SEMI-ABELIAN CATEGORIES, SEMI-LOCALISATIONS AND TORSION THEORIES

INTRODUCTION

1950 S. MAC LANE, DUALITY FOR GROUPS, BULL. AMER. MATH. SOCIETY

1955 D. BUCHSBAUM, EXACT CATEGORIES AND DUALITY, TRANS. AMER. MATH. SOCIETY

1957 A. GROTHENDIECK, SUR QUELQUES POINTS D'ALGEBRE HOMOLOGIQUE, TOHOKU MATH. J.

The notion of ABELIAN CATEGORY has become very important in HOMOLOGICAL ALGEBRA.

What can be said about the STRUCTURAL PROPERTIES of the NON-ABELIAN category GRP ?

Would it be possible to find a "4th proportional" in: Ab : ABELIAN CATEGORY = GRP : ?

AIM: FIND AN "AXIOMATIC CONTEXT" FOR

- ISOMORPHISM THEOREMS
- NON-ABELIAN HOMOLOGICAL ALGEBRA
- RADICAL AND TORSION THEORIES
- COMMUTATOR THEORY

SEVERAL PROPOSALS OF "NON-ABELIAN" CONTEXTS:

1954	AMITSUR	RADICAL THEORY
1959	KUROSH	
1956	HIGGINS	NON-ABELIAN HOMOLOGY ISOMORPHISM THEOREMS
1961	FRÖLICH	
1970	GERSTENHABER	
1971	WYLER	
1968	HUQ	COMMUTATOR THEORY

IN 2001 THE NOTION OF SEMI-ABELIAN CATEGORY WAS INTRODUCED BY G. JANELIDZE, L. MÁRKI AND W. THOLEN.

TERMINOLOGY:

$\mathcal{C}$ IS ABELIAN $\iff$ $\mathcal{C}$ IS SEMI-ABELIAN + $\mathcal{C}^{op}$ IS SEMI-ABELIAN

OUTLINE

I) REGULAR AND HOMOLOGICAL CATEGORIES

DEFINITIONS, EXAMPLES AND PROPERTIES

II) SEMI-ABELIAN CATEGORIES

- DEFINITION, EXAMPLES, RELATIONSHIP WITH ABELIAN CATEGORIES
- TORSION THEORIES, EXAMPLES, PROPERTIES
- REFLECTIVE SUBCATEGORIES
- CLOSURE OPERATORS

III) SEMI-LOCALISATIONS

- FACTORISATION SYSTEMS, SEMI-LEFT-EXACT REFLECTORS
- ABSTRACT CHARACTERISATION OF SEMI-LOCALISATIONS
- CATEGORICAL GALOIS THEORY

AIM OF THIS "MINI-COURSE":

1) EXPLAIN WHAT A SEMI-ABELIAN CATEGORY IS, AND STUDY SOME OF ITS BASIC PROPERTIES.

2) INTRODUCE THE NOTION OF TORSION THEORY IN THE SEMI-ABELIAN CONTEXT, AND EXAMINE SOME NEW NON-ABELIAN EXAMPLES.

3) RELATE TORSION THEORIES TO:

- SEMI-LEFT-EXACT REFLECTIONS
- CLOSURE OPERATORS
- FACTORISATION SYSTEMS.

REGULAR CATEGORIES

A finitely complete category $\mathcal{C}$ is **REGULAR** if

1) any arrow $f : A \longrightarrow B$ has a factorisation

$$
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, two heads, "p"'] & & B \\
& I \arrow[ur, tail, "i"'] &
\end{tikzcd}
$$

where p is a REGULAR EPI and i is a MONOMORPHISM.

2) these regular epi-mono factorisations are pullback-stable:

$$
\begin{tikzcd}
E \times_B A \arrow[rr, "\pi_2"] \arrow[dr, dashed, two heads, "\bar{p}"] \arrow[dd, "\pi_1"'] & & A \arrow[dr, two heads, "p"] \arrow[dd, "f"' near start] & \\
& E \times_B I \arrow[rr] \arrow[dl, tail, "\bar{i}"] & & I \arrow[dl, tail, "i"] \\
E \arrow[rr] & & B &
\end{tikzcd}
$$

EXAMPLES

SET

The regular epi / mono factorisation of a map $X \xrightarrow{f} Y$ is given by

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \\ & \mathrm{Im}(f) \arrow[ur, "i"'] & \end{tikzcd}$$

where $\mathrm{Im}(f) = \{ f(x) \mid x \in X \}$.

In SET: regular epis $\equiv$ surjective maps

Surjective maps are easily seen to be PULLBACK STABLE:

GRP category of groups

Ab " " abelian groups

Mon " " monoids

Lattices " " lattices

GRP(TOP) " " topological groups

GRP(COMP) " " compact groups

sSET, sGRP

COUNTER-EXAMPLES

TOP category of topological spaces

CAT category of categories

$\Big\}$ NOT REGULAR

RELATIONS

$$\begin{tikzcd}
 & & R \times_Y S \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & \\
 & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & & S \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & \\
X & & Y & & Z
\end{tikzcd}$$

The composite $S \circ R$ of S and R is given by the regular image of the unique arrow

$$\begin{tikzcd}
R \times_Y S \arrow[rr, "\langle r_1 \cdot \pi_1,\, s_2 \cdot \pi_2 \rangle"] \arrow[dr, two heads, "p"'] & & X \times Z \\
 & S \circ R \arrow[ur, tail, "i"'] &
\end{tikzcd}$$

Accordingly:

$$\begin{tikzcd}
 & S \circ R \arrow[d, tail, "i"] \arrow[dl, dashed, bend right] \arrow[dr, dashed, bend left] & \\
X & X \times Z \arrow[l, "p_1"] \arrow[r, "p_2"'] & Z
\end{tikzcd}$$

is a relation from X to Z.

SET

$$R \times_Y S = \{((x,y)\in R, (u,z)\in S) \mid r_2(x,y) = s_1(u,z)\}$$

$$\cong \{(x,y,z) \in X \times Y \times Z \mid (x,y)\in R, (y,z)\in S\}$$

$$S \circ R = \{(x,z) \in X \times Z \mid \exists\, (a,b,c) \in R \times_Y S \text{ WITH } a = x, c = z\}$$

$$= \{(x,z) \in X \times Z \mid \exists\, b \in Y \text{ with } (x,b)\in R, (b,z)\in S\}$$

What is a RELATION on a group $(X, \cdot, 1)$
in the category GRP ?

It is a diagram

$$\begin{tikzcd} R \arrow[r, tail, "(r_1, r_2)"] \arrow[dr, shift right, "r_1"'] \arrow[dr, shift left, "r_2"] & X \times X \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] \\ & X \end{tikzcd}$$

determining a relation on the underlying set of X,
with the property that R is a SUBGROUP of $X \times X$:

- $(1,1) \in R$
- if $(x,y) \in R \Rightarrow (x^{-1}, y^{-1}) \in R$
- if $\left.\begin{array}{l}(x,y) \in R \\ (u,v) \in R\end{array}\right\} \Rightarrow (xu, yv) \in R$

EXERCICE

Let $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X$ be a REFLEXIVE RELATION in GRP,
so that $r_1 \cdot e = 1_X = r_2 \cdot e$.

Prove that R is then SYMMETRIC and TRANSITIVE,
thus an EQUIVALENCE RELATION.

DEFINITION (CARBONI, LAMBEK, PEDICCHIO, 1991)

A category $\mathcal{C}$ with finite limits is a MAL'TSEV category if any reflexive relation in $\mathcal{C}$ is an equivalence relation.

EXAMPLES

- any ABELIAN CATEGORY
- GRP, RNG, LIE_K, GRP(TOP), GRP(COMP), BOOLE

REMARK

The categories SET, SSET, TOP, MON, LATTICES are <u>not</u> Mal'tsev categories

THEOREM (CARBONI, LAMBEK, PEDICCHIO)

For a regular category $\mathcal{C}$ the following are equivalent:

1) $\mathcal{C}$ is a Mal'tsev category
2) $R \circ S = S \circ R$ for any $R \in Eq_X(\mathcal{C})$, $S \in Eq_X(\mathcal{C})$
3) any reflexive relation in $\mathcal{C}$ is SYMMETRIC
4) any reflexive relation in $\mathcal{C}$ is TRANSITIVE

Given an arrow $f: A \longrightarrow B$ in $\mathcal{C}$ its **KERNEL PAIR** is the relation $(Eq(f), p_1, p_2)$ in the pullback

$$\begin{tikzcd} Eq(f) \arrow[r, "p_2"] \arrow[d, "p_1"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

SET

$$Eq(f) = \{ (a_1, a_2) \in A \times A \mid f(a_1) = f(a_2) \}$$

is the **EQUIVALENCE RELATION** on A obtained by identifying two elements in A when

$$f(a_1) = f(a_2).$$

DEFINITION

A regular category $\mathcal{C}$ is **EXACT** if any equivalence relation in $\mathcal{C}$ is **EFFECTIVE**, i.e. a **KERNEL PAIR**:

$$\begin{tikzcd} R \times_A R \arrow[r, shift left=2, "\pi_1"] \arrow[r, dashed] \arrow[r, shift right=2, "\pi_2"'] & R \arrow[loop above, "\sigma"] \arrow[r, shift left=2, "r_1"] \arrow[r, shift right=2, "r_2"'] & A \arrow[l, dashed, "e"'] \arrow[r, dashed, "\exists f"] & B \end{tikzcd}$$

such that $R \cong Eq(f)$

EXAMPLES

GRP, RNG, MON, LATTICES, BOOLE, ANY ABELIAN CATEGORY

HOMOLOGICAL CATEGORIES

DEFINITION (BORCEUX - BOURN, 2004)

A REGULAR CATEGORY $\mathcal{C}$ is HOMOLOGICAL if

1) $\mathcal{C}$ is POINTED : 0

2) $\mathcal{C}$ is PROTOMODULAR : the Split Short Five Lemma holds in $\mathcal{C}$, i.e. given

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, tail] \arrow[d, "u"'] & A \arrow[r, "f"'] \arrow[d, "v"'] & B \arrow[r] \arrow[l, bend right, "s"'] \arrow[d, "w"] & 0 \\
0 \arrow[r] & K' \arrow[r, tail] & A' \arrow[r, "f'"] & B' \arrow[r] \arrow[l, bend left, "s'"] & 0
\end{tikzcd}$$

u, w ISOS $\Rightarrow$ v ISO

EXAMPLES

- any ABELIAN CATEGORY

- GRP, RNG, LIE_K, GRP(TOP), GRP(COMP)

- X-MOD

OBJECTS $\quad A \xrightarrow{\alpha} B \qquad \alpha({}^{b}a) = b\,\alpha(a)\,b^{-1}$

$${}^{\alpha(a)}a_1 = a\,a_1\,a^{-1}$$

ARROWS

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, dashed, "f_1"'] & B \arrow[d, dashed, "f_0"] \\
A' \arrow[r, "\alpha'"'] & B'
\end{tikzcd}
\qquad f_1({}^{b}a) = {}^{f_0(b)}f_1(a)$$

X-MOD ≅ GRPD(GRP)

GRPD(GRP)

- <u>objects</u>: INTERNAL GROUPOIDS IN GROUPS
- <u>arrows</u>: INTERNAL FUNCTORS $(f_0, f_1) : \mathbb{X} \longrightarrow \mathbb{Y}$

$$
\begin{tikzcd}
X_1 \times_{X_0} X_1 \arrow[r, "m_X"] \arrow[d, dashed] & X_1 \arrow[loop above, "\sigma_X"] \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] \arrow[d, "f_1"] & X_0 \arrow[l, "e" description] \arrow[d, "f_0"] \\
Y_1 \times_{Y_0} Y_1 \arrow[r, "m_Y"'] & Y_1 \arrow[r, shift left=2, "\delta"] \arrow[r, shift right=2, "\gamma"'] & Y_0 \arrow[l, "\varepsilon" description]
\end{tikzcd}
$$

The equivalence is provided by the

NORMALISATION FUNCTOR:

given

$$
\begin{tikzcd}
\mathbb{X} : X_1 \times_{X_0} X_1 \arrow[r, shift left=3, "\pi_1"] \arrow[r, "m_X" description] \arrow[r, shift right=3, "\pi_2"'] & X_1 \arrow[loop above, "\sigma"] \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & X_0 \arrow[l, "e" description]
\end{tikzcd}
$$

one associates the crossed module

$$
K[d] \xrightarrow{\ker(d)} X_1 \xrightarrow{\ c\ } X_0
$$

where $\quad {}^{x}k = e(x) \cdot k \cdot e(x)^{-1} \qquad k \in K[d],\ x \in X_0$

<u>FACT</u>: If $\mathcal{E}$ is homological, then GRPD($\mathcal{E}$) is homological!

<u>REMARK</u>

ANY HOMOLOGICAL CATEGORY IS A MAL'TSEV CATEGORY.
(SEE BORCEUX - BOURN, 2004)

PROPOSITION ① (BOURN, 1991)

$\mathcal{C}$ finitely complete and POINTED. TFCAE:

1) the SPLIT SHORT FIVE LEMMA holds in $\mathcal{C}$

2) given a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "1."] & C \arrow[r] \arrow[d, shift right, "p"'] \arrow[dr, phantom, "2."] & E \arrow[d] \\
B \arrow[r] & D \arrow[r] \arrow[u, shift right, "s"'] & F
\end{tikzcd}$$

where $p \cdot s = 1_D$, 1. is a PULLBACK, 1.+2. is a PULLBACK $\Big\} \Rightarrow$ 2 is a PULLBACK

PROOF

2) ⇒ 1) Given a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, tail, "k"] \arrow[d, "u"'] & A \arrow[r, shift right, "f"'] \arrow[d, "v"'] & B \arrow[l, shift right, "s"'] \arrow[r] \arrow[d, "w"] & 0 \\
0 \arrow[r] & K' \arrow[r, tail, "k'"'] & A' \arrow[r, shift left, "f'"] & B' \arrow[l, shift left, "s'"] \arrow[r] & 0
\end{tikzcd}$$

where u and w are ISOS, one forms the cube

$$\begin{tikzcd}[row sep=scriptsize, column sep=scriptsize]
K \arrow[rr, "k"] \arrow[dr] \arrow[dd, "\simeq u"'] & & A \arrow[dr, "f"] \arrow[dd, "v" near start] & \\
& 0 \arrow[rr, crossing over] & & B \arrow[dd, "\simeq w"] \\
K' \arrow[rr, "k'" near end] \arrow[dr] & & A' \arrow[dr, "f'"] & \\
& 0 \arrow[rr] \arrow[from=uu, equal, crossing over] & & B'
\end{tikzcd}$$

$$\Rightarrow \quad \begin{tikzcd}
K \arrow[r, "k"] \arrow[d] \arrow[dr, phantom, "1."] & A \arrow[r, "v"] \arrow[d, shift right, "f"'] \arrow[dr, phantom, "2."] & A' \arrow[d, "f'"] \\
0 \arrow[r] & B \arrow[r, "w"'] \arrow[u, shift right] & B'
\end{tikzcd}$$

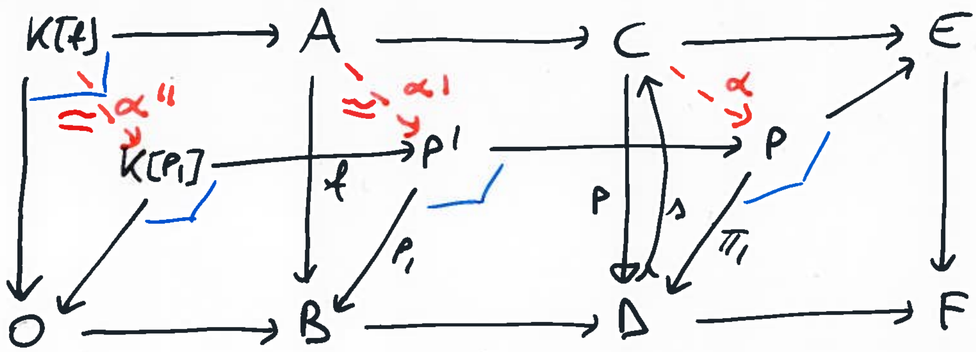
1. IS A PULLBACK, 1.+2. IS A PULLBACK $\Big\} \Rightarrow$ 2. IS A PULLBACK

1) $\Rightarrow$ 2) Assume that the SPLIT SHORT FIVE LEMMA holds, and consider the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "1."] & C \arrow[r] \arrow[d, shift right, "p"'] \arrow[dr, phantom, "2."] & E \arrow[d] \\
B \arrow[r] & D \arrow[r] \arrow[u, shift right, "s"'] & F
\end{tikzcd}$$

where 1. and 1.+2. are PULLBACKS

One then forms the diagram



where α' is an iso (by assumption) $\Rightarrow$ α'' is an ISO

One then gets the diagram.

$$\begin{tikzcd}
0 \arrow[r] & K[p] \arrow[r] \arrow[d, dashed, "\alpha''"] & C \arrow[r, shift right, "p"'] \arrow[d, dashed, "\alpha"] & D \arrow[r] \arrow[l, shift right, "s"'] \arrow[d, equal] & 0 \\
0 \arrow[r] & K[\pi_1] \arrow[r] & P \arrow[r, shift left, "\pi_1"] & D \arrow[r] \arrow[l, shift left, "\alpha \cdot s"] & 0
\end{tikzcd}$$

where α'' is an <u>ISO</u> $\Rightarrow$ α is an ISO

$\Rightarrow$ 2. is a PULLBACK

$\square$

COROLLARY ② (BOURN, 1991)

In a POINTED PROTOMODULAR category $\mathcal{C}$

$[f : X \longrightarrow Y$ is a MONO$] \Longleftrightarrow [K[f] \cong 0]$

PROOF

$\Rightarrow$) If $f : X \rightarrowtail Y$ is a MONO, then the square

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & X \arrow[d, tail, "f"] \\ 0 \arrow[r] & Y \end{tikzcd}$$

is a PULLBACK, and $K[f] \cong$

$\Leftarrow$) Conversely, when $K[f] \cong 0$ form the diagram

$$\begin{tikzcd} K[f] \cong 0 \arrow[r, "\ker(f)"] \arrow[d] \arrow[dr, phantom, "1."] & X \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "2."] & X \arrow[d, "f"] \\ 0 \arrow[r] & X \arrow[r, "f"'] & Y \end{tikzcd}$$

where 1. and 1. + 2. are PULLBACKS.

From PROTOMODULARITY it follows that 2. is a pullback, and $f : X \longrightarrow Y$ is a MONO

$\square$

PROPOSITION 3

Let $\mathcal{C}$ be a REGULAR POINTED CATEGORY.

Then $\mathcal{C}$ is HOMOLOGICAL

$$\Updownarrow$$

for any commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "1."] & C \arrow[r] \arrow[d, "p"] \arrow[dr, phantom, "2."] & E \arrow[d] \\ B \arrow[r] & D \arrow[r] & F \end{tikzcd}$$

where p is a REGULAR EPI
1. is a PULLBACK
1.+2. is a PULLBACK

$\Longrightarrow$ 2. is a PULLBACK

PROOF

EXERCICE (HINT: USE THE BARR-KOCK THM)

DEFINITION

A pair of arrows $A \xrightarrow{\alpha} C \xleftarrow{\beta} B$

is JOINTLY EXTREMAL EPIMORPHIC if,

for every MONO $J \overset{j}{\rightarrowtail} C$ such that

$$\begin{tikzcd} & J \arrow[d, tail, "j"] & \\ A \arrow[ur, dashed, bend left, "a"] \arrow[r, "\alpha"'] & C & B \arrow[l, "\beta"] \arrow[ul, dashed, bend right, "b"'] \end{tikzcd}$$

then j is an ISO.

EXERCICE

In a category with equalisers

[JOINTLY EXTREMAL EPIMORPHIC] ⇒ [jointly EPIMORPHIC]

where jointly EPIMORPHIC means

$$\begin{tikzcd} A \arrow[r, "\alpha"] & C \arrow[d, shift right, "u"'] \arrow[d, shift left, "v"] & B \arrow[l, "\beta"'] \\ & D & \end{tikzcd}$$

$$\left.\begin{array}{l} u \cdot \alpha = v \cdot \alpha \\ u \cdot \beta = v \cdot \beta \end{array}\right\} \Rightarrow \quad u = v$$

LEMMA (4)

Consider a pullback

$$\begin{tikzcd} E \times_B A \arrow[r, "p_2"] \arrow[d, "p_1"'] & A \arrow[d, shift right, "p"'] \\ E \arrow[r, "f"'] & B \arrow[u, shift right, "s"'] \end{tikzcd}$$

along a split epi $p : A \to B$ in a HOMOLOGICAL CATEGORY. Then the pair (p_2, s) is JOINTLY EXTREMAL EPIMORPHIC.

PROOF

1ST STEP The property holds for the special case $E = 0$:

$$\begin{tikzcd} & & J \arrow[d, "j"'] \\ K[p] \arrow[r, tail, "\mathrm{Ker}(p)"] \arrow[d] \arrow[urr, dashed, "a"] & A \arrow[d, shift right, "p"'] & \\ 0 \arrow[r] & B \arrow[u, shift right, "s"'] \arrow[uur, dashed, "b"'] & \end{tikzcd}$$

Assume that $(\mathrm{Ker}(p), s)$ factor through a mono $J \overset{j}{\rightarrowtail} A$. One can then form the diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "a"] \arrow[d, equal] & J \arrow[r, "p \cdot j"'] \arrow[d, tail, "j"] & B \arrow[r] \arrow[d, equal] \arrow[l, bend right, "b"'] & 0 \\ 0 \arrow[r] & K \arrow[r] & A \arrow[r, "p"] & B \arrow[r] \arrow[l, bend left, "s"] & 0 \end{tikzcd}$$

The SPLIT SHORT FIVE LEMMA implies that j is an ISO.

2ND STEP Consider then any pullback along a split epi

$$\begin{tikzcd}
E \times_B A \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[rr, dashed, bend left, "a"] & A \arrow[d, shift right, "p"'] & J \arrow[l, "j"'] \\
E \arrow[r, "f"'] & B \arrow[u, shift right, "s"'] \arrow[ur, dashed, bend right, "b"'] &
\end{tikzcd}$$

and assume that p_2 and s factor through j. Complete the diagram by taking the kernel of p_1:

$$\begin{tikzcd}
K[p_1] \arrow[r, tail, "\mathrm{Ker}(p_1)"] \arrow[d] & E \times_B A \arrow[r, "p_2"] \arrow[d, "p_1"] \arrow[rr, dashed, bend left, "a"] & A \arrow[d, shift right, "p"'] & J \arrow[l, "j"'] \\
0 \arrow[r] & E \arrow[r] & B \arrow[u, shift right, "s"'] \arrow[ur, dashed, bend right, "b"'] &
\end{tikzcd}$$

Clearly $K[p_1] \cong K[p]$ and

$$K[p] \xrightarrow{p_2 \cdot \mathrm{Ker}(p_1)} A \xleftarrow{s} B$$

factor through $J \rightarrowtail^{j} A$. Since the pair $(p_2 \cdot \mathrm{Ker}(p_1), s)$ is JOINTLY EXTREMAL EPIMORPHIC one concludes that $J \overset{j}{\rightarrowtail} A$ is an iso, as desired.

$\square$

EXERCICE SHOW THAT THE PROPERTY USED IN LEMMA 4 IS ACTUALLY EQUIVALENT TO PROTOMODULARITY.

Given a SPLIT EPI $X \underset{f}{\overset{s}{\rightleftarrows}} Y$ in a HOMOLOGICAL CATEGORY, when we consider its KERNEL $k : K \to X$,

$$\begin{tikzcd} K \arrow[r, tail, "k"] \arrow[d] & X \arrow[d, shift right, "f"'] \\ 0 \arrow[r] & Y \arrow[u, shift right, "s"'] \end{tikzcd}$$

the pair (k, s) is JOINTLY EXTREMAL EPIMORPHIC.

This means that X is then the SUPREMUM of $K \overset{k}{\rightarrowtail} X$ and $Y \overset{s}{\rightarrowtail} X$ as SUBOBJECTS OF X.

This shows a difference with the ADDITIVE CONTEXT.

Indeed, in the ADDITIVE CONTEXT one has an ISOMORPHISM:

$$\begin{tikzcd} & & K \oplus Y \arrow[dl, "\cong"', "{[k,s]}"] \\ K \arrow[r, tail, "k"'] \arrow[d] \arrow[urr, bend left] & X \arrow[d, shift right, "f"'] & \\ 0 \arrow[r] & Y \arrow[u, shift right, "s"'] \arrow[uur, bend right] & \end{tikzcd}$$

PROPOSITION (5)

In a pointed protomodular category any REGULAR EPI is a NORMAL EPI.

PROOF

Let $A \xrightarrow{f} B$ be a regular epi and $K \overset{k}{\rightarrowtail} A$ its KERNEL. We are going to show that

$$f = \mathrm{coker}(k).$$

Form the diagram

$$\begin{tikzcd}
K \times K \arrow[rr, dashed, "\hat{k}"] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] & & Eq(f) \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] \\
K \arrow[rr, tail, "k"] \arrow[d] & & A \arrow[d, "f"] \arrow[u, dashed, bend right, "\Delta = (1,1)"'] \arrow[dr, dashed, bend left, "\alpha"] \\
0 \arrow[rr] \arrow[rrr, dashed, bend right] & & B \arrow[r, dashed] & C
\end{tikzcd}$$

and consider an arrow $A \xrightarrow{\alpha} C$ such that $\alpha \cdot k = 0$

Then: $(\alpha \cdot p_1) \cdot \hat{k} = \alpha \cdot k \cdot p_1 = 0 = (\alpha \cdot p_2) \cdot \hat{k}$

$(\alpha \cdot p_1) \cdot \Delta = \alpha = (\alpha \cdot p_2) \cdot \Delta$

$(\hat{k}, \Delta)$ JOINTLY EPIMORPHIC

$\Longrightarrow$ $\alpha \cdot p_1 = \alpha \cdot p_2$.

Since the regular epi f is the coequaliser of its kernel pair $Eq(f) \overset{p_1}{\underset{p_2}{\rightrightarrows}} A$, it follows that there is a unique $\varphi : B \longrightarrow C$ such that

$$\varphi \cdot f = \alpha \qquad \square$$

REMARK

In a HOMOLOGICAL CATEGORY the notion of SHORT EXACT SEQUENCE reduces to a regular epi

$$A \xrightarrow{f} B$$

equipped with its kernel $K[f]$:

$$0 \longrightarrow K(f) \xrightarrow{\kappa} A \xrightarrow{f} B \longrightarrow 0$$

As we have just shown, any REGULAR EPI is the COKERNEL OF ITS KERNEL!

PROPOSITION 5 is the CATEGORICAL VERSION of the FIRST ISOMORPHISM THEOREM in GRP:

if $G \xrightarrow{f} G'$ is a SURJECTIVE HOMOMORPHISM,

then

$$\begin{tikzcd} 0 \arrow[r] & K[f] \arrow[r, tail] & G \arrow[r, "f"] & G' \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & 0 \\ & & & G/K(f) & \end{tikzcd}$$

THEOREM 6 (BOURN, 2001)

$\mathcal{C}$ REGULAR and POINTED. Then:

$\mathcal{C}$ IS HOMOLOGICAL $\Leftrightarrow$ THE SHORT FIVE LEMMA HOLDS IN $\mathcal{C}$

PROOF:

$\Leftarrow$) CLEAR, any SPLIT EPI is a NORMAL EPI in a HOMOLOGICAL CATEGORY.

$\Rightarrow$) APPLY PROPOSITION 3 TO THE "CUBE" DERIVED FROM THE COMMUTATIVE DIAGRAM

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "k"] \arrow[d, "u"'] & A \arrow[r, "f"] \arrow[d, "v"'] & B \arrow[r] \arrow[d, "w"] & 0 \\
0 \arrow[r] & K' \arrow[r, "k'"'] & A' \arrow[r, "f'"'] & B' \arrow[r] & 0
\end{tikzcd}$$

WHERE u AND w ARE ISOMORPHISM

$\square$

The Short Five Lemma holds in any HOMOLOGICAL CATEGORY, and is crucial to establish other "homological lemmas":

- NOETHER'S ISOMORPHISM THEOREMS
- 3×3 - LEMMA
- FIVE LEMMA
- SNAKE LEMMA

The proofs are not too difficult; however, they are different from the ones in the ABELIAN CASE, simply because the notion of HOMOLOGICAL CATEGORY is not SELF-DUAL!

For more details see the book "MAL'CEV, PROTOMODULAR, HOMOLOGICAL AND SEMI-ABELIAN CATEGORIES" BY F. BORCEUX AND D. BOURN (2004)

EXERCICE In a HOMOLOGICAL CATEGORY PROVE THE FIVE LEMMA: GIVEN

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d, "u_1"] & B' \arrow[r] \arrow[d, "u_2"', "\cong"] & C' \arrow[r] \arrow[d, "u_3"] & D' \arrow[r] \arrow[d, "u_4"', "\cong"] & E' \arrow[r] \arrow[d, "u_5"] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & D \arrow[r] & E \arrow[r] & 0 \end{tikzcd}$$

$\left.\begin{array}{l} u_2, u_4 \text{ ISOS} \\ u_1 \text{ EPI REG}, u_5 \text{ MONO} \end{array}\right\} \Rightarrow u_3 \text{ ISO}$