathcal{C} \rightarrow \mathcal{A}$  is **SEMI-LEFT EXACT**.

PROOF

1)  $\Rightarrow$  2) When  $(\tau, \mathcal{A})$  is a torsion theory we have observed that  $\eta_x: x \rightarrow x/T(x)$  is the unit. Accordingly,  $\mathcal{A}$  is **(NORMAL EPI)-REFLECTIVE** in  $\mathcal{C}$ .

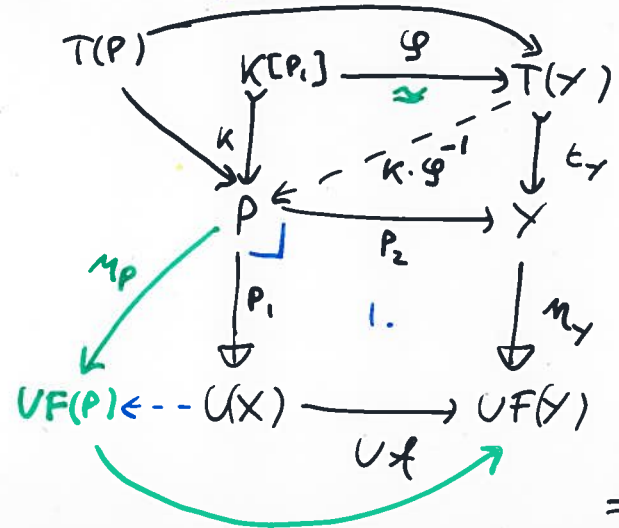
Moreover, 
$$\tau = \{ x \in \mathcal{C} \mid F(x) = x/T(x) = 0 \}$$



It follows that  $\epsilon_{T(x)}: T(T(x)) \rightarrow T(x)$  is an ISO.

2)  $\Rightarrow$  3)

Consider the **PULLBACK**



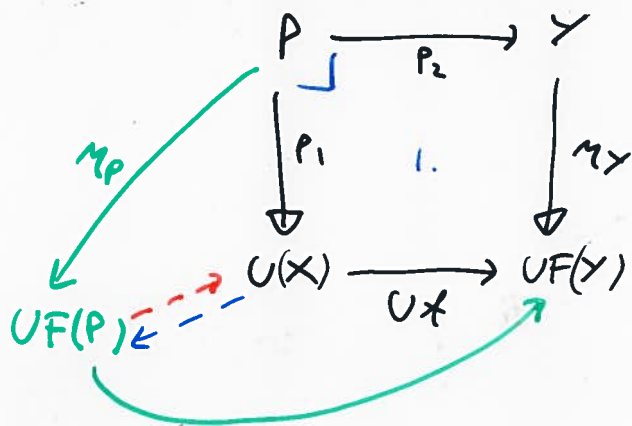
$k \cdot \varphi^{-1} = \text{Ker}(P_i)$

The IDEMPOTENCY of  $T$  implies

$T(X) = T(T(Y)) \leq T(P)$   
(as SUBOBJECTS OF  $P$ )

$\Rightarrow \eta_P$  FACTORS THROUGH  $P_i$

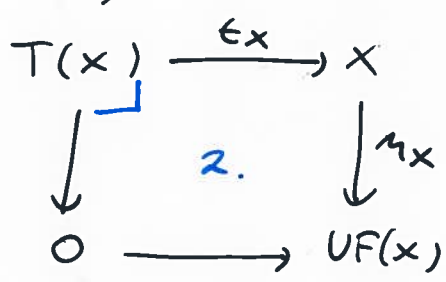
Obviously  $P_1$  FACTORS THROUGH  $\eta_P$  :



Accordingly,  $P_1 \cong \eta_P$ , and the pullback 1. is PRESERVED

3)  $\Rightarrow$  2) Assume that the reflector  $F: \mathcal{C} \rightarrow \mathcal{A}$

is SEMI-LEFT EXACT, so that the PULLBACK 2.



is PRESERVED by  $F: \mathcal{C} \rightarrow \mathcal{A}$ . It follows that

$$F(T(x)) \cong 0, \text{ and } \tau T(x) \xrightarrow{\epsilon_{T(x)}} T(x) \text{ is an ISO.}$$

2)  $\Rightarrow$  1)

One defines  $\tau = \{ Y \in \mathcal{C} \mid Y \cong T(x), \text{ for some } x \in \mathcal{C} \}$

and checks that  $(\tau, \mathcal{A})$  is a TORSION THEORY

DEFINITION (MANTOVANI, 1998)

A full reflective subcategory  $\mathcal{F}$  of a finitely complete category  $\mathcal{C}$

$$\mathcal{F} \begin{array}{c} \xleftarrow{F} \\ \perp \\ \xrightarrow{\quad} \end{array} \mathcal{C}$$

is called a **SEMI-LOCALISATION** of  $\mathcal{C}$  if  $F: \mathcal{C} \rightarrow \mathcal{F}$  is **SEMI-LEFT-EXACT**.

REMARK

The previous Theorem says in particular that the **torsion-free subcategories** of a **homological category** are precisely the **SEMI-LOCALISATIONS** with the property that the unit of the adjunction is a **normal epi**.

TERMINOLOGY

In category theory one calls **LOCALISATION** a full reflective subcategory

$$\mathcal{F} \begin{array}{c} \xleftarrow{F} \\ \perp \\ \xrightarrow{\quad} \end{array} \mathcal{C}$$

where the reflector  $F: \mathcal{C} \rightarrow \mathcal{F}$  preserves **FINITE LIMITS**.



# EXAMPLES OF SEMI-LOCALISATIONS

$$\text{Ab} \xrightleftharpoons[\cup]{\cap} \text{Ab}$$

$\epsilon \cdot 1$

$$A \xrightarrow{\mu_A} A/T(A),$$

where  $T(A) = \{a \in A \mid \exists m \in \mathbb{N}^*, ma = 0\}$

$$\text{Norm Mono} \xrightleftharpoons[\cup]{\cap} X\text{-Mod}$$

$$\begin{array}{ccc} A & \xrightarrow{\mu'_\alpha} & \alpha(A) \\ \downarrow \alpha & & \downarrow \\ B & \xrightarrow{\mu''_\alpha} & B \end{array}$$

$$\text{Dis}(\mathcal{E}) \xrightleftharpoons[\cup]{\cap} \text{GRPD}(\mathcal{E})$$

$$\begin{array}{ccc} X_1 & \xrightarrow{\mu^*_X} & \pi_0(X) \\ d \uparrow \downarrow c & & \uparrow \downarrow \\ X_0 & \xrightarrow{\mu^*_X} & \pi_0(X) \end{array}$$

$$\text{RED CRNG} \xrightleftharpoons[\cup]{\cap} \text{CRNG}$$

$$A \xrightarrow{\mu_A} A/\text{Nil}(A)$$

where  $\text{Nil}(A) = \{a \in A \mid \exists m \in \mathbb{N}^*, a^m = 0\}$

$$\text{GRP}(\text{Haus}) \xrightleftharpoons[\cup]{\cap} \text{GRP}(\text{Top})$$

$$G \xrightarrow{\mu_G} G/\overline{\{0\}}_G$$

$\overline{\{0\}}$  is the closure of 0 in G

$$\text{GRP}(\text{TotDis}) \xrightleftharpoons[\cup]{\cap} \text{GRP}(\text{Haus})$$

$$H \xrightarrow{\mu_H} H/\Gamma_0(H)$$

where  $\Gamma_0(H) =$  CONNECTED COMPONENT OF 0

# SEMI-LOCALISATIONS AND CLOSURE OPERATORS

DEFINITION (BOURN-GRAM, 2006)

- An **IDEMPOTENT CLOSURE OPERATOR** on NORMAL MONOS consists in giving, for any normal monos  $S \xrightarrow{f} X$ , another normal monos  $\bar{S} \xrightarrow{\bar{f}} X$  in such a way that:

- 1)  $S \subset \bar{S}$ .
- 2)  $S \subset T \Rightarrow \bar{S} \subset \bar{T}$
- 3)  $\overline{f^{-1}(S)} \subset f^{-1}(\bar{S})$  ,  $\forall \gamma \xrightarrow{f} X$
- 4)  $\overline{\bar{S}} = \bar{S}$

- Such an operator is called **HOMOLOGICAL** if, moreover,

$$5) \overline{f^{-1}(S)} = f^{-1}(\bar{S}) \quad \forall \text{ NORMAL EPI } f: \gamma \twoheadrightarrow X$$

We are going to show that, in a **HOMOLOGICAL CATEGORY**  $\mathcal{C}$ , there is a bijection between

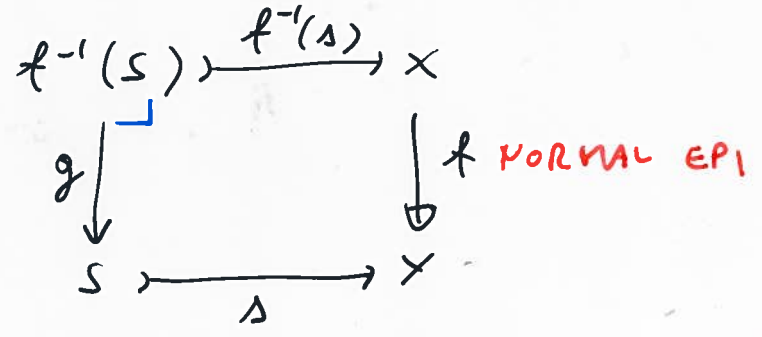
HOMOLOGICAL  
CLOSURE  
OPERATORS



(NORMAL EPI)-REFLECTIVE  
SUBCATEGORIES  
OF  $\mathcal{C}$

LEMMA (12)

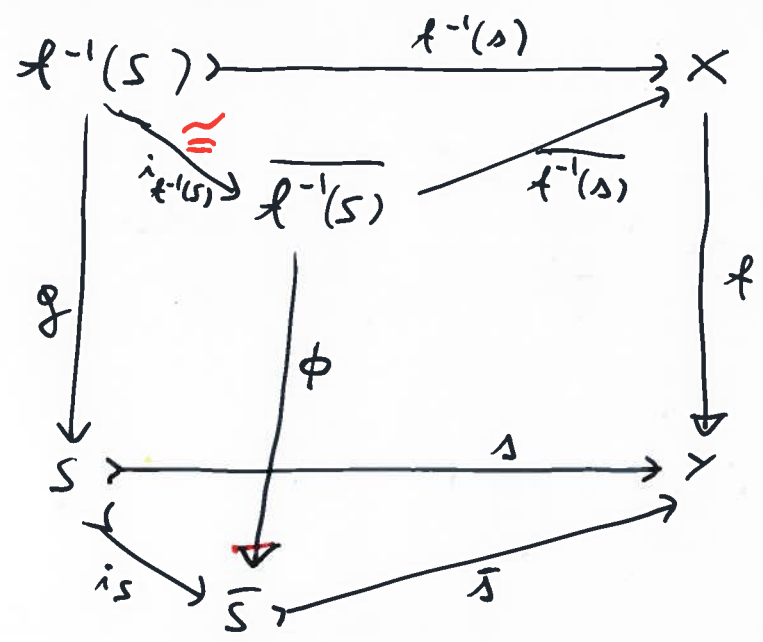
Given a **HOMOLOGICAL CLOSURE OPERATOR** on **normal mon.** in a **HOMOLOGICAL CATEGORY  $\mathcal{C}$** , the following are equivalent:



- 1.  $\Delta$  is **CLOSED**  $\iff f^{-1}(\Delta)$  is **CLOSED**
- 2.  $\Delta$  is **DENSE**  $\iff f^{-1}(\Delta)$  is **DENSE**

PROOF 1. If  $\Delta: S \rightarrow Y$  is closed, then  $\overline{f^{-1}(S)} \stackrel{(5)}{=} f^{-1}(\overline{S}) = f^{-1}(S)$ , showing that  $f^{-1}(S)$  is closed.

Conversely, assume that  $f^{-1}(S)$  is closed and consider



The assumption says that  $i_{f^{-1}(S)}$  is an ISO, and (5) that  $\phi$  is a regular epi. Then  $i_S \circ g = \phi \cdot i_{f^{-1}(S)}$  is a **REGULAR EPI**  $\implies i_S$  IS AN ISO.  $\square$

THEOREM (13) (BOURN - GRAN, 2006)

Let  $\mathcal{C}$  be a **HOMOLOGICAL CATEGORY**. There is a BIJECTION between:

- 1) **(NORMAL-EPI)-REFLECTIVE SUBCATEGORIES OF  $\mathcal{C}$**
- 2) **HOMOLOGICAL CLOSURE OPERATORS**

PROOF

Given a homological closure operator  $(\bar{\quad})$  one defines  $\mathcal{F} = \{ F \in \mathcal{C} \mid 0 \rightarrow F \text{ is closed} \}$ .

The category  $\mathcal{F}$  is (NORMAL EPI)-REFLECTIVE: the square

$$\begin{array}{ccc}
 \bar{0}_x & \xrightarrow{k_x} & x \\
 \downarrow & & \downarrow \eta_x = \text{coker}(k_x) \\
 0 & \xrightarrow{\quad} & x/\bar{0}_x
 \end{array}$$

is a PUSHOUT and a PULLBACK. By Lemma 12 we deduce that  $0 \rightarrow x/\bar{0}_x$  is closed (since  $\bar{0}_x \rightarrow x$  is no). Accordingly,  $x/\bar{0}_x \in \mathcal{F}$ . Let us check the UNIVERSAL PROPERTY.

$$\begin{array}{ccccc}
 \bar{0}_x & \xrightarrow{k_x} & x & \xrightarrow{\eta_x} & x/\bar{0}_x \\
 \vdots & & \downarrow \forall f & & \\
 0 & \dashrightarrow & F \in \mathcal{F} & & 
 \end{array}$$

Given  $x \xrightarrow{f} F \in \mathcal{F}$ ,  $\bar{0}_F = 0$ , and then  $\bar{0}_x \subset \overbrace{f^{-1}(0)}^{(2)} \subset \overbrace{f^{-1}(\bar{0}_F)}^{(3)}$

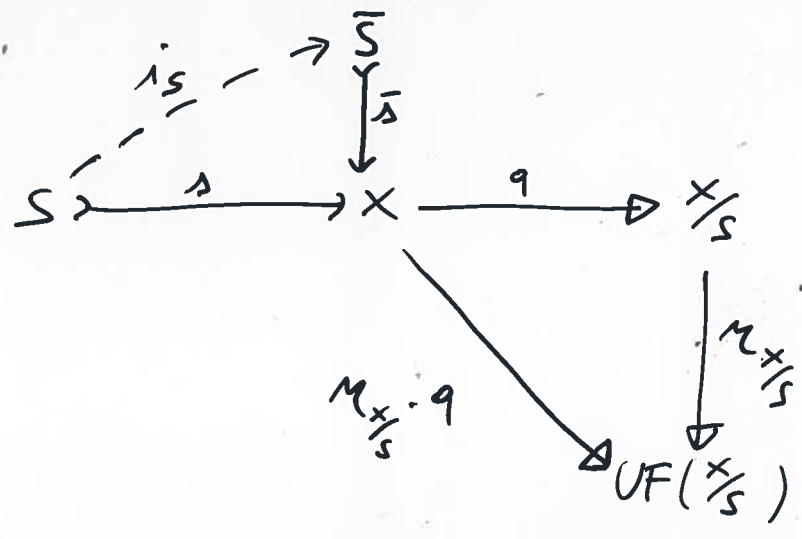
gives  $\bar{0}_x \subset f^{-1}(0_F) = \text{Ker}(f)$   
 $\Rightarrow \exists! F(f): x/\bar{0}_x \rightarrow F$  s.t.  $F(f) \cdot \eta_x = f$

This shows that  $\mathcal{F} \xrightleftharpoons[\cup]{\perp} \mathcal{C}$  is (NORMAL EPI)-REFLECTIVE

Conversely, starting from a (normal epi)-reflective subcategory  $\mathcal{F}$  of  $\mathcal{C}$

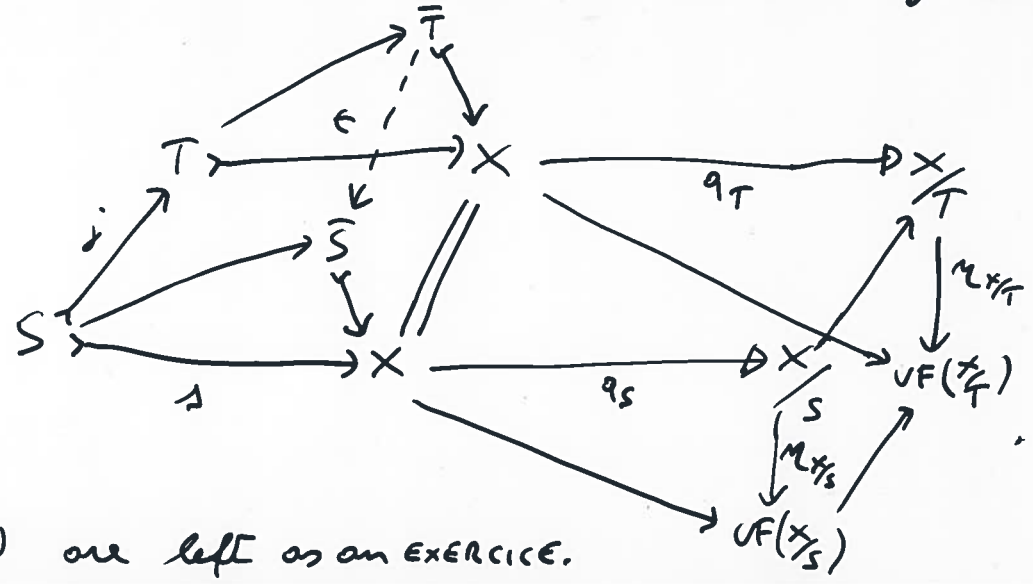
$$\mathcal{F} \begin{array}{c} \xleftarrow{F} \\ \perp \\ \xrightarrow{U} \end{array} \mathcal{C}$$

one defines the closure  $\bar{S} \rightarrow X$  of a normal mono  $S \rightarrow X$  in  $\mathcal{C}$  as follows:



$$\bar{S} = \text{Ker}(m_{X/S} \cdot q)$$

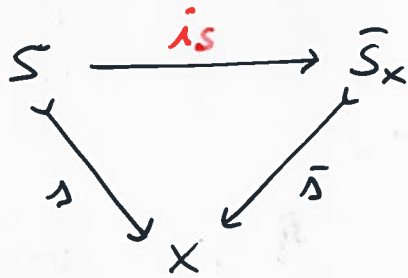
- (1) is valid by construction,
- (4)  $\bar{\bar{S}} = \bar{S}$  holds because any normal epi is the cokernel of its kernel
- (2)  $S < T \Rightarrow \bar{S} < \bar{T}$  can be checked by looking at



(3) and (5) are left as an EXERCISE. □

DEFINITION

A homological closure operator is **WEAKLY HEREDITARY** if the canonical inclusions



are **DENSE**:  $\overline{S_{\bar{S}_X}} = \bar{S}_X$

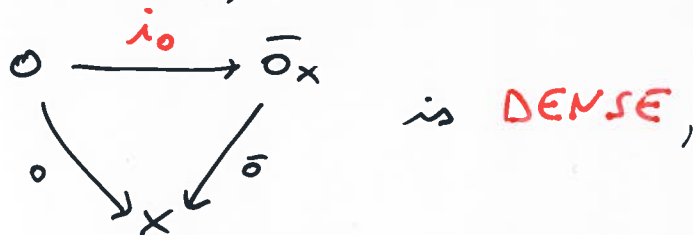
THEOREM (14)

In a HOMOLOGICAL CATEGORY  $\mathcal{C}$  there is a bijection between:  
 1) **TORSION THEORIES** in  $\mathcal{C}$   
 2) **WEAKLY HEREDITARY CLOSURE OPERATORS**

PROOF: It suffices to check that, in the bijection of THEOREM (13),

$\bar{(\ )}$  is **WEAKLY HEREDITARY**  $\Leftrightarrow$  the reflector  $F: \mathcal{C} \rightarrow \mathcal{F}$  is **SEMI-LEFT-EXACT**

$\Rightarrow$ ) When  $\bar{(\ )}$  is **WEAKLY HEREDITARY**, the induced radical  $T(X) \rightarrow X$  associated with  $F: \mathcal{C} \rightarrow \mathcal{F}$  is **IDEMPOTENT**: indeed,  $\bar{0}_X = T(X)$  and



so that

$$T(T(X)) = \bar{0}_{\bar{0}_X} = \bar{0}_X = T(X)$$

$\Leftarrow$ ) Conversely, when the reflector  $F: \mathcal{C} \rightarrow \mathcal{A}$  is SEMI-LEFT-EXACT, one has  $T(T(x)) \simeq T(x)$ .

Given any NORMAL MONO  $S \xrightarrow{\delta} X$ , form the diagram

$$\begin{array}{ccccc}
 & & \delta & & \\
 & & \curvearrowright & & \\
 S & \xrightarrow{i_S} & \bar{S}_X & \xrightarrow{\bar{\delta}} & X \\
 \downarrow & & \downarrow & \lrcorner & \downarrow q \\
 0 & \xrightarrow{i_0} & \overline{0_{X/S}} & \xrightarrow{\quad} & X/S
 \end{array}$$

1.                      2.

and observe that 2. is a PULLBACK since

$$q^{-1}(\overline{0_{X/S}}) \stackrel{(5)}{=} \overline{q^{-1}(0)_X} = \bar{S}_X$$

It follows that 1. is a PULLBACK and is DENSE,

since  $0 \xrightarrow{i_0} \overline{0_{X/S}}$  is DENSE (see LEMMA (12))

□

## EXERCISE

A torsion theory  $(\tau, \mathcal{A})$  in  $\mathcal{C}$  is HEREDITARY when the subcategory  $\tau$  is stable under SUBOBJECTS:

$$[\text{if } M \xrightarrow{m} X \text{ is a MONO and } X \in \tau] \Rightarrow [M \in \tau]$$

PROVE THAT:  $(\tau, \mathcal{A})$  is HEREDITARY



$F: \mathcal{C} \rightarrow \mathcal{A}$  preserves MONOMORPHISMS

EXAMPLES

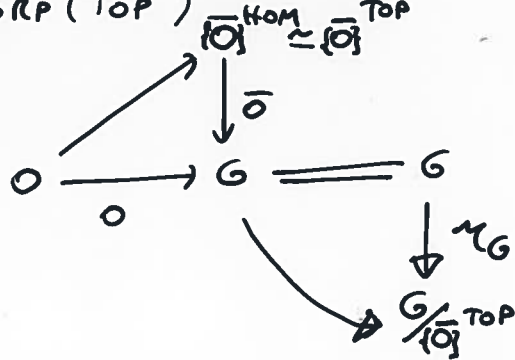
1) Consider the reflection

$$\text{GRP}(\text{HAUS}) \xrightleftharpoons[\cup]{\cap} \text{GRP}(\text{TOP})$$

By definition of  $\overline{(\quad)}^{\text{HOM}}$  one has

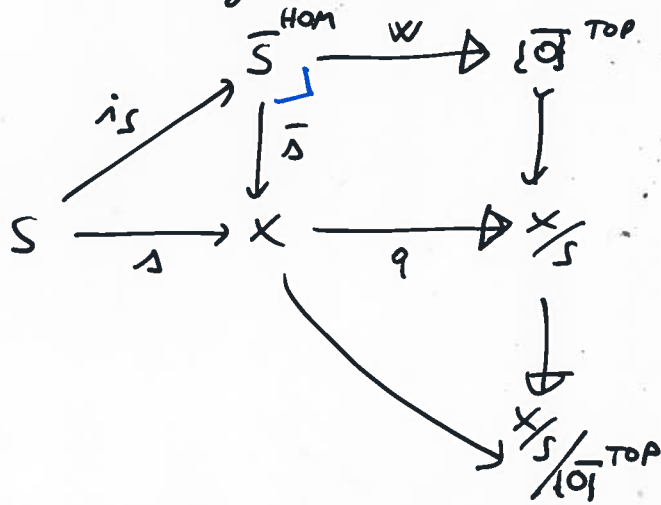
$$\overline{\{0\}}^{\text{HOM}} = \overline{\{0\}}^{\text{TOP}}$$

since  $\forall G \in \text{GRP}(\text{TOP})$



Let  $S \xrightarrow{\Delta} X$  be any NORMAL SUBGROUP of  $X \in \text{GRP}(\text{TOP})$

Then:



One gets an exact sequence

$$0 \longrightarrow S \xrightarrow{is} \overline{S}^{\text{HOM}} \xrightarrow{W} \overline{\{0\}}^{\text{TOP}} \longrightarrow 0$$

$S$  is closed for  $\overline{(\quad)}^{\text{HOM}} \iff is : S \longrightarrow \overline{S}^{\text{HOM}}$  is an ISO

$\iff \overline{\{0\}}^{\text{TOP}} = \{0\}$

$\iff X/S \in \text{GRP}(\text{HAUS})$

$\iff S = \overline{S}^{\text{TOP}}$

CONCLUSION:  $\overline{(\quad)}^{\text{HOM}} = \overline{(\quad)}^{\text{TOP}}$



2)

$$\text{GRP (PROF)} \begin{array}{c} \xleftarrow{F} \\ \xrightarrow{U} \\ \perp \end{array} \text{GRP (COMP)}$$

The reflection of a compact group  $G$  is

$$G \xrightarrow{\pi_G} G/\Gamma_0(G)$$

The closure of a NORMAL SUBGROUP  $S \rightarrow X$

$$\begin{aligned} \text{is given by } \bar{S} &= S \cdot \Gamma_0(X) \\ &= \{s \cdot x \mid s \in S, x \in \Gamma_0(X)\} \end{aligned}$$

3)

$$\text{RED CRNG} \begin{array}{c} \xleftarrow{F} \\ \xrightarrow{U} \\ \perp \end{array} \text{CRNG}$$

Given  $S \rightarrow A$  an ideal one computes its closure

$$\begin{array}{ccc} \bar{S} & \longrightarrow & \text{Nil}(A/S) = \overline{0}_{A/S} \\ \downarrow & \lrcorner & \downarrow \\ S & \xrightarrow{\pi} & A/S \end{array}$$

$$\bar{S} = \pi^{-1}(\text{Nil}(A/S))$$

$$= \{a \in A \mid \exists m \in \mathbb{N}^* \text{ WITH } a^m = \overline{0}_{A/S}\}$$

$$= \{a \in A \mid \exists m \in \mathbb{N}^* \text{ WITH } a^m \in S\}$$

$$= \sqrt{S}, \text{ the } \underline{\text{radical}} \text{ of } S$$

4)

$$Ab \xrightleftharpoons[\nu]{\mu} Ab$$

e.f.

Here, for an  $A \in Ab$ ,  $\bar{0}_A = \{a \in A \mid \exists m \in \mathbb{N}^*, ma = 0\}$

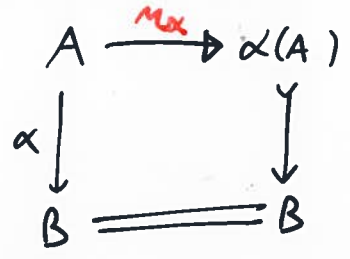
Given a subgroup  $K \rightarrow A$ ,

$$\bar{K} = \{a \in A \mid \exists m \in \mathbb{N}^* \text{ s.t. } ma \in K\}$$

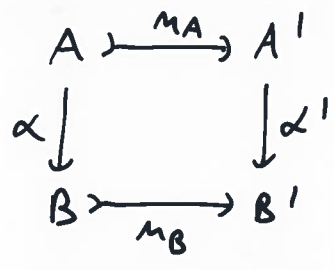
5)

$$Norm Mono \xrightleftharpoons[\nu]{\mu} X-Mod$$

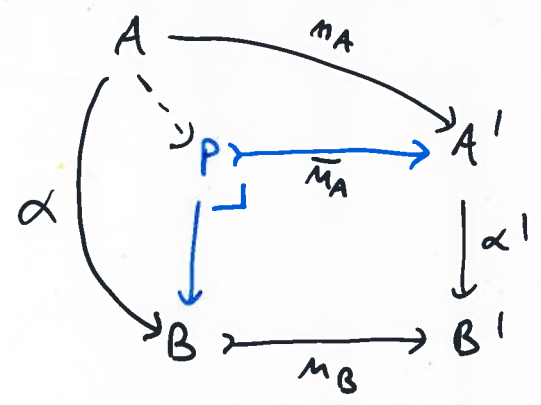
If  $A \xrightarrow{\alpha} B \in X-Mod$  then the unit  $\mu_\alpha$  is given by



One can check that the closure of a NORMAL MONO



is obtained by the PULLBACK



# SEMI-LOCALISATIONS OF SEMI-ABELIAN CATEGORIES

Torsion-free subcategories of an abelian category are not abelian, in general.

Indeed, the classical torsion-free subcategory

$$\text{Ab}_{\text{t.f.}} \begin{array}{c} \xleftarrow{F} \\ \xrightarrow{U} \\ \text{Ab} \end{array}$$

is not **EXACT**, since one can find **EQUIVALENCE RELATIONS** which are not **EFFECTIVE**:

let  $R_2 \begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{p_2} \end{array} \mathcal{C}$  be the equivalence

relation on  $\mathcal{C}$  defined by  $(M, N) \in R_2$

$$\begin{array}{c} \Downarrow \\ \exists K \in \mathcal{C} \text{ s.t. } M - N = 2K \end{array}$$

This equivalence relation  $R_2 \begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{p_2} \end{array} \mathcal{C}$  can not be the **KERNEL PAIR** of its coequaliser:

$$R_2 \begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{p_2} \end{array} \mathcal{C} \xrightarrow{q} \mathcal{C}/R_2 \cong 0 \text{ in } \text{Ab}_{\text{t.f.}}!$$

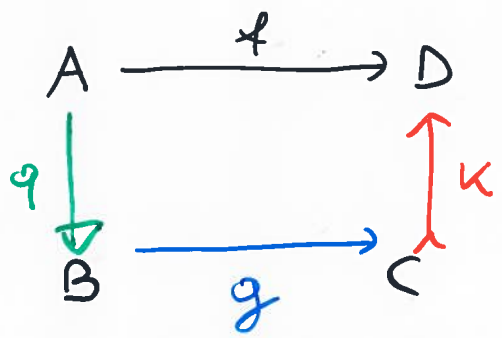
Although  $\text{Ab}_{\text{t.f.}}$  is **REGULAR** and **ADDITIVE**, it is not **EXACT**  $\Rightarrow$  not **ABELIAN**!

A **torsion-free** subcategory of an **abelian**  $\mathcal{C}$  inherits some interesting exactness properties:

THEOREM (RUMP, 2001)

For a category  $\mathcal{F}$  the following are equivalent

- 1)  $\mathcal{F}$  is a **torsion-free** subcategory of an **abelian category**  $\mathcal{C}$
- 2) a)  $\mathcal{F}$  is **additive**
- b) any morphism in  $\mathcal{F}$  has a factorisation



where  $q$  is a **NORMAL EPI**  
 $g$  is a **BIMORPHISM**  
 $k$  is a **NORMAL MONO**

c) **NORMAL EPIS** are pullback stable.

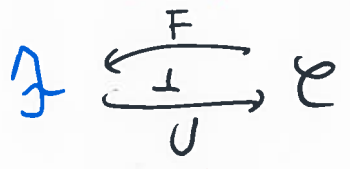
W. RUMP has shown that many categories of interest satisfy these conditions ( $Ab(TOP)$ ,  $Ab(Haus)$ ,  $BAN$ , ETC.).

QUESTION

Can one find an intrinsic characterization of torsion-free subcategories of a semi-abelian category?

REMARK

Any (normal epi)-reflective subcategory



of a semi-abelian category  $\mathcal{C}$  is

- HOMOLOGICAL
- WITH BINARY COPRODUCTS

QUESTION (2<sup>ND</sup> VERSION)

Among homological categories can one characterise the ones occurring as torsion-free subcategories of a semi-abelian category?

The answer to this question was recently found in collaboration with **S. LACK**, by using some remarkable results on **semi-localisations of EXACT CATEGORIES** due to **S. MANTOVANI (1998)**.

### DEFINITION

A **COKERNEL**  $q: B \rightarrow Q$  of a mono  $m: A \rightarrow B$  is **STABLE** if, given any  $f: D \rightarrow Q$  as in

$$\begin{array}{ccc} & & D \\ & & \downarrow f \\ A \xrightarrow{m} B & \xrightarrow{q} & Q \end{array}$$

with pullback

$$\begin{array}{ccccc} & & B \times_D Q & \xrightarrow{\pi_2} & D \\ & m' \nearrow & \downarrow \pi_1 & & \downarrow f \\ A \xrightarrow{m} & B & \xrightarrow{q} & Q \end{array}$$

one then has  $\pi_2 = \text{coker}(m')$

## THEOREM (GRAN-LACK, 2014)

For a category  $\mathcal{A}$  the following are equivalent:

- $\mathcal{A}$  is a SEMI-LOCALISATION OF A SEMI-ABELIAN CATEGORY
- $\mathcal{A}$  is a TORSION-FREE SUBCATEGORY OF A SEMI-ABELIAN CATEGORY
- $\mathcal{A}$  is HOMOLOGICAL, has BINARY COPRODUCTS, and STABLE COKERNELS OF BOURN-NORMAL HOMOMORPHISMS
- $\mathcal{A}$  is HOMOLOGICAL, has BINARY COPRODUCTS, and STABLE COEQUALISERS OF EQUIVALENCE RELATIONS

## EXAMPLES

All torsion-free subcategories of semi-abelian categories we have encountered so far.

Also  $\text{GRP}(\text{TOP})$ ,  $\text{RNG}(\text{TOP})$ ,  $\text{RED}(\text{RNG})$ ,  $\text{GRP}_{\text{T.F.}}$ ,  $\text{NORM}(\text{TOP})$  ...

## REMARK

The result above provides in particular a **new proof** of RUMP'S result.