DEFINITION (JANELIDZE, MÁRKI, THOLEN, 2001)

A category $\mathcal{C}$ is SEMI-ABELIAN if

- $\mathcal{C}$ is HOMOLOGICAL
- $\mathcal{C}$ is EXACT
- $\mathcal{C}$ has BINARY COPRODUCTS

REMARK

ABELIAN $\Rightarrow$ SEMI-ABELIAN

The classical result by Tierney:

ABELIAN $\equiv$ EXACT + ADDITIVE

tells us that it suffices to check that the SPLIT SHORT FIVE LEMMA holds in any abelian category. This is clear, since any split short exact sequence is of the form

$$0 \longrightarrow K \longrightarrow K \oplus B \rightleftarrows B \longrightarrow 0$$

LEMMA ⑦ If $\mathcal{C}$ and $\mathcal{C}^{op}$ are HOMOLOGICAL, then the canonical arrow $\alpha : A+B \longrightarrow A\times B$ is an isomorphism.

PROOF

The pair $(1,0)$ and $(0,1)$ in the diagram

$$\begin{tikzcd}
A \arrow[r, "{(1,0)}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A\times B \arrow[d, shift right, "p_2"'] \\
0 \arrow[r] & B \arrow[u, shift right, "{(0,1)}"']
\end{tikzcd}$$

is JOINTLY EXTREMAL EPIMORPHIC. This implies that $\alpha : A+B \longrightarrow A\times B$ is EXTREMAL EPIC: given a mono $J \overset{j}{\rightarrowtail} A\times B$ such that $\exists$ a

$$\begin{tikzcd}
 & & J \arrow[d, "j"] & & \\
 & & A\times B & & \\
A \arrow[rr, "i_1"'] \arrow[urr, "{(1,0)}"] \arrow[uurr, dashed, bend left] & & A+B \arrow[u, "\alpha"] \arrow[uu, bend left, "a"] & & B \arrow[ll, "i_2"] \arrow[ull, "{(0,1)}"'] \arrow[uull, dashed, bend right]
\end{tikzcd}$$

with $j \cdot a = \alpha$, then j is an ISO.

The arrow $\alpha : A+B \longrightarrow A\times B$ is also an EPI in $\mathcal{C}^{op}$ (since this latter is homological).

Accordingly, $\alpha : A+B \longrightarrow A\times B$ is a MONO + EXTREMAL EPI

$\Rightarrow$ α is an ISOMORPHISM

$\square$

Proposition ⑧ $\mathcal{C}$ pointed protomodular + binary products are biproducts. Then $\mathcal{C}$ is additive.

Proof (sketch)

Given two arrows $A \underset{g}{\overset{f}{\rightrightarrows}} B$ one defines

$$f+g = A \xrightarrow{(1_A,1_A)} A\times A \xrightarrow{f\times g} B\times B \xrightarrow{[1_B,1_B]} B$$

$$= [1_B,1_B]\cdot f\oplus g\cdot(1_A,1_A)$$

+ commutative :

$$\begin{tikzcd}
A \arrow[r, "{(1_A,1_A)}"] \arrow[dr, bend right, "{(1_A,1_A)}"'] & A\oplus A \arrow[r, "f\oplus g"] \arrow[d, "{(\pi_2,\pi_1)}"'] & B\oplus B \arrow[r, "{[1_B,1_B]}"] \arrow[d, "{(\pi_1,\pi_2)}"'] & B \\
 & A\oplus A \arrow[r, "g\oplus f"'] & B\oplus B \arrow[ur, "{[1_B,1_B]}"'] &
\end{tikzcd}$$

0 :

$$\begin{tikzcd}
A \arrow[r, "{(1_A,1_A)}"] \arrow[dr, equal] & A\oplus A \arrow[r, "f\oplus 0"] \arrow[d, "\pi_1"'] & B\oplus B \arrow[r, "{[1_B,1_B]}"] & B \\
 & A \arrow[r, "f"'] & B \arrow[u, "i_1"'] \arrow[ur, equal] &
\end{tikzcd}$$

+ associative :

$$\begin{tikzcd}
 & A\oplus A \arrow[r, "(f+g)\oplus h"] \arrow[d, "{(1,1)\oplus 1}"] & B\oplus B \arrow[ddr, bend left] & \\
A \arrow[ur] \arrow[r] \arrow[dr] \arrow[ddr] & (A\oplus A)\oplus A \arrow[r, "(f\oplus g)\oplus h"'] \arrow[d, "\cong"] & (B\oplus B)\oplus B \arrow[u, "{[1,1]\oplus 1}"'] \arrow[d, "\cong"] \arrow[dr] & \\
 & A\oplus(A\oplus A) \arrow[r, "f\oplus(g+h)"'] & B\oplus(B\oplus B) \arrow[r] \arrow[d, "{1\oplus[1,1]}"] & B \\
 & A\oplus A \arrow[u, "{1\oplus(1,1)}"'] \arrow[r, "f\oplus(g+h)"'] & B\oplus B \arrow[ur, bend right] &
\end{tikzcd}$$

+ DISTRIBUTIVE WITH RESPECT TO COMPOSITION:

Given $A \underset{g}{\overset{f}{\rightrightarrows}} B$ and $E \xrightarrow{e} A$, the commutativity of

$$\begin{tikzcd} A \arrow[r, "{(1,1)}"] & A\oplus A \arrow[r, "f\oplus g"] & B\oplus B \arrow[r, "{[1,1]}"] & B \\ E \arrow[u, "e"] \arrow[r, "{(1,1)}"'] & E\oplus E \arrow[u, "e\oplus e"'] \arrow[ur, "(f\cdot e)\oplus(g\cdot e)"'] & & \end{tikzcd}$$

shows that $(f+g)\cdot e = f\cdot e + g\cdot e$.

Similarly one checks the other distributivity.

+ HAS AN "OPPOSITE"

One checks that any identity 1_A has an opposite -1_A (using PROTOMODULARITY) and then $f\cdot(-1_A) = -f$

$\square$

COROLLARY 9

$\mathcal{C}$ ABELIAN $\Longleftrightarrow$ $\mathcal{C}$ SEMI-ABELIAN + $\mathcal{C}^{op}$ SEMI-ABELIAN

PROOF

It follows immediately from Tierney's result ABELIAN = EXACT + ADDITIVE and PROPOSITION 8

$\square$

Remark

Any SEMI-ABELIAN CATEGORY $\mathcal{C}$ contains a full subcategory $Ab(\mathcal{C})$ of ABELIAN OBJECTS, which is a full reflective subcategory of $\mathcal{C}$:

$$\begin{tikzcd} Ab(\mathcal{C}) \arrow[r, shift right, hook, "U"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift right, "ab"'] \end{tikzcd} \qquad (*)$$

the left adjoint $ab : \mathcal{C} \longrightarrow Ab(\mathcal{C})$ sends an element $X \in \mathcal{C}$ to the quotient by the largest commutator:

$$X \xrightarrow[\eta_X]{} X/[X,X]$$

$Ab(\mathcal{C})$ is an ABELIAN CATEGORY, and the adjunction $(*)$ has been studied in relationship with the categorical theory of CENTRAL EXTENSIONS (Bourn-Gran, 2002, Gran - Van der Linden, 2008)

TORSION THEORIES

Originally defined by DICKSON (1966) in the realm of ABELIAN CATEGORIES, torsion theories have been recently investigated in a NON-ADDITIVE CONTEXT:

DEFINITION

A pair of full (replete) subcategories $(\mathcal{T}, \mathcal{F})$ of a HOMOLOGICAL CATEGORY $\mathcal{C}$ is a TORSION THEORY if

1) $\forall x \in \mathcal{C}$ $\exists$ a short exact sequence

$$0 \longrightarrow T(x) \xrightarrow{\varepsilon_x} x \xrightarrow{\eta_x} F(x) \longrightarrow 0$$

with $T(x) \in \mathcal{T}$, $F(x) \in \mathcal{F}$

2) $\mathcal{C}(T, F) = \{ 0_{T,F} \}$ $\quad \forall T \in \mathcal{T}$, $\forall F \in \mathcal{F}$

$$\begin{tikzcd} T \arrow[rr] \arrow[dr, dashed] & & F \\ & 0 \arrow[ur, dashed] & \end{tikzcd}$$

EXAMPLES

1) Any torsion theory $(\mathcal{T}, \mathcal{F})$ in an ABELIAN CATEGORY. In particular, $(Ab_{t.}, Ab_{t.f})$, or (Ab_{DIV}, Ab_{RED}) in Ab.

2) Let X-MOD be the category of CROSSED MODULES, NormMono its full subcategory of NORMAL MONOS, Ab the full subcategory whose objects are

$$\begin{array}{c} A \in Ab \\ \downarrow \\ 0 \end{array}$$

Then, for any crossed module $A \xrightarrow{\alpha} B$ one has an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \pi_1(\alpha) \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\alpha"] & \alpha(A) \arrow[r] \arrow[d, tail] & 0 \\
0 \arrow[r] & 0 \arrow[r] & B \arrow[r, equal] & B \arrow[r] & 0
\end{tikzcd}$$

$\in Ab$ $\qquad\qquad \in$ NormMono

Moreover, any arrow

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d] & N \arrow[d, tail, "n"] \\
0 \arrow[r, "f_0"'] & B
\end{tikzcd}$$

is the 0-ARROW, since $n : N \rightarrowtail B$ is INJECTIVE

3) In X-Mod consider the full subcategories

$\mathcal{T}$ = CExt whose objects are CENTRAL EXTENSIONS

$\mathcal{F}$ = Grp whose objects are $0 \longrightarrow G$, with $G \in$ Grp.

Then any crossed module $A \xrightarrow{\alpha} B$ yields an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, equal] \arrow[d] & A \arrow[r] \arrow[d, "\alpha"] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underset{\in \text{ CExt}}{\alpha(A)} \arrow[r] & B \arrow[r] & \underset{\in \text{ Grp}}{B/\alpha(A)} \arrow[r] & 0
\end{tikzcd}$$

Furthermore, an arrow

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "f"'] & 0 \arrow[d] \\
\underset{\in \text{ CExt}}{B} \arrow[r, "f_0"'] & G \in \text{ Grp}
\end{tikzcd}$$

is necessarily the 0-ARROW, since f is SURJECTIVE

MORE GENERALLY: $\mathcal{C}$ HOMOLOGICAL $\Rightarrow$ Grpd($\mathcal{C}$) HOMOL.

(ConnGrpd($\mathcal{C}$), Dis($\mathcal{C}$)) is a TORSION THEORY in Grpd($\mathcal{C}$)

4) Let CRng be the category of COMMUTATIVE RINGS

NilCRng its subcategory of NILPOTENT RINGS

RedCRng " " " REDUCED RINGS

The pair (NilCRng, RedCRng) is a TORSION THEORY in CRng.

Indeed, given a ring A, let $Nil(A) = \{a \in A \mid \exists n \in \mathbb{N}^* \; a^n = 0\}$.

It is an IDEAL of A, and

$$0 \longrightarrow \underset{\in NilCRng}{Nil(A)} \rightarrowtail A \longrightarrow \underset{\in RedCRng}{A/Nil(A)} \longrightarrow 0$$

One easily sees that, for any arrow

$$\begin{tikzcd} N \arrow[rr, "f"] \arrow[dr] & & R \\ & 0 \arrow[ur] & \end{tikzcd}$$

from a nilpotent ring N to a reduced R,

$$f(x) = 0 \quad \forall x \in N$$

5) Let GRP(TOP) be the category of TOPOLOGICAL GROUPS,
GRP(IND) the category of INDISCRETE GROUPS,
GRP(HAUS) the category of HAUSDORFF GROUPS.

Then (GRP(IND), GRP(HAUS)) is a torsion theory in GRP(TOP).
Indeed, if we denote by $\overline{\{0\}}_G$ the closure of $\{0\}$ in a topological group G, one has an exact sequence in GRP(TOP):

$$0 \longrightarrow \overline{\{0\}}_G \rightarrowtail G \longrightarrow G/\overline{\{0\}}_G \longrightarrow 0$$

Furthermore, if $I \in$ GRP(IND) and $H \in$ GRP(HAUS)

$$\begin{tikzcd} I \arrow[rr, "f"] \arrow[dr] & & H \\ & 0 \arrow[ur] & \end{tikzcd}$$

This follows from the fact that $\overline{\{0\}}_H = \{0\}$ and $f^{-1}\left(\overline{\{0\}}_H\right) = \overline{\{0\}}_I = I$.

6) In the category GRP(COMP) of compact Hausdorff groups, the pair (GRP(CONN), GRP(PROF)) is a torsion theory. Here GRP(CONN) is the category of <u>connected</u> compact groups, GRP(PROF) the category of <u>profinite</u> groups.

7) In the category $\mathbf{Grp}$ consider the full subcategories

$\mathbf{Perf}$ of PERFECT GROUPS : $G \in \mathbf{Grp}$ SUCH THAT $G=[G,G]$

$\mathbf{HypoAb}$ of HYPOABELIAN GROUPS : these are the groups whose largest PERFECT SUBGROUP is TRIVIAL.

Then $(\mathbf{Perf}, \mathbf{HypoAb})$ is a torsion theory in $\mathbf{Grp}$.

Another torsion theory in $\mathbf{Grp}$ is given by $(\mathbf{Grp}_{T.}, \mathbf{Grp}_{T.F.})$, where $\mathbf{Grp}_{T.}$ is the category of TORSION GROUPS (= generated by torsion elements) and $\mathbf{Grp}_{T.F.}$ the category of TORSION-FREE GROUPS.

THEOREM (10)

Let $(\mathcal{T}, \mathcal{F})$ be a TORSION THEORY in a HOMOLOGICAL CATEGORY $\mathcal{C}$. Then the TORSION-FREE subcategory $\mathcal{F}$ is reflective in $\mathcal{C}$:

$$\begin{tikzcd} \mathcal{F} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend right, "F"'] \end{tikzcd} \qquad (*)$$

PROOF:

Let us show that the canonical quotient

$$X \xrightarrow[\eta_X]{} F(X) = X/T(X)$$

is the UNIT of the adjunction $(*)$.

Given an object $F \in \mathcal{F}$ and an arrow $X \xrightarrow{f} F$ one gets the diagram

$$\begin{tikzcd} 0 \arrow[r] & T(X) \arrow[r, tail] \arrow[dr, dashed] & X \arrow[r, "\eta_X"] \arrow[ddr, "f"] & F(X) \arrow[r] \arrow[dd, dashed, "\exists!\,\hat{f}"] & 0 \\ & & 0 \arrow[dr, dashed] & & \\ & & & F & \end{tikzcd}$$

and a UNIQUE $\hat{f}$ such that $\hat{f} \cdot \eta_X = f$.

$\square$

Any torsionfree subcategory $\mathcal{F}$ of a HOMOLOGICAL CATEGORY $\mathcal{C}$ is then a (NORMAL EPI) - REFLECTION :

$$\begin{tikzcd} \mathcal{F} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend right, "F"'] \end{tikzcd}$$

FULL, REFLECTIVE,

$\forall X \in \mathcal{C}$ $\quad X \xrightarrow{\eta_X} UF(X)$

IS A NORMAL EPI

We are going to prove that torsion theories correspond to a special kind of (NORMAL EPI)-REFLECTION, namely those ones where the REFLECTOR $F : \mathcal{C} \to \mathcal{F}$ is SEMI-LEFT-EXACT (CASSIDY, HEBERT, KELLY, 1985)

DEFINITION

A full reflective subcategory $\mathcal{F}$ of a finitely complete category $\mathcal{C}$

$$\begin{tikzcd} \mathcal{F} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend right, "F"'] \end{tikzcd}$$

has a SEMI-LEFT-EXACT reflector if F preserves the pullbacks of the form

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, "\eta_Y"] \\ UX \arrow[r, "Uf"'] & UY \end{tikzcd}$$

where η_Y is the Y-component of the UNIT, and f lies in $\mathcal{F}$.

THEOREM (11) Let $\mathcal{C}$ be a HOMOLOGICAL CATEGORY, $\mathcal{F}$ a full replete subcategory of $\mathcal{C}$. TFCAE:

1) $\mathcal{F}$ is a TORSION-FREE subcategory of $\mathcal{C}$;

2) $\mathcal{F}$ is (NORMAL EPI)-REFLECTIVE in $\mathcal{C}$ and the induced radical T is IDEMPOTENT: $\forall X \in \mathcal{C}$, $T(T(X)) \cong T(X)$;

3) $\mathcal{F}$ is (NORMAL EPI)-REFLECTIVE in $\mathcal{C}$ and the reflector $F : \mathcal{C} \longrightarrow \mathcal{F}$ is SEMI-LEFT EXACT.

PROOF

1) $\Rightarrow$ 2) When $(\mathcal{T}, \mathcal{F})$ is a torsion theory we have observed that $\eta_X : X \longrightarrow X/T(X)$ is the unit. Accordingly, $\mathcal{F}$ is (NORMAL EPI)-REFLECTIVE in $\mathcal{C}$.

Moreover,
$$\mathcal{T} = \{ X \in \mathcal{C} \mid F(X) = X/T(X) = 0 \}$$

since the unit
$$\begin{tikzcd} T(X) \arrow[rr, "\eta_{TX}"] \arrow[dr] & & F(T(X)) \\ & 0 \arrow[ur, "\cong"'] & \end{tikzcd}$$
is a NORMAL EPI.

It follows that $\epsilon_{T(X)} : T(T(X)) \longrightarrow T(X)$ is an ISO.

2) $\Rightarrow$ 3)

Consider the PULLBACK
$$\begin{tikzcd} T(P) \arrow[dr] \arrow[rr, bend left] & K[P_1] \arrow[r, "\varphi", "\cong"'] \arrow[d, "K"'] & T(Y) \arrow[d, "\epsilon_Y"] \arrow[dl, dashed, "K \cdot \varphi^{-1}"'] \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dl, "\eta_P"'] \arrow[dr, phantom, "1."] & Y \arrow[d, "\eta_Y"] \\ UF(P) \arrow[rr, bend right] & U(X) \arrow[l, dashed] \arrow[r, "Uf"'] & UF(Y) \end{tikzcd}$$

$K \cdot \varphi^{-1} = Ker(P_1)$

The IDEMPOTENCY of T implies

$T(Y) = T(T(Y)) \leq T(P)$

(AS SUBOBJECTS OF P)

$\Rightarrow \eta_P$ FACTORS THROUGH P_1

Obviously p_1 FACTORS THROUGH η_P :

$$\begin{tikzcd}
 & P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dl, "\eta_P"'] \arrow[dr, phantom, "1."] & Y \arrow[d, "\eta_Y"] \\
UF(P) \arrow[r, dashed, shift left] \arrow[rr, bend right, ""] & U(X) \arrow[r, "Uf"'] \arrow[l, dashed, shift left] & UF(Y)
\end{tikzcd}$$

Accordingly, $p_1 \cong \eta_P$, and the pullback 1. is PRESERVED

3) ⇒ 2) Assume that the reflector $F : \mathcal{C} \longrightarrow \mathcal{F}$ is SEMI-LEFT EXACT, so that the PULLBACK 2.

$$\begin{tikzcd}
T(x) \arrow[r, "\epsilon_x"] \arrow[d] \arrow[dr, phantom, "2."] & x \arrow[d, "\eta_x"] \\
0 \arrow[r] & UF(x)
\end{tikzcd}$$

is PRESERVED by $F : \mathcal{C} \longrightarrow \mathcal{F}$. It follows that

$F(T(x)) \cong 0$, and $TT(x) \xrightarrow{\epsilon_{T(x)}} T(x)$ is an iso.

2) ⇒ 1)

One defines $\mathcal{T} = \{ Y \in \mathcal{C} \mid Y \cong T(x) \text{, for an } x \in \mathcal{C} \}$

and checks that $(\mathcal{T}, \mathcal{F})$ is a TORSION THEORY

DEFINITION (MANTOVANI, 1998)

A full reflective subcategory $\mathcal{F}$ of a finitely complete category $\mathcal{C}$

$$\begin{tikzcd} \mathcal{F} \arrow[r, bend right, hook, "U"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend right, "F"'] \end{tikzcd}$$

is called a SEMI-LOCALISATION of $\mathcal{C}$ if $F: \mathcal{C} \to \mathcal{F}$ is SEMI-LEFT-EXACT.

REMARK

The previous Theorem says in particular that the torsion-free subcategories of a homological category are precisely the SEMI-LOCALISATIONS with the property that the unit of the adjunction is a normal epi.

TERMINOLOGY

In category theory one calls LOCALISATION a full reflective subcategory

$$\begin{tikzcd} \mathcal{F} \arrow[r, bend right, hook, "U"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend right, "F"'] \end{tikzcd}$$

where the reflector $F: \mathcal{C} \to \mathcal{F}$ preserves FINITE LIMITS.

Examples of semi-localisations

$$\begin{tikzcd} \mathsf{Ab}_{t.f} \arrow[r, hook, shift right=1.5, "U"'] \arrow[r, phantom, "\perp"] & \mathsf{Ab} \arrow[l, shift right=1.5, "F"'] \end{tikzcd}$$

$$A \xrightarrow{\eta_A} A/T(A),$$

where $T(A) = \{a \in A \mid \exists m \in \mathbb{N}^*,\ ma = 0\}$

$$\begin{tikzcd} \mathsf{NormMono} \arrow[r, hook, shift right=1.5, "U"'] \arrow[r, phantom, "\perp"] & X\text{-}\mathsf{Mod} \arrow[l, shift right=1.5, "F"'] \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "\eta^1_\alpha"] \arrow[d, "\alpha"] & \alpha(A) \arrow[d, tail] \\ B \arrow[r, equal, "\eta^0_\alpha"'] & B \end{tikzcd}$$

$$\begin{tikzcd} \mathsf{Dis}(\mathcal{C}) \arrow[r, hook, shift right=1.5, "U"'] \arrow[r, phantom, "\perp"] & \mathsf{Grpd}(\mathcal{C}) \arrow[l, shift right=1.5, "\pi_0"'] \end{tikzcd}$$

$$\begin{tikzcd} X_1 \arrow[r, "\eta^0_{\mathbb{X}}"] \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] & \pi_0(\mathbb{X}) \arrow[d, shift right=2, "1"'] \arrow[d, shift left=2, "1"] \\ X_0 \arrow[r, "\eta^0_{\mathbb{X}}"'] \arrow[u] & \pi_0(\mathbb{X}) \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd} \mathsf{RedCRng} \arrow[r, hook, shift right=1.5, "U"'] \arrow[r, phantom, "\perp"] & \mathsf{CRng} \arrow[l, shift right=1.5, "F"'] \end{tikzcd}$$

$$A \xrightarrow{\eta_A} A/\mathrm{Nil}(A)$$

where $\mathrm{Nil}(A) = \{a \in A \mid \exists m \in \mathbb{N}^*,\ a^m = 0\}$

$$\begin{tikzcd} \mathsf{Grp}(\mathsf{Haus}) \arrow[r, hook, shift right=1.5, "U"'] \arrow[r, phantom, "\perp"] & \mathsf{Grp}(\mathsf{Top}) \arrow[l, shift right=1.5, "F"'] \end{tikzcd}$$

$$G \xrightarrow{\eta_G} G/\overline{\{0\}}_G$$

$\overline{\{0\}}$ is the closure of 0 in G

$$\begin{tikzcd} \mathsf{Grp}(\mathsf{TotDis}) \arrow[r, hook, shift right=1.5, "V"'] \arrow[r, phantom, "\perp"] & \mathsf{Grp}(\mathsf{Haus}) \arrow[l, shift right=1.5, "G"'] \end{tikzcd}$$

$$H \xrightarrow{\eta_H} H/\Gamma_0(H)$$

where $\Gamma_0(H) =$ connected component of 0

SEMI-LOCALISATIONS AND CLOSURE OPERATORS

DEFINITION (BOURN-GRAN, 2006)

- An IDEMPOTENT CLOSURE OPERATOR on NORMAL MONOS consists in giving, for any normal mono $S \overset{s}{\rightarrowtail} X$, another normal mono $\bar{S} \overset{\bar{s}}{\rightarrowtail} X$ in such a way that:

1) $S \subset \bar{S}$.
2) $S \subset T \Rightarrow \bar{S} \subset \bar{T}$
3) $\overline{f^{-1}(S)} \subset f^{-1}(\bar{S})$, $\forall\, Y \xrightarrow{f} X$.
4) $\bar{\bar{S}} = \bar{S}$

- Such an operator is called HOMOLOGICAL if, moreover,

5) $\overline{f^{-1}(S)} = f^{-1}(\bar{S})$ $\forall$ NORMAL EPI $f : Y \twoheadrightarrow X$

We are going to show that, in a HOMOLOGICAL CATEGORY $\mathcal{C}$, there is a bijection between

HOMOLOGICAL CLOSURE OPERATORS $\Longleftrightarrow$ (NORMAL EPI)-REFLECTIVE SUBCATEGORIES OF $\mathcal{C}$

LEMMA (12)

Given a HOMOLOGICAL CLOSURE OPERATOR on normal monos in a HOMOLOGICAL CATEGORY $\mathcal{C}$, the following are equivalent:

$$\begin{tikzcd}
f^{-1}(S) \arrow[r, tail, "f^{-1}(s)"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
S \arrow[r, tail, "s"'] & Y
\end{tikzcd}$$

f NORMAL EPI

1. s is CLOSED $\iff$ $f^{-1}(s)$ is CLOSED
2. s is DENSE $\iff$ $f^{-1}(s)$ is DENSE

PROOF 1. If $s : S \longrightarrow Y$ is closed, then

$$\overline{f^{-1}(S)} \overset{(5)}{=} f^{-1}(\overline{S}) = f^{-1}(S),$$

showing that $f^{-1}(s)$ is closed.

Conversely, assume that $f^{-1}(s)$ is closed and consider

$$\begin{tikzcd}
f^{-1}(S) \arrow[rrr, tail, "f^{-1}(s)"] \arrow[dr, "i_{f^{-1}(S)}"', "\cong"] \arrow[dd, "g"'] & & & X \arrow[dd, "f"] \\
& \overline{f^{-1}(S)} \arrow[urr, "\overline{f^{-1}(s)}"'] \arrow[dd, "\phi"] & & \\
S \arrow[rrr, tail, "s"] \arrow[dr, tail, "i_S"'] & & & Y \\
& \overline{S} \arrow[urr, tail, "\overline{s}"'] & &
\end{tikzcd}$$

The assumption says that $i_{f^{-1}(S)}$ is an iso, and (5) that ϕ is a regular epi. Then $i_S \circ g = \phi \cdot i_{f^{-1}(S)}$ is a REGULAR EPI $\Rightarrow$ i_S IS AN ISO. □

THEOREM (13) (Bourn - Gran, 2006)

Let $\mathcal{E}$ be a HOMOLOGICAL CATEGORY. There is a BIJECTION between :

1) (NORMAL-EPI) - REFLECTIVE SUBCATEGORIES OF $\mathcal{E}$
2) HOMOLOGICAL CLOSURE OPERATORS

PROOF Given a homological closure operator $\overline{(\ )}$ one defines $\mathcal{F} = \{ F \in \mathcal{E} \mid 0 \longrightarrow F \text{ is closed} \}$.

The category $\mathcal{F}$ is (NORMAL EPI) - REFLECTIVE : the square

$$\begin{tikzcd}
\overline{0}_X \arrow[r, "k_X"] \arrow[d] & X \arrow[d, "\eta_X = \mathrm{coker}(k_X)"] \\
0 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X/\overline{0}_X
\end{tikzcd}$$

is a PUSHOUT and a PULLBACK. By Lemma 12 we deduce that $0 \rightarrowtail X/\overline{0}_X$ is closed (since $\overline{0}_X \rightarrowtail X$ is so).

Accordingly, $X/\overline{0}_X \in \mathcal{F}$. Let us check the UNIVERSAL PROPERTY

$$\begin{tikzcd}
\overline{0}_X \arrow[r, tail, "k_X"] \arrow[d, dashed] & X \arrow[r, "\eta_X"] \arrow[d, "\forall f"] & X/\overline{0}_X \\
0 \arrow[r, dashed] & F \in \mathcal{F} &
\end{tikzcd}$$

Given $X \xrightarrow{f} F \in \mathcal{F}$, $\overline{0}_F = 0$, and then $\overline{0}_X \underset{(2)}{\subset} \overline{f^{-1}(0)} \underset{(3)}{\subset} f^{-1}(\overline{0}_F)$

gives $\overline{0}_X \subset f^{-1}(0_F) = \mathrm{Ker}(f)$

$$\Rightarrow \exists !\ F(f) : X/\overline{0}_X \longrightarrow F \text{ s.t. } F(f) \cdot \eta_X = f.$$

This shows that $\mathcal{F} \underset{U}{\overset{F}{\leftrightarrows}} \mathcal{E}$ ($\perp$) is (NORMAL EPI) - REFLECTIVE

Conversely, starting from a (normal epi)-reflective subcategory $\mathcal{X}$ of $\mathcal{C}$

$$\begin{tikzcd} \mathcal{X} \arrow[r, shift right, "U"'] & \mathcal{C} \arrow[l, shift right, "F"'] \arrow[l, phantom, "\perp"] \end{tikzcd}$$

one defines the closure $\bar{S} \rightarrowtail X$ of a normal mono $S \rightarrowtail X$ in $\mathcal{C}$ as follows:

$$\begin{tikzcd} & \bar{S} \arrow[d, tail, "\bar{s}"] & \\ S \arrow[ur, dashed, "i_S"] \arrow[r, tail, "s"] & X \arrow[r, "q"] \arrow[dr, "\eta_{X/S} \cdot q"'] & X/S \arrow[d, "\eta_{X/S}"] \\ & & UF(X/S) \end{tikzcd}$$

$$\boxed{\bar{s} = \mathrm{Ker}\left(\eta_{X/S} \cdot q\right)}$$

(1) is valid by construction,

(4) $\bar{\bar{S}} = \bar{S}$ holds because any normal epi is the cokernel of its kernel

(2) $S \subset T \Rightarrow \bar{S} \subset \bar{T}$ can be checked by looking at

$$\begin{tikzcd} & & \bar{T} \arrow[d] \arrow[dl, dashed, "\in"'] & & \\ & T \arrow[ur] \arrow[r, tail] & X \arrow[rr, "q_T"] \arrow[rrd] \arrow[dd, equal] & & X/T \arrow[d, "\eta_{X/T}"] \\ & \bar{S} \arrow[dr] & & & UF(X/T) \\ S \arrow[uur, "j"] \arrow[ur] \arrow[rr, tail, "s"'] & & X \arrow[r, "q_S"'] \arrow[dr] & X/S \arrow[uur] \arrow[d, "\eta_{X/S}"] & \\ & & & UF(X/S) \arrow[uur] & \end{tikzcd}$$

(3) and (5) are left as an EXERCICE. $\square$

DEFINITION

A homological closure operator is WEAKLY HEREDITARY if the canonical inclusions

$$\begin{tikzcd} S \arrow[rr, "i_S"] \arrow[dr, "s"'] & & \bar{S}_X \arrow[dl, "\bar{s}"] \\ & X & \end{tikzcd}$$

are DENSE : $\overline{S}_{\bar{S}_X} = \bar{S}_X$

THEOREM (14)

In a HOMOLOGICAL CATEGORY $\mathcal{C}$ there is a bijection between : 1) TORSION THEORIES in $\mathcal{C}$

2) WEAKLY HEREDITARY CLOSURE OPERATORS

PROOF : It suffices to check that, in the bijection of THEOREM (13),

$\overline{(\ )}$ is WEAKLY HEREDITARY $\Leftrightarrow$ the reflector $F : \mathcal{C} \to \mathcal{F}$ is SEMI-LEFT-EXACT

$\Rightarrow$) When $\overline{(\ )}$ is WEAKLY HEREDITARY, the induced radical $T(X) \rightarrowtail X$ associated with $F : \mathcal{C} \to \mathcal{F}$ is IDEMPOTENT : indeed, $\bar{O}_X = T(X)$ and

$$\begin{tikzcd} O \arrow[rr, "i_O"] \arrow[dr, "o"'] & & \bar{O}_X \arrow[dl, "\bar{o}"] \\ & X & \end{tikzcd}$$

is DENSE,

so that

$$T(T(X)) = \overline{O}_{\bar{O}_X} = \bar{O}_X = T(X).$$

$\Leftarrow$) Conversely, when the reflector $F: \mathcal{C} \to \mathcal{F}$ is SEMI-LEFT-EXACT, one has $T(T(X)) \simeq T(X)$.

Given any NORMAL MONO $S \overset{s}{\rightarrowtail} X$, form the diagram

$$\begin{tikzcd}
S \arrow[r, "i_S"] \arrow[d] \arrow[rr, bend left, "s"] \arrow[dr, phantom, "1."] & \overline{S}_X \arrow[r, "\overline{s}"] \arrow[d] \arrow[dr, phantom, "2."] & X \arrow[d, "q"] \\
0 \arrow[r, "i_0"'] & \overline{0}_{X/S} \arrow[r] & X/S
\end{tikzcd}$$

and observe that 2. is a PULLBACK since

$$q^{-1}(\overline{0}_{X/S}) \overset{(5)}{=} \overline{q^{-1}(0)}_X = \overline{S}_X$$

It follows that 1. is a PULLBACK and is DENSE, since $0 \xrightarrow[i_0]{} \overline{0}_{X/S}$ is DENSE (see LEMMA 12)

$\square$

EXERCISE

A torsion theory $(\mathcal{T}, \mathcal{F})$ in $\mathcal{C}$ is HEREDITARY when the subcategory $\mathcal{T}$ is stable under SUBOBJECTS:

[if $M \overset{m}{\rightarrowtail} X$ is a MONO and $X \in \mathcal{T}$] $\Rightarrow$ [$M \in \mathcal{T}$]

PROVE THAT: $(\mathcal{T}, \mathcal{F})$ is HEREDITARY

$$\Updownarrow$$

$F: \mathcal{C} \to \mathcal{F}$ preserves MONOMORPHISMS

EXAMPLES

1) Consider the reflection

$$\begin{tikzcd} \mathsf{Grp}(\mathsf{Haus}) \arrow[r, shift right, hook, "U"'] \arrow[r, phantom, "\perp"] & \mathsf{Grp}(\mathsf{Top}) \arrow[l, shift right, "F"'] \end{tikzcd}$$

By definition of $\overline{(\ )}^{\mathrm{Hom}}$ one has

$$\overline{\{0\}}^{\mathrm{Hom}} = \overline{\{0\}}^{\mathrm{Top}}$$

since $\forall G \in \mathsf{Grp}(\mathsf{Top})$

$$\begin{tikzcd}
& \overline{\{0\}}^{\mathrm{Hom}} \simeq \overline{\{0\}}^{\mathrm{Top}} \arrow[d, "\bar{0}"] & \\
0 \arrow[ur] \arrow[r, "0"'] & G \arrow[r, equal] \arrow[dr, bend right] & G \arrow[d, "\eta_G"] \\
& & G/\overline{\{0\}}^{\mathrm{Top}}
\end{tikzcd}$$

Let $S \overset{s}{\rightarrowtail} X$ be any NORMAL SUBGROUP of $X \in \mathsf{Grp}(\mathsf{Top})$

Then:

$$\begin{tikzcd}
& \overline{S}^{\mathrm{Hom}} \arrow[r, "w"] \arrow[d, "\bar{s}"] \arrow[dr, phantom, "\lrcorner", very near start] & \overline{\{0\}}^{\mathrm{Top}} \arrow[d] \\
S \arrow[ur, "i_S"] \arrow[r, "s"'] & X \arrow[r, "q"'] \arrow[dr, bend right] & X/S \arrow[d] \\
& & (X/S)/\overline{\{0\}}^{\mathrm{Top}}
\end{tikzcd}$$

One gets an exact sequence

$$0 \longrightarrow S \xrightarrow{i_S} \overline{S}^{\mathrm{Hom}} \xrightarrow{w} \overline{\{0\}}^{\mathrm{Top}} \longrightarrow 0$$

S is closed for $\overline{(\ )}^{\mathrm{Hom}}$ $\iff$ $i_S : S \longrightarrow \overline{S}^{\mathrm{Hom}}$ is an iso

$\iff$ $\overline{\{0\}}^{\mathrm{Top}} = \{0\}$

$\iff$ $X/S \in \mathsf{Grp}(\mathsf{Haus})$

$\iff$ $S = \overline{S}^{\mathrm{Top}}$

CONCLUSION: $\overline{(\ )}^{\mathrm{Hom}} = \overline{(\ )}^{\mathrm{Top}}$

2)

$$\mathrm{GRP}(\mathrm{PROF}) \underset{U}{\overset{F}{\rightleftarrows}} \mathrm{GRP}(\mathrm{COMP}) \qquad F \dashv U$$

The reflection of a compact group G is

$$G \xrightarrow[\eta_G]{} G/\Gamma_0(G)$$

The closure of a NORMAL SUBGROUP $S \rightarrowtail X$ is given by

$$\overline{S} = S \cdot \Gamma_0(X)$$
$$= \{ s \cdot x \mid s \in S,\ x \in \Gamma_0(X) \}$$

3)

$$\mathrm{RED\,CRNG} \underset{U}{\overset{F}{\rightleftarrows}} \mathrm{CRNG} \qquad F \dashv U$$

Given $S \rightarrowtail A$ an ideal one computes its closure

$$\begin{tikzcd} & \overline{S} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Nil}(A/S) = \overline{0}_{A/S} \arrow[d, tail] \\ S \arrow[ur] \arrow[r, tail, "s"'] & A \arrow[r, "\pi"'] & A/S \end{tikzcd}$$

$$\overline{S} = \pi^{-1}\left(\mathrm{Nil}(A/S)\right)$$
$$= \{ a \in A \mid \exists\, n \in \mathbb{N}^* \text{ WITH } \overline{a^n} = \overline{0}_{A/S} \}$$
$$= \{ a \in A \mid \exists\, n \in \mathbb{N}^* \text{ WITH } a^n \in S \}$$
$$= \sqrt{S}\text{, the radical of } S$$

4)

$$\begin{tikzcd} Ab_{t.f.} \arrow[r, bend right, "U"', hook] & Ab \arrow[l, bend right, "F"'] \arrow[l, phantom, "\perp"] \end{tikzcd}$$

Here, for an $A \in Ab$, $\overline{0}_A = \{a \in A \mid \exists m \in \mathbb{N}^*, ma = 0\}$

Given a subgroup $K \rightarrowtail A$,

$$\overline{K} = \{a \in A \mid \exists m \in \mathbb{N}^* \text{ s.t. } ma \in K\}$$

5)

$$\begin{tikzcd} \text{NormMono} \arrow[r, bend right, "U"', hook] & X\text{-Mod} \arrow[l, bend right, "F"'] \arrow[l, phantom, "\perp"] \end{tikzcd}$$

If $A \xrightarrow{\alpha} B \in X\text{-Mod}$ then the unit η_α is given by

$$\begin{tikzcd} A \arrow[r, "\eta_\alpha"] \arrow[d, "\alpha"'] & \alpha(A) \arrow[d, tail] \\ B \arrow[r, equal] & B \end{tikzcd}$$

One can check that the closure of a normal mono

$$\begin{tikzcd} A \arrow[r, tail, "m_A"] \arrow[d, "\alpha"'] & A' \arrow[d, "\alpha'"] \\ B \arrow[r, tail, "m_B"'] & B' \end{tikzcd}$$

is obtained by the PULLBACK

$$\begin{tikzcd} A \arrow[dr, dashed] \arrow[drr, bend left, "m_A"] \arrow[ddr, bend right, "\alpha"'] & & \\ & P \arrow[r, tail, "\overline{m}_A"'] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[d, "\alpha'"] \\ & B \arrow[r, tail, "m_B"'] & B' \end{tikzcd}$$

SEMI-LOCALISATIONS OF SEMI-ABELIAN CATEGORIES

Torsion-free subcategories of an abelian category are not abelian, in general.

Indeed, the classical torsion-free subcategory

$$\begin{tikzcd} Ab_{t.f.} \arrow[r, shift left, "U"] \arrow[r, phantom, "\perp"] & Ab \arrow[l, shift left, "F"'] \end{tikzcd}$$

is not EXACT, since one can find EQUIVALENCE RELATIONS which are not EFFECTIVE:

let $R_2 \overset{p_1}{\underset{p_2}{\rightrightarrows}} \mathbb{Z}$ be the equivalence relation on $\mathbb{Z}$ defined by

$$(m, n) \in R_2$$
$$\Updownarrow$$
$$\exists k \in \mathbb{Z} \text{ s.t. } m - n = 2k$$

This equivalence relation $R_2 \rightrightarrows \mathbb{Z}$ can not be the KERNEL PAIR of its coequaliser:

$$R_2 \overset{p_1}{\underset{p_2}{\rightrightarrows}} \mathbb{Z} \xrightarrow[q]{} \mathbb{Z}/R_2 \cong 0 \text{ in } Ab_{t.f.}\,!$$

Although $Ab_{t.f.}$ is REGULAR and ADDITIVE, it is not EXACT $\Longrightarrow$ not ABELIAN!

A torsion-free subcategory of an abelian $\mathcal{C}$ inherits some interesting exactness properties:

<u>THEOREM</u> (RUMP, 2001)

For a category $\mathcal{F}$ the following are equivalent

1) $\mathcal{F}$ is a torsion-free subcategory of an abelian category $\mathcal{C}$.

2) a) $\mathcal{F}$ is additive

b) any morphism in $\mathcal{F}$ has a factorisation

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "q"'] & D \\ B \arrow[r, "g"'] & C \arrow[u, "k"'] \end{tikzcd}$$

where q is a NORMAL EPI
g is a BIMORPHISM
k is a NORMAL MONO

c) NORMAL EPIS are pullback stable.

W. RUMP has shown that many categories of interest satisfy these conditions (Ab(TOP), Ab(HAUS), BAN, ETC.).

QUESTION

Can one find an intrinsic characterisation of torsion-free subcategories of a semi-abelian category?

REMARK

Any (normal epi)-reflective subcategory

$$\begin{tikzcd} \mathcal{F} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend right, "F"'] \end{tikzcd}$$

of a semi-abelian category $\mathcal{C}$ is

- HOMOLOGICAL
- WITH BINARY COPRODUCTS

QUESTION (2ND VERSION)

Among homological categories can one characterise the ones occuring as torsion-free subcategories of a semi-abelian category?

The answer to this question was recently found in collaboration with S. LACK, by using some remarkable results on semi-localisations of EXACT CATEGORIES due to S. MANTOVANI (1998).

DEFINITION

A COKERNEL $q : B \twoheadrightarrow Q$ of a mono $m : A \rightarrowtail B$ is STABLE if, given any $f : D \to Q$ as in

$$\begin{tikzcd} & & D \arrow[d, "f"] \\ A \arrow[r, tail, "m"'] & B \arrow[r, two heads, "q"'] & Q \end{tikzcd}$$

with pullback

$$\begin{tikzcd} & B \times_Q D \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[d, "f"] \\ A \arrow[r, tail, "m"'] \arrow[ur, dashed, "m'"] & B \arrow[r, two heads, "q"'] & Q \end{tikzcd}$$

one then has $\pi_2 = \mathrm{coker}(m')$

Theorem (Gran-Lack, 2014)

For a category $\mathcal{F}$ the following are equivalent:

a) $\mathcal{F}$ is a SEMI-LOCALISATION OF A SEMI-ABELIAN CATEGORY

b) $\mathcal{F}$ is a TORSION-FREE SUBCATEGORY OF A SEMI-ABELIAN CATEGORY

c) $\mathcal{F}$ is HOMOLOGICAL, has BINARY COPRODUCTS, and STABLE COKERNELS OF BOURN-NORMAL MONOMORPHISMS

d) $\mathcal{F}$ is HOMOLOGICAL, has BINARY COPRODUCTS, and STABLE COEQUALISERS OF EQUIVALENCE RELATIONS

Examples

All torsion-free subcategories of semi-abelian categories we have encountered so far.

Also $\mathrm{Grp(Top)}$, $\mathrm{Rng(Top)}$, $\mathrm{RedCRng}$, $\mathrm{Grp}_{T.F.}$, $\mathrm{NormMon}$ ...

Remark

The result above provides in particular a new proof of Rump's result.