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Abstract

This paper aims to study both univariate and multivariate versions of the Laplace transform order. The importance of this stochastic order relation in actuarial sciences is enhanced by outlining several possible applications.

Key words and phrases: utility theory, risk theory, Laplace transform, stochastic order, life distribution, dependence

1 Introduction

1.1 Integral stochastic orders

It is well established that stochastic order relations constitute an important tool in the analysis of various actuarial problems. A large number of applications of stochastic orderings in insurance and related topics are provided by the comprehensive books of Goovaerts *et al.* (1990) and Kaas *et al.* (1994). The interested reader is referred e.g. to Shaked and Shanthikumar (1994) for a general overview of this topic in probability and statistics.

Most stochastic order relations used in actuarial sciences are of integral form. Let us recall that an integral stochastic ordering $\preceq_{\mathcal{F}}$ is a stochastic order relation generated by a class $\mathcal{F}$ of measurable functions. To be specific, given two random variables (or two random vectors) X and Y, X is said to precede Y in the $\preceq_{\mathcal{F}}$-sense, written as $X \preceq_{\mathcal{F}} Y$, if, and only if,

$$E\phi(X) \leq E\phi(Y) \text{ for all the functions } \phi \in \mathcal{F}, \tag{1.1}$$

provided the expectations involved in (1.1) exist. Such stochastic orderings have been studied e.g. by Marshall (1991) and by Müller (1997a) in a very general setting. Note that $X \preceq_{\mathcal{F}} Y$ in fact means that the distribution functions F_X and F_Y corresponding to X and Y are ordered, and not the particular versions of these random variables. The notations $X \preceq_{\mathcal{F}} Y$ and $F_X \preceq_{\mathcal{F}} F_Y$ will be used interchangeably in the remainder.

The reason why orderings of the form (1.1) are so appreciated in actuarial sciences originates from the classical expected utility theory. In this framework, each decision-maker is assumed to possess a utility function ϕ expressing his preferences; $\phi(x)$ describes "how much the decision-maker appreciates a fortune of \$ x". The decision-maker with utility ϕ is then said to base his preferences on the "expected utility hypothesis" if, for all uncertain future fortunes modeled by the random variables X and Y, he prefers Y to X if, and only if, $E\phi(X) \leq E\phi(Y)$, provided the expectations exist. Coming back to (1.1), utility theory gives an intuitive interpretation for $\preceq_{\mathcal{F}}$, by considering the common properties shared by all the elements of $\mathcal{F}$ viewed as utility functions. For instance, if $\mathcal{F}$ is the class of the non-decreasing functions on the real line $\mathbb{R}$, $\preceq_{\mathcal{F}}$ is the stochastic dominance $\preceq_{st}$; the latter expresses the common preferences of all the decision-makers thinking that more money is better. If one further assumes that $\mathcal{F}$ contains all the non-decreasing and concave functions on $\mathbb{R}$, $\preceq_{\mathcal{F}}$ is the increasing concave order $\preceq_{icv}$ (termed second degree stochastic dominance in economics); it expresses the common preferences of all the profit-seeking decision-makers who are risk averse (i.e. those who always prefer a constant fortune to a random fortune with the same mean, in view of Jensen's inequality).

Contrary to economists, actuaries are mainly concerned with the comparison of risks (and not of random fortunes). Risks are non-negative random variables representing the random amounts of money the insurance company will have to pay out in order to indemnify the policyholder and/or the third party. This explains the duality existing between the stochastic order relations used in economics (of concave-type) and those used in actuarial sciences (of convex-type). To be specific, let ϕ be a non-decreasing and concave utility of wealth function for a decision-maker. The function $\widetilde{\phi}$ defined by

$$\widetilde{\phi}(x) = -\phi(-x), \quad x \in \mathbb{R}, \tag{1.2}$$

is clearly non-decreasing and convex. According to Denuit, Dhaene and Van Wouwe (1999), $\widetilde{\phi}$ will be said to be the pain function of the decision-maker, because $\widetilde{\phi}(x)$ may be viewed as the pain associated with a debt of amount \$ x. Therefore, in order to compare risks, actuaries are mostly interested in integral stochastic orderings generated by classes of pain functions, whereas economists consider integral stochastic orderings generated by classes of utility functions. For instance, the dual version of $\preceq_{icv}$ is the stop-loss order $\preceq_{s\ell}$ generated through (1.1) with for $\mathcal{F}$ the class of the non-decreasing and convex functions.

1.2 Laplace transform order

Let us now focus on the common preferences of all the decision-makers with a utility function ϕ of the form

$$\phi(x) = 1 - \exp(-tx), \quad t > 0; \tag{1.3}$$

this gives rise to the Laplace transform order. To be specific, given two non-negative random variables X and Y, X is said to precede Y in the Laplace transform order, denoted as $X \preceq_{Lt} Y$, when the inequality

$$E\exp(-tX) \leq E\exp(-tY) \text{ holds for all } t \geq 0. \tag{1.4}$$

It is worth mentioning that the Laplace transforms involved in (1.4) always exist (since X and Y are non-negative). In other words, $X \preceq_{Lt} Y$ means that the income X is preferred to the income Y by all the decision-makers with a utility function of the form (1.3), that is, those people with a constant risk aversion. From a probabilistic point of view, the Laplace transform order boils down to requiring that the Laplace transforms of two non-negative random variables are uniformly ordered.

It is worth to mention that the definition (1.4) is adopted from Klefsjö (1983). It differs from the definition given e.g. in Alzaid, Kim and Proschan (1991) and followed by Shaked and Shanthikumar (1994, formula (3.B.1)), where the reverse inequality is required in (1.4). This stochastic order has been introduced by Rolski and Stoyan (1976) to obtain some inequalities connected with queueing problems; see Shaked and Shanthikumar (1994, Section 3B) for an extensive presentation. The order $\preceq_{Lt}$ is a highly valued tool in reliability theory (it is worth mentioning that many concepts of reliability theory have found interesting applications in actuarial sciences, following the works of Straub (1971) and Heilmann (1983); for recent developments in that direction, see e.g. Cai and Garrido (1998)). It has also been widely used in economics; see e.g. Thistle (1993) and the references therein. Note that because of the definition (1.4) of $\preceq_{Lt}$, $X \preceq_{Lt} Y$ in fact means that X is "larger" than Y (since the expectation of a decreasing function is larger with Y than with X). Defining the class of functions $\mathcal{U}_{Lt} = \{x \mapsto \exp(-tx), \ t > 0\}$, the ordering $\preceq_{Lt}$ is an integral stochastic order relation, since it satisfies (1.1) with $\mathcal{F} = \mathcal{U}_{Lt}$. We will exploit this characteristic of the Laplace transform order in the sequel.

To end with, let us also mention several related works of independent interest. Shaked and Wong (1997) defined two stochastic order relations based on ratios of Laplace transforms. Bartoszewicz (1999) derived characterizations of these orderings. Belzunce, Ortega and Ruiz (1999) have introduced an ordering based on the Laplace transform order of residual lives.

It is easily seen that given two non-negative random variables X and Y, the implications

$$X \preceq_{st} Y \Rightarrow X \preceq_{icv} Y \Rightarrow Y \preceq_{Lt} X \tag{1.5}$$

are valid, since any element ϕ of $\mathcal{U}_{Lt}$ is such that $-\phi$ is non-decreasing and concave. Moreover, if $EX = EY$, it can be shown that

$$X \preceq_{s\ell} Y \Leftrightarrow Y \preceq_{icv} X, \tag{1.6}$$

where $\preceq_{s\ell}$ has to be interpreted as "less variable than".

In the framework of expected utility theory, it is possible to find an intuitive meaning for a ranking of risks in the $\preceq_{Lt}$-sense if we adopt the point of view of the insured person (or the point of view of the ceding company in a reinsurance agreement). The risks X and Y they passed to the insurer (or to the reinsurer) then represent random future incomes. Therefore, $X \preceq_{Lt} Y$ means that every insured with constant risk aversion $t \geq 0$ prefers an indemnity of amount X to an indemnity of amount Y.

In addition to its interpretation in terms of expected utility, $\preceq_{Lt}$ also possesses some other intuitive meanings, namely when life distributions are compared. We review some of them next.

In actuarial mathematics, exponential mixtures are often candidates for modeling the amount of insurance loss or the duration of disability. For an application in risk theory, see e.g. Babier and Chan (1992) and for a general study, we refer the interested reader to Hesselager, Wang and Willmot (1997). The latter authors considered an exponential mixture with a conditional survival function

$$\overline{F}(x|\Theta = \theta) = \exp(-\theta x), \;\; x \geq 0,$$

where the parameter Θ has a distribution function G with support in the half positive real line $\mathbb{R}^+$. The resulting exponential mixture has survival function

$$\overline{F}(x) = \int_{\theta=0}^{+\infty} \exp(-\theta x) dG(\theta), \;\; x \geq 0, \tag{1.7}$$

which thus appears as the Laplace transform of the mixing distribution G. Therefore, given two mixing distributions $G_1(\theta) = P[\Theta_1 \leq \theta]$ and $G_2(\theta) = P[\Theta_2 \leq \theta]$, with corresponding exponential mixtures $\overline{F}_1$ and $\overline{F}_2$ given by (1.7), the implication

$$\Theta_1 \preceq_{Lt} \Theta_2 \Rightarrow \overline{F}_1(x) \leq \overline{F}_2(x) \text{ for all } x \geq 0, \tag{1.8}$$

holds, that is, the exponential mixtures associated to Θ_1 and Θ_2 are ordered in the stochastic dominance $\preceq_{st}$.

Laplace transform order also possesses a nice interpretation in life insurance, when time-until-death random variables need to be compared. Let X and Y be two remaining lifetimes. Considering (1.4), it is easily seen that $X \preceq_{Lt} Y$ then means that the whole life premium relating to X is always smaller than the whole life premium relating to Y, whatever the interest rate is; X thus models a "longer" lifelength than Y. Using integration by parts, one finds for $t \geq 0$

that

$$\begin{aligned} E\exp(-tX) &= F_X(0)+\int_{x=0+}^{+\infty}\exp(-tx)dF_X(x)\\ &= F_X(0)+[\exp(-tx)F_X(x)]_{x=0+}^{+\infty}-\int_{x=0}^{+\infty}F_X(x)d\{\exp(-tx)\}\\ &= t\int_{x=0}^{+\infty}\exp(-tx)F_X(x)dx. \end{aligned} \tag{1.9}$$

This yields

$$\frac{1-E\exp(-tX)}{t}=\int_{x=0}^{+\infty}\exp(-tx)P[X>x]dx,\ \ t>0. \tag{1.10}$$

Considering (1.10), $X \preceq_{Lt} Y$ implies that the net single premium of a continuous annuity will be higher for X than for Y, whatever the interest rate is. A smaller random future lifetime in the $\preceq_{Lt}$-sense thus intuitively means that the insured person will live longer. Technically speaking, (1.10) can also be considered as the expected value of $\min\{X, E_t\}$, where E_t is a random variable distributed according to the exponential distribution with mean $1/t$ and independent of X. Therefore,

$$X \preceq_{Lt} Y \Leftrightarrow E\min\{X,E_t\}\geq E\min\{Y,E_t\} \text{ for all } t>0, \tag{1.11}$$

where E_t is independent of both X and Y. Similarly, using a standard conditioning argument, (1.9) also yields

$$1-E\exp(-tX)=P[X>E_t],$$

whence it follows that

$$X \preceq_{Lt} Y \Leftrightarrow P[X>E_t]\geq P[Y>E_t] \text{ for all } t>0. \tag{1.12}$$

Formulas (1.11) and (1.12) provide other intuitive explanations of a $\preceq_{Lt}$-ranking for claim amounts, by reference to exponentially distributed risks. Note that broad classes of stochastic orderings can be obtained by substituting another parametric family for $\{E_t,\ t>0\}$ in (1.11) or in (1.12). For instance, considering non-negative constants in (1.12) yields $\preceq_{st}$.

To end with, let us mention that several classes of probability distributions have been defined as $\min\{X_\theta, E_t\}$, for some parametric family $\{X_\theta,\ \theta\in\mathcal{T}\}$, $\mathcal{T}\subseteq\mathbb{R}^p$. For instance, Davis and Feldstein (1979) considered the family of probability distributions with hazard rate function of the form

$$\lambda(x)=\beta+\frac{\alpha}{x+\lambda},\ \ \alpha,\beta,\lambda>0;$$

each member of this family can be represented as $\min\{X_{(\alpha,\lambda)}, E_\beta\}$, where $X_{(\alpha,\lambda)}$ is distributed according to the Pareto law with parameters α and λ.

1.3 Exponential order

Let us now turn to a companion of the $\preceq_{Lt}$ order, namely the exponential order. The rationale underlying the introduction of this ordering is mainly inspired from utility theory. The common preferences of all the decision-makers with pain functions of the form

$$\widetilde{\phi}(x)=\exp(tx)-1,\ \ t>0, \tag{1.13}$$

generate the exponential order (note that (1.3) and (1.13) are related through (1.2)). To be specific, the risk X is smaller than the risk Y in the exponential order, denoted as $X \preceq_{exp} Y$, when the inequality

$$E\exp(tX) \leq E\exp(tY) \text{ holds for all } t \geq 0, \tag{1.14}$$

provided that the expectations involved in (1.14) exist. This order has been examined in Goovaerts *et al.* (1990, Section 4.3 page 45ff) and in Kaas *et al.* (1994, Section 3, page 53ff). It has been used by Kaas and Gerber (1994) in order to compare different approximations for the total claims of a risk portfolio. This time, $X \preceq_{exp} Y$ means that X is "smaller" than Y, since the expectation of an increasing function is larger with Y than with X.

From a probabilistic point of view, the exponential order thus boils down to requiring that the moment generating functions of the non-negative random variables X and Y are uniformly ordered. The ordering $\preceq_{exp}$ also expresses the common preferences among risks of all the decision-makers with a utility function of the form (1.3) or, equivalently, with a pain function of the form (1.13).

If there exist $h > 0$ such that $E\exp(tX)$ exists and is finite for $t < h$ then the Taylor expansion of the exponential function yields

$$E\exp(tX) = 1 + \sum_{k=1}^{+\infty} \frac{t^k}{k!} EX^k \text{ for } t < h.$$

From this series expansion, it is clear that the exponential order is weaker than the moment order $\preceq_{mom}$, defined as $X \preceq_{mom} Y \Leftrightarrow EX^k \leq EY^k$ for every positive integer k, provided the moments exist. It is well known that if any moment of a distribution is infinite, the moment generating function does not exist. However, it is conceivable that there might exist distributions with moments of all orders and, yet, the moment generating function does not exist in any neighborhood around 0. In fact, the Log-Normal distribution is one such example. The order $\preceq_{exp}$ is thus incapable of comparing a Log-Normally distributed random variable with any other risk.

In an actuarial framework, $\preceq_{exp}$ also possesses several other nice interpretations. We briefly review them now. Consider an insurance company setting its price list according to the exponential premium calculation principle. In such a case, the amount of premium $\pi_t[X]$ relating to the risk X is given by

$$\pi_t[X] = \frac{1}{t} \ln E\exp(tX). \tag{1.15}$$

From (1.14) and (1.15), we obviously have that

$$X \preceq_{exp} Y \Leftrightarrow \pi_t[X] \leq \pi_t[Y] \text{ for all } t > 0.$$

Therefore, $X \preceq_{exp} Y$ is equivalent to X needing smaller premiums than Y in case the exponential premium calculation principle is used, whatever the risk aversion coefficient t is. We thus see that the exponential order is closely related to the classical exponential premium calculation principle, as the stop-loss order is linked to the stop-loss premiums.

Another interpretation of $\preceq_{exp}$ is provided by the collective risk theory. In the classical risk model, individual claim amounts $X_1, X_2, X_3, \ldots$ are assumed to be independent and identically distributed, with common distribution function $F_X(x) = P[X \le x]$ such that $F_X(0) = 0$. The number of claims in the interval $[0, t]$ is assumed to be independent of the individual claim amounts and to form a Poisson process $\{N_t,\ t \in I\!R^+\}$ with constant rate λ. Let the premium rate $c > 0$ be such that the inequality $c > \lambda EX$ holds. Further, let $\psi_X(z)$ be the ultimate ruin probability with an initial capital z, i.e. the probability that the process $Z_t = z + ct - X_1 - X_2 - \ldots - X_{N_t}$, $t \ge 0$, describing the wealth of the insurance company, ever falls below zero. If the moment generating function of X exists, Lundberg's inequality provides an exponential upper bound on ψ_X, namely $\psi_X(z) \le \exp(-\kappa_X z)$, where κ_X is the Lundberg's adjustment coefficient satisfying the integral equation

$$E\exp(\kappa_X X) = 1 + \frac{c}{\lambda}\kappa_X. \tag{1.16}$$

Now, consider a second risk process with independent and identically distributed claim amounts $Y_1, Y_2, Y_3, \ldots$, with common distribution function $F_Y(x) = P[Y \le x]$. The two risk processes only differ in their respective claim severities. Combining (1.14) and (1.16), we clearly see that $X \preceq_{exp} Y \Rightarrow \kappa_X \ge \kappa_Y$. Therefore, the smaller the claim amount X is (in the $\preceq_{exp}$-sense), the larger Lundberg's coefficient κ_X is and the smaller Lundberg's upper bound $\exp(-\kappa_X z)$ is; we then expect the ruin probability ψ_X itself to be smaller.

Despite its nice interpretation, $\preceq_{exp}$ suffers from serious theoretical drawbacks. Whereas the Laplace transform always exists and is finite for non-negative random variables, this is not always the case for the moment generating function. The set $\mathcal{E} = \{t > 0 | E\exp(tX) < +\infty\}$ can be $I\!R^+$, a finite interval or even the empty set. Let $t_{\max} = \sup \mathcal{E}$. If $\mathcal{E} \neq \emptyset$, we see that $t \mapsto E\exp(tX)$ is a well-defined continuous and strictly increasing function of $t \in [0, t_{\max}[$, with value 1 at the origin. If the moment generating function of X is finite for a value $t_0 > 0$ then there exists a constant $b > 0$ such that for all $x \ge 0$, $P[X > x] \le b\exp(-t_0 x)$. In other words, X has an exponentially bounded tail. The exponential order is therefore useless for comparing pairs of random variables with heavy tails (like Pareto distributions), for which $E\exp(tX) = +\infty$ for all $t > 0$. For these reasons, we will focus on the single $\preceq_{Lt}$ in the remainder of this paper. We give a detailed account of the properties of this stochastic ordering in Section 2.

1.4 Multivariate Laplace transform order

In an insurance portfolio, the aggregate claims during a reference period is given by the sum S of the claims relating to the individual risks. A standard assumption is that the summands in S are independent. Of course, this hypothesis relies on computational convenience rather than realism: a fine example is a catastrophe hitting many insureds simultaneously. Some interesting results about the effects of the dependence in actuarial sciences are available. Recently, a number of articles have been devoted to the study of the impact of a possible dependence among insured risks in setting premium rates and base their analysis on multivariate stochastic orderings: see e.g. Dhaene and Denuit (1999) and the references therein.

In this paper, we study the multivariate extension of $\preceq_{Lt}$ and we show that this order relation is relevant for actuarial applications, namely to take into account some dependencies in an insurance portfolio. Before going further, let us clarify the notations used in the remainder of this work. A point of the n-dimensional real space $I\!R^n$ is denoted by a bold lowercase letter such as $\mathbf{x}$; x_i then stands for the ith component of $\mathbf{x}$, $i = 1, 2, \ldots, n$. An n-dimensional random vector is denoted by a bold uppercase letter such as $\mathbf{X}$; X_i then stands for the ith component of $\mathbf{X}$, $i = 1, 2, \ldots, n$. The notation $\mathbf{x} \leq \mathbf{y}$ means $x_i \leq y_i$ for all $i = 1, 2, \ldots, n$, whereas $\mathbf{x} < \mathbf{y}$ means that $\mathbf{x} \leq \mathbf{y}$ and $x_i < y_i$ for at least one i. All the classical operations have to be interpreted componentwise ($\mathbf{x} \pm \mathbf{y}$ is a vector with ith component $x_i \pm y_i$, $t\mathbf{x}$ is a vector with ith component tx_i, and so on). All the vectors are tacitly assumed to be column vectors and the superscript "prime" denotes the usual transposition. Finally, $\mathbf{0} = (0, 0, \ldots, 0)'$, $\mathbf{1} = (1, 1, \ldots, 1)'$, and so on.

The multivariate Laplace transform order, introduced by Shaked and Shanthikumar (1994) and recently used by Wong (1997) in the context of multivariate shock models, is defined as follows: let $\mathbf{X}$ and $\mathbf{Y}$ be two non-negative n-dimensional random vectors; $\mathbf{X}$ is said to be smaller than $\mathbf{Y}$ in the Laplace transform order, written as $\mathbf{X} \preceq_{Lt} \mathbf{Y}$, if the inequality

$$E \exp\left(-\mathbf{t}'\mathbf{X}\right) \leq E \exp\left(-\mathbf{t}'\mathbf{Y}\right) \text{ holds for all } \mathbf{t} \geq \mathbf{0}. \tag{1.17}$$

Definition (1.17) naturally extends (1.4) to higher dimension. Henceforth, when we write $\preceq_{Lt}$, we do not specify the dimension of the elements to be compared, since this is easily deduced from the context.

Note that $\preceq_{Lt}$ is a multivariate integral stochastic ordering of product-type (in the sense of Bergmann (1991)). To be specific, such an ordering is defined by (1.1) with the aid of a class $\mathcal{F}$ of measurable functions $\phi : I\!R^n \to I\!R$ that can be written as a product $\prod_{i=1}^n \phi_i$ of univariate measurable function $\phi_i : I\!R \to I\!R$. Thus, $\preceq_{Lt}$ is defined with the help of (1.1) using the class

$$\mathcal{U}_{Lt} = \left\{ \mathbf{x} \mapsto \prod_{i=1}^{n} \exp(-t_i x_i), \ \ \mathbf{t} \geq \mathbf{0} \right\}.$$

In Section 3, we establish the main features of the multivariate $\preceq_{Lt}$.

Section 4 contains some applications of $\preceq_{Lt}$ in actuarial sciences. To be specific, $\preceq_{Lt}$ is first used to compare elements of the class of multivariate counting distributions defined by Ambagaspityia (1998). The second illustration deals with the comparison of Markov chains using $\preceq_{Lt}$ and makes a foray into credibility theory. Finally, Marceau's model for dependent risks is considered and analyzed with the aid of $\preceq_{Lt}$.

2 Properties of the univariate $\preceq_{Lt}$

2.1 Completely monotone functions

As pointed out by Denuit, Lefèvre and Mesfioui (1999), $\preceq_{Lt}$ can be seen as a limiting form of the s-increasing concave orders as $s \to +\infty$. Let us briefly expand on this aspect. Let $\mathcal{U}_{cm}$ be the class of the completely monotone functions, that is the class of the functions $\phi : (0, +\infty) \to I\!R^+$ satisfying $(-1)^k \phi^{(k)} \geq 0$ for all $k \geq 1$, where $\phi^{(k)}$ denotes the kth derivative of ϕ. As $x \to 0$, the derivatives of $\phi \in \mathcal{U}_{cm}$ approach finite or infinite limits denoted as $\phi^{(k)}(0)$. Typical examples of completely monotone functions are $x \mapsto 1/x$ and $x \mapsto \exp(-x)$. It is easily seen that the Laplace transform of any non-negative random variable X is completely monotone. A classical result from real analysis, known as Bernstein's Theorem, states that conversely every completely monotone function ϕ such that $\phi(0) = 1$ is the Laplace transform of some non-negative random variable. To be specific, given $\phi \in \mathcal{U}_{cm}$, there exists a measure μ on $\mathbb{R}^+$, not necessarily finite, such that

$$\phi(x) = \int_{t=0}^{+\infty} \exp(-tx) d\mu(t), \quad x \in \mathbb{R}^+; \tag{2.1}$$

see, e.g. Theorem 1a of Feller (1966, page 416). Let $\mathcal{U}^*_{cm}$ be the class of all the continuous functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that the restriction ϕ^* of ϕ to $(0, +\infty)$ belongs to $\mathcal{U}_{cm}$. The following result can be found in Lin (1998a, Lemma 2); its proof is mainly based on the representation (2.1).

Property 2.1 *Let X and Y be two non-negative random variables; then, $X \preceq_{Lt} Y$ if, and only if,*

(i) *(1.1) holds with $\mathcal{F} = \mathcal{U}^*_{cm}$;*

(ii) *$E\phi(X) \geq E\phi(Y)$ for each continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}$ such that $\phi^{(1)} \in \mathcal{U}_{cm}$, provided the expectations exist.*

Applying Property 2.1(ii) to $\phi(x) = x$ yields $X \preceq_{Lt} Y \Rightarrow EX \geq EY$. Another elementary remark is as follows: given two non-negative independent random variables X and Y, the stochastic inequality

$$X + Y \preceq_{Lt} X \tag{2.2}$$

holds (this directly follows from the fact that $X \leq X + Y$ almost surely).

The following result is taken from Cai and Wu (1997, Theorem 3.11). A function ϕ is said to be strictly completely monotone if ϕ admits a representation (2.1) with μ strictly increasing.

Property 2.2 *Let X and Y be two non-negative random variables. If $X \preceq_{Lt} Y$ and $E\phi_0(X) = E\phi_0(Y)$ for some strictly completely monotone function ϕ_0, then X and Y are identically distributed (written as $X =_d Y$).*

For instance, $x \mapsto x^{-\alpha}$, $\alpha > 0$, is strictly completely monotone (use $\mu(t) = t^\alpha \Gamma(\alpha)/\alpha$ in (2.1)), and therefore

$$X \preceq_{Lt} Y \text{ and } EX^{-\alpha} = EY^{-\alpha} \text{ for some } \alpha > 0 \Rightarrow X \text{ and } Y \text{ are identically distributed.}$$

2.2 Closure poperties

Let us now examine the invariance and closure properties of $\preceq_{Lt}$. The properties given in this section can be found in Shaked and Shanthikumar (1994). For any random variable Z and event A, we denote below by $[Z|A]$ any random variable whose distribution function is the conditional distribution of Z given A.

Property 2.3 **(i)** *Closure under mixture: let X and Y be two non-negative random variables, and let Θ be a random variable valued in $\mathcal{T} \subseteq I\!R$; then*

$$[X|\Theta = \theta] \preceq_{Lt} [Y|\Theta = \theta] \text{ for all } \theta \in \mathcal{T} \Rightarrow X \preceq_{Lt} Y.$$

(ii) *Functional invariance (type I): let X and Y be two non-negative random variables, and let $\varphi : I\!R^+ \to I\!R^+$ be a continuous function such that $\varphi^{(1)} \in \mathcal{U}_{cm}$; then*

$$X \preceq_{Lt} Y \Rightarrow \varphi(X) \preceq_{Lt} \varphi(Y).$$

In particular, $\preceq_{Lt}$ is shift and scale invariant.

(iii) *Functional invariance (type II): let $X_1, X_2, \ldots, X_n$ (resp., $Y_1, Y_2, \ldots, Y_n$) be independent non-negative random variables and let $\Phi : I\!R_n^+ \to I\!R$ be such that $x_i \mapsto \partial\Phi/\partial x_i \in \mathcal{U}_{cm}$ for $i = 1, 2, \ldots, n$; then*

$$X_i \preceq_{Lt} Y_i \text{ for } i = 1, 2, \ldots, n \Rightarrow \Phi(X_1, X_2, \ldots, X_n) \preceq_{Lt} \Phi(Y_1, Y_2, \ldots, Y_n).$$

In particular, $\preceq_{Lt}$ is closed under convolution.

(iv) *Closure under compounding: let $\{X_1, X_2, X_3, \ldots\}$ and $\{Y_1, Y_2, Y_3, \ldots\}$ be two sequences of non-negative independent and identically distributed random variables and let N and M be positive integer-valued random variables, independent of the X_i's and of the Y_j's; then,*

$$\left.\begin{array}{l} X_i \preceq_{Lt} Y_i \text{ for all } i \\ N \preceq_{Lt} M \end{array}\right\} \Rightarrow \sum_{i=1}^{N} X_i \preceq_{Lt} \sum_{i=1}^{M} Y_i.$$

2.3 Stationary excess operator

The stationary excess operator is a classical tool in applied probability; it naturally appears in risk theory and has recently been used by Hesselager, Wang and Willmot (1997).

Let X be a non-negative random variable with distribution function F_X and finite mean. The stationary excess operator maps F_X to its equilibrium distribution $F_{\widetilde{X}}$ (also called limiting distribution or backward recurrence time in a renewal process with inter-arrival distribution F_X) given by

$$F_{\widetilde{X}}(x) = \int_{y=0}^{x} \frac{1 - F_X(y)}{EX} dy = P[\widetilde{X} \leq x], \;\; x \in I\!R^+. \tag{2.3}$$

This is e.g. the distribution of the capital at ruin starting from initial capital 0. The next property holds.

Property 2.4 *Let X and Y be two non-negative random variables such that $EX = EY$; then*

$$X \preceq_{Lt} Y \Leftrightarrow \widetilde{Y} \preceq_{Lt} \widetilde{X}.$$

Proof. Using (1.9), we have for $t \geq 0$ that

$$\begin{aligned}
E\exp(-t\widetilde{X}) &= t\int_{x=0}^{+\infty}\exp(-tx)\left\{\int_{y=0}^{x}\frac{1-F_X(y)}{EX}dy\right\}dx \\
&= \frac{t}{EX}\int_{y=0}^{+\infty}\left\{\int_{x=y}^{+\infty}\exp(-tx)dx\right\}P[X>y]dy \\
&= \frac{1}{EX}\int_{y=0}^{+\infty}\exp(-ty)P[X>y]dy = \frac{1-E\exp(-tX)}{tEX},
\end{aligned}$$

by (1.10). This ends the proof. $\diamond$

2.4 Comparing arithmetic random variables

An integral stochastic ordering (1.1) designed to compare pairs of random variables valued in the set $\mathbb{N}$ of the non-negative integers (think for instance of claim frequencies) should preferably use functions ϕ with domain $\mathbb{N}$ rather than the real line $\mathbb{R}$. Fishburn and Lavalle (1995), Lefèvre and Utev (1996) and Denuit and Lefèvre (1997), among others, studied such orderings. Let $\phi : \mathbb{N} \to \mathbb{R}^+$. Recall that the forward difference operator Δ is defined as $\Delta\phi(i) = \phi(i+1) - \phi(i)$, $i \in \mathbb{N}$, and then $\Delta^k = \Delta \circ \Delta^{k-1}$ for $k \geq 2$; $\Delta^k\phi$ is the kth "derivative" of a function ϕ defined on an arithmetic grid.

Considering Property 2.1, we could say that, given two random variables M and N valued in $\mathbb{N}$, N precedes M in the discrete Laplace transform order, denoted as $N \preceq_{Lt}^{\mathbb{N}} M$, when (1.1) holds with for $\mathcal{F}$ the class $\mathcal{U}_{Lt}^{\mathbb{N}}$ of all the functions $\phi : \mathbb{N} \to \mathbb{R}^+$ such that $(-1)^k\Delta^k\phi(i) \geq 0$ for all $k, i \in \mathbb{N}$. The natural question is then: are $\preceq_{Lt}$ and $\preceq_{Lt}^{\mathbb{N}}$ equivalent? The answer is given in the next result.

Proposition 2.5 *Given two random variables N and M valued in $\mathbb{N}$, we have that*

$$N \preceq_{Lt}^{\mathbb{N}} M \Leftrightarrow N \preceq_{Lt} M.$$

Proof. Obviously, $\preceq_{Lt}^{\mathbb{N}}$ is stronger than $\preceq_{Lt}$ since the restriction to $\mathbb{N}$ of any element of $\mathcal{U}_{Lt}$ belongs to $\mathcal{U}_{Lt}^{\mathbb{N}}$. Widder (1946, Theorem 14b on page 164) showed that, conversely, given $\phi \in \mathcal{U}_{Lt}^{\mathbb{N}}$, there exist a function $\varphi \in \mathcal{U}_{Lt}$ such that $\phi(n) = \varphi(n)$ for all $n \in \mathbb{N}$. $\diamond$

In conclusion, $\preceq_{Lt}$ is also a powerful tool to compare arithmetic random variables.

2.5 Ordering of mixtures

Let us consider a family of non-negative random variables $\{X_\theta,\ \theta \in \mathcal{T}\}$, indexed by a single parameter $\theta \in \mathcal{T} \subseteq \mathbb{R}^+$. Given a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, define the function $\phi^* : \mathcal{T} \to \mathbb{R}^+$ by $\phi^*(\theta) = E\phi(X_\theta)$. We study in this section the validity of the implication

$$\phi \in \mathcal{U}_{cm} \Rightarrow \phi^* \in \mathcal{U}_{cm}. \tag{2.4}$$

Similar problems have already been investigated in the literature, namely by Shaked and Shanthikumar (1994) and Denuit, Lefèvre and Utev (1999). At first sight, this seems very theoretical. Nevertheless, if (2.4) holds, we immediately find that

$$\Theta_1 \preceq_{Lt} \Theta_2 \Rightarrow X_{\Theta_1} \preceq_{Lt} X_{\Theta_2}, \tag{2.5}$$

where Θ_1 and Θ_2 are both valued in $\mathcal{T}$ and X_{Θ_i} $(i = 1, 2)$ is distributed according to

$$P[X_{\Theta_i} \leq x] = \int_{\theta \in \mathcal{T}} P[X_\theta \leq x] dP[\Theta_i \leq \theta], \;\; i = 1, 2, \;\; x \in \mathbb{R}^+.$$

Property 2.6 **(i)** *(2.4) is satisfied if, and only if, the implication $\phi \in \mathcal{U}_{Lt} \Rightarrow \phi^* \in \mathcal{U}_{cm}$ holds.*

(ii) *Given a non-negative random variable Y, the family $\{X_\theta, \;\; \theta \in \mathbb{R}^+\}$ defined as either $X_\theta = \theta Y$ or $X_\theta = Y + \theta$ satisfies (2.4). In other words, families with a location or a scale parameter fulfill (2.4).*

(iii) *Given a sequence $\{Y_1, Y_2, Y_3, \ldots\}$ of non-negative independent and identically distributed random variables, the family $\{X_\theta, \;\; \theta \in \mathbb{N}\}$ defined as $X_\theta = \sum_{k=1}^{\theta} Y_k$ satisfies (2.4).*

Proof. (i) is easily deduced from the representation (2.1). Then, (ii) follows from (i) together with the fact that both

$$\theta \mapsto E\exp(-tX_\theta) = E\exp(-\theta t Y)$$

and

$$\theta \mapsto E\exp(-tX_\theta) = \exp(-t\theta) E\exp(-tY)$$

belong to $\mathcal{U}_{cm}$. Finally, to get (iii), it suffices to note that

$$\theta \mapsto E\exp(-tX_\theta) = \{E\exp(-tY_1)\}^\theta = \exp\{\theta \ln E\exp(-tY_1)\}$$

belongs to $\mathcal{U}_{cm}$. $\diamond$

Let us now point out several interesting examples.

Example 2.7 **(i)** *From Property 2.6 (ii), (2.4) is fulfilled for X_θ exponentially distributed with mean θ; it suffices to take $Y = X_1$. Note that the apparent contradiction of (2.5) with (1.8) considering (1.5) is due to the different parameterization of the exponential distribution; despite the theoretical interest of the validity of (2.4) for the exponential family, (1.8) is stronger than (2.5).*

(ii) *From Property 2.6(i), (2.4) is valid when X_θ is Poisson distributed with mean θ, since*

$$\theta \mapsto E\exp(-tX_\theta) = \exp(\theta(\exp(-t) - 1)) \in \mathcal{U}_{cm}.$$

(iii) *From Property 2.6 (iii), we also have that (2.4) is fulfilled for the family of the binomial distributions with parameter $n \in \mathbb{N}$ and with mean np, as well as for the family of the Negative Binomial with parameter $n \in \mathbb{N}$ and with mean $n\theta$.*

2.6 The $\mathcal{L}$-class of life distributions

Exponential distributions have been widely used in actuarial sciences. Many concepts of sup- and sub-exponentiality have been discussed in the literature. This section expands on this issue by considering the $\mathcal{L}$-class of life distributions, defined by Klefsjö (1983). To be specific, let X be a non-negative random variable with a finite mean EX; X is said to belong to the $\mathcal{L}$-class if, and only if, $X \preceq_{Lt} E_{EX}$, where E_{EX} denotes an exponentially distributed random variable with mean EX, that is, if, and only if, its Laplace transform satisfies

$$E\exp(-tX) \leq \frac{1}{1+tEX} \text{ for all } t \in I\!R^+.$$

From (1.10), this is equivalent to require that

$$\int_{x=0}^{+\infty} \exp(-tx)P[X > x]dx \geq \frac{EX}{1+tEX} \text{ for } t > 0. \tag{2.6}$$

The $\mathcal{L}$-class is strictly larger than the HNBUE (Harmonic New Better than Used in Expectation) class consisting of those random variables X for which $E_{EX} \preceq_{icv} X \Leftrightarrow X \preceq_{s\ell} E_{EX}$. Hence, the $\mathcal{L}$-class also contains the smaller NBUE (New Better than Used in Expectation), NBU (New Better than Used), IFRA (Increasing Failure Rate in Average) and IFR (Increasing Failure Rate) classes of distributions.

If the reverse inequality holds in (2.6), then X is said to belong to the $\overline{\mathcal{L}}$-class; that is, $X \in \overline{\mathcal{L}}$ if, and only if, $E_{EX} \preceq_{Lt} X$. The $\overline{\mathcal{L}}$-class is strictly larger than the HNWUE (Harmonic New Worse than Used in Expectation) class consisting of those random variables X for which $X \preceq_{icv} E_{EX} \Leftrightarrow E_{EX} \preceq_{s\ell} X$. The $\overline{\mathcal{L}}$-class also contains the smaller NWUE (New Worse than Used in Expectation), NWU (New Worse than Used), DFRA (Decreasing Failure Rate in Average) and DFR (Decreasing Failure Rate) classes of distributions.

We summarize below some interesting properties of the $\mathcal{L}$- and $\overline{\mathcal{L}}$-classes.

Property 2.8 **(i)** *$X, Y \in \mathcal{L} \Rightarrow X + Y \in \mathcal{L}$; this is not true for the $\overline{\mathcal{L}}$-class.*

(ii) *If $E[X|\Theta = \theta] = E[X|\Theta = \theta']$ for all $\theta, \theta' \in \mathcal{T}$ then $[X|\Theta = \theta] \in \mathcal{L}$ for all $\theta \in \mathcal{T}$ implies that $X \in \mathcal{L}$. The implication $[X|\Theta = \theta] \in \overline{\mathcal{L}}$ for all $\theta \in \mathcal{T} \Rightarrow X \in \overline{\mathcal{L}}$ holds without any requirement on the conditional means.*

(iii) *The coefficient of variation of a distribution belonging to the $\mathcal{L}$-class (resp. $\overline{\mathcal{L}}$-class) is ≤ 1 (resp. ≥ 1).*

(iv) *Let X be a positive random variable in $\mathcal{L}$, and let $\widetilde{X}$ be defined as in (2.3). Then,*

$$X \in \mathcal{L} \Leftrightarrow X \preceq_{Lt} \widetilde{X}.$$

The $\mathcal{L}$-class thus consists of all life distributions such that the equilibrium transformation (2.3) is larger than the original one in the $\preceq_{Lt}$-order.

(v) *Let $X \in \mathcal{L}$; then $\alpha_t \leq P[X > t]$ for $t \leq EX$ and $P[X > t] \leq 1 - \alpha_t$ for $t > EX$, where*

$$\alpha_t = \inf\left\{\alpha | \inf_{s>0} (\exp(st/EX) - (1+s)(1 - \alpha + \alpha \exp(-s(1-t/EX)/\alpha)) \geq 0\right\}.$$

The bounds are sharp.

Let $X \in \overline{\mathcal{L}}$; then

$$P[X > t] \leq \begin{cases} \inf_{s>0} \frac{s}{(s+t/EX)(1-\exp(-s))} & \text{if } t < 2EX \\ EX/t \text{ if } t \geq 2EX. \end{cases}$$

The bounds are sharp.

Proof. The proofs of these assertions can be found in Klefsjö (1983, Section 3.2) for (i), in Klefsjö (1983, Section 3.3) for (ii), in Bhattacharjee and Sengupta (1996) for (iii), in Lin (1998a, Theorem 4) and in Sengupta (1995) for (v). $\diamond$

It is interesting to mention that Sengupta (1995) provides bounds for the distribution functions of $\preceq_{Lt}$-ordered random variables, thus generalizing (v). Extensions of the concept of the $\mathcal{L}$-class can be obtained either by replacing $\preceq_{Lt}$ by another stochastic ordering or by substituting another parametric family of distributions for the exponential one. In that respect, Lin (1998b) defined an analog of the $\mathcal{L}$-class by requiring the life distribution to be smaller than a gamma distribution in the $\preceq_{Lt}$-sense; see also Bartoszewicz (2000).

Let us now prove the following result.

Proposition 2.9 *Let $\{X_\theta,\ \theta \in I\!R_0^+\}$ be a family of exponentially distributed random variables with mean θ. Then, given any non-negative random variable Θ, X_Θ belongs to the $\overline{\mathcal{L}}$-class.*

Proof. Since $\Theta \preceq_{icv} E\Theta$ (where $E\Theta$ denotes a random variable equal to $E\Theta$ almost surely), we have from (1.6) that $E\Theta \preceq_{Lt} \Theta$. From Example 2.7(i), we then get that $X_{E\Theta} \preceq_{Lt} X_\Theta$, where $X_{E\Theta}$ is exponentially distributed with mean $E\Theta = EX_\Theta$, hence the announced result. $\diamond$

We thus have shown that any mixture of exponential distributions belong to the $\overline{\mathcal{L}}$-class. As a consequence, the Pareto distribution belongs to the $\overline{\mathcal{L}}$-class, but also all the distributions studied in Hesselager, Wang and Willmot (1997), as well as the Pareto mixtures of exponentials considered in Rolski, Schmidli, Schmidt and Teugels (1999, pp. 62). Another argument yielding Proposition 2.9 consists in noting that any mixture of exponentials is DFR, and thus belongs to the $\overline{\mathcal{L}}$-class.

3 Properties of the multivariate $\preceq_{Lt}$

3.1 Multivariate and univariate versions of $\preceq_{Lt}$

From their definition, the multivariate and univariate versions of $\preceq_{Lt}$ are closely related. This is formalized in the next result.

Proposition 3.1 *Given two non-negative random vectors* $\mathbf{X}$ *and* $\mathbf{Y}$*,*

$$\mathbf{X} \preceq_{Lt} \mathbf{Y} \Leftrightarrow \mathbf{t}'\mathbf{X} \preceq_{Lt} \mathbf{t}'\mathbf{Y} \text{ for all } \mathbf{t} \geq \mathbf{0}.$$

Proof. Let us first show the "$\Rightarrow$"-part. It is easily seen that for any $v \geq 0$

$$E \exp\left(-v\mathbf{t}'\mathbf{X}\right) = E \exp\left(-\tilde{\mathbf{t}}'\mathbf{X}\right) \leq E \exp\left(-\tilde{\mathbf{t}}'\mathbf{Y}\right) = E \exp\left(-v\mathbf{t}'\mathbf{Y}\right),$$

where $\tilde{\mathbf{t}}$ stands for $v\mathbf{t}$. The "$\Leftarrow$"-part is obvious. $\diamond$

Thus, saying that two random vectors $\mathbf{X}$ and $\mathbf{Y}$ are ordered in the $\preceq_{Lt}$-sense boils down to require that any non-negative linear combinations of the components of $\mathbf{X}$ and $\mathbf{Y}$ are ordered in the univariate $\preceq_{Lt}$-sense. In that respect, many properties of the multivariate extension of $\preceq_{Lt}$ can be obtained from the results derived in Section 2.

As pointed out by Dhaene and Goovaerts (1996), a relevant problem in actuarial sciences is to find an order relation between two random vectors such that the stop-loss order holds between the sums of their components. In view of the works of Dhaene and Goovaerts (1996) and Müller (1997b), the supermodular order satisfies this requirement. Proposition 3.1 shows that the multivariate extension (1.17) of $\preceq_{Lt}$ is such that this property holds with the univariate $\preceq_{Lt}$ substituted for the stop-loss order.

3.2 Links with other orderings

Let us now carefully examine the links existing between $\preceq_{Lt}$ and other classical multivariate stochastic orderings that can be found in the literature. To be specific, we consider the orderings defined in Table 1: since they are all integral stochastic orderings, we have listed their generating cone of functions $\mathcal{F}$ involved in (1.1). Let us recall that for a function $\phi : I\!R^n \to I\!R$, the componentwise difference operator is defined as follows:

$$\Delta_i^{\epsilon}\phi(\mathbf{x}) = \phi(\mathbf{x} + \epsilon\mathbf{e}_i) - \phi(\mathbf{x}), \quad i = 1, 2, \ldots, n, \tag{3.1}$$

where $\epsilon > 0$ and where $\mathbf{e}_i$ denotes the ith unit vector of $I\!R^n$.

Proposition 3.2 *The following implications hold true for* $\preceq_{Lt}$*: given two non-negative random vectors* $\mathbf{X}$ *and* $\mathbf{Y}$*,* $\mathbf{X} \preceq_{st} \mathbf{Y} \Rightarrow \mathbf{Y} \preceq_{Lt} \mathbf{X}$*,* $\mathbf{X} \preceq_{ccx} \mathbf{Y} \Rightarrow \mathbf{X} \preceq_{Lt} \mathbf{Y}$*,* $\mathbf{X} \preceq_{sm} \mathbf{Y} \Rightarrow \mathbf{X} \preceq_{Lt} \mathbf{Y}$*,* $\mathbf{X} \preceq_{\ell o} \mathbf{Y} \Rightarrow \mathbf{X} \preceq_{Lt} \mathbf{Y}$*.*

Proof. It suffices to note that $\phi \in \mathcal{U}_{Lt} \Rightarrow -\phi \in \mathcal{U}_{st}$, and that $\mathcal{U}_{Lt} \subset \mathcal{U}_{ccx}$, $\mathcal{U}_{sm}$ and $\mathcal{U}_{\ell o}$. $\diamond$

In the light of the preceding result, $\preceq_{Lt}$ thus appears as a very weak multivariate order, since a $\preceq_{Lt}$-ranking is induced by a corresponding ranking for most integral stochastic orderings used in actuarial sciences.

Ordering	Generator $\mathcal{F}$ and usual denomination
$\preceq_{st}$	$\mathcal{U}_{st} = \{\phi : I\!R^n_+ \to I\!R \mid \phi \text{ non-decreasing}\}$ Stochastic dominance
$\preceq_{ccx}$	$\mathcal{U}_{ccx} = \{\phi : I\!R^n_+ \to I\!R \mid \phi \text{ componentwise convex}\}$ Componentwise convex order
$\preceq_{sm}$	$\mathcal{U}_{sm} = \{\phi : I\!R^n_+ \to I\!R \mid \Delta_i^\epsilon \Delta_j^\delta \phi \geq 0 \text{ for all } i \neq j, \epsilon, \delta\}$ Supermodular order
$\preceq_{\ell o}$	$\mathcal{U}_{\ell o} = \left\{\phi : I\!R^n_+ \to I\!R \mid \phi = \prod \phi_i \text{ with } \phi_i : I\!R^+ \to I\!R^+ \text{ non-increasing}\right\}$ Lower orthant order

Table 1: Some classical multivariate integral stochastic orderings defined by (1.1) among non-negative n-dimensional random vectors.

3.3 An alternative generator

In the introduction, we pointed out that $\preceq_{Lt}$ is the integral stochastic ordering by the class $\mathcal{U}_{Lt}$. We aim here to determine another class of functions generating $\preceq_{Lt}$.

Proposition 3.3 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two non-negative random vectors; then,*

$$\mathbf{X} \preceq_{Lt} \mathbf{Y} \Leftrightarrow E\prod_{i=1}^n \phi_i(X_i) \leq E\prod_{i=1}^n \phi_i(Y_i) \text{ for all } \phi_i \in \mathcal{U}_{cm},$$

provided that the expectations exist.

Proof. The "$\Leftarrow$"-part is obvious since $\mathcal{U}_{Lt} \in \mathcal{U}_{cm}$. For the "$\Rightarrow$"-part, we use the fact that given a function $\phi_i \in \mathcal{U}_{cm}$, $i = 1, 2, \ldots, n$, there exists a measure μ_i on $I\!R^+$ such that the representation (2.1) holds. Then, by Fubini's theorem,

$$E\prod_{i=1}^n \phi_i(X_i) = \int_{t_1=0}^{+\infty}\int_{t_2=0}^{+\infty}\ldots\int_{t_n=0}^{+\infty} E\exp\left(-\mathbf{t}'\mathbf{X}\right) d\mu_1(t_1)d\mu_2(t_2)\ldots d\mu_n(t_n),$$

whence the desired result follows. $\diamond$

Let us mention the following immediate consequence of Proposition 3.3. Let $\varphi_1, \varphi_2, \ldots, \varphi_n : I\!R^+ \to I\!R^+$ be continuous functions such that $\varphi_j^{(1)} \in \mathcal{U}_{cm}$ for $j = 1, 2, \ldots, n$. Then, given two non-negative random vectors $\mathbf{X}$ and $\mathbf{Y}$,

$$\mathbf{X} \preceq_{Lt} \mathbf{Y} \Rightarrow \left(\varphi_1(X_1), \varphi_2(X_2), \ldots, \varphi_n(X_n)\right) \preceq_{Lt} \left(\varphi_1(Y_1), \varphi_2(Y_2), \ldots, \varphi_n(Y_n)\right).$$

To establish the validity of the latter implication, it suffices to invoke Criterion 2 of Feller (1966, page 417) ensuring that $\phi \circ \varphi_j \in \mathcal{U}_{cm}$ for any $\phi \in \mathcal{U}_{cm}$.

3.4 Closure properties

Let us now detail the main properties of $\preceq_{Lt}$.

Proposition 3.4 **(i)** *Closure under mixture: let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two non-negative random vectors, and let* $\mathbf{\Theta}$ *be a random vector valued in* $\mathcal{T} \subseteq \mathbb{R}^n$*; then*

$$[\mathbf{X}|\mathbf{\Theta} = \theta] \preceq_{Lt} [\mathbf{Y}|\mathbf{\Theta} = \theta] \text{ for all } \theta \in \mathcal{T} \Rightarrow \mathbf{X} \preceq_{Lt} \mathbf{Y}.$$

(ii) *Functional invariance: let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two non-negative random vectors; then,*

$$\mathbf{X} \preceq_{Lt} \mathbf{Y} \Rightarrow \varphi(\mathbf{t}'\mathbf{X}) \preceq_{Lt} \varphi(\mathbf{t}'\mathbf{Y})$$

for any $\mathbf{t} \geq \mathbf{0}$ *and* φ *such that* $\varphi^{(1)} \in \mathcal{U}_{cm}$.

(iii) *Closure under convolution: let* $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ *(resp.* $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n$*) be non-negative independent random vectors; then*

$$\mathbf{X}_i \preceq_{Lt} \mathbf{Y}_i \text{ for } i = 1, 2, \ldots, n \Rightarrow \sum_{i=1}^{n} \mathbf{X}_i \preceq_{Lt} \sum_{i=1}^{n} \mathbf{Y}_i.$$

(iv) *Closure under marginalization: let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two non-negative random vectors; then*

$$\mathbf{X} \preceq_{Lt} \mathbf{Y} \Rightarrow X_i \preceq_{Lt} Y_i \text{ for } i = 1, 2, \ldots, n. \tag{3.2}$$

The reverse implication in (3.2) also holds when the random vectors $\mathbf{X}$ *and* $\mathbf{Y}$ *have independent components. Further, for any integers* $1 \leq i_1 \leq i_2 \leq \ldots \leq i_k \leq n$ *and* $k \leq n$,

$$\mathbf{X} \preceq_{Lt} \mathbf{Y} \Rightarrow (X_{i_1}, X_{i_2}, \ldots, X_{i_k}) \preceq_{Lt} (Y_{i_1}, Y_{i_2}, \ldots, Y_{i_k}).$$

(v) *Closure under compounding: let* $\mathbf{N}$ *be a random vector with non-negative integer-valued components and let* $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \ldots$ *and* $\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3, \ldots$ *be two sequences of non-negative independent random vectors;* $\mathbf{N}$*, the* $\mathbf{X}_i$*'s and the* $\mathbf{Y}_j$*'s being mutually independent. Then,*

$$\mathbf{X}_i \preceq_{Lt} \mathbf{Y}_i \text{ for all } i \Rightarrow \left(\sum_{i=1}^{N_1} X_{i1}, \sum_{i=1}^{N_2} X_{i2}, \ldots, \sum_{i=1}^{N_n} X_{in}\right) \preceq_{Lt} \left(\sum_{i=1}^{N_1} Y_{i1}, \sum_{i=1}^{N_2} Y_{i2}, \ldots, \sum_{i=1}^{N_n} Y_{in}\right)$$

with the convention that the empty sum is zero, where X_{ij} *(resp.* Y_{ij}*) denotes the* j*th component of* $\mathbf{X}_i$ *(resp.* $\mathbf{Y}_i$*). In particular, given a non-negative integer-valued random variable* N,

$$\mathbf{X}_i \preceq_{Lt} \mathbf{Y}_i \text{ for all } i \Rightarrow \sum_{i=1}^{N} \mathbf{X}_i \preceq_{Lt} \sum_{i=1}^{N} \mathbf{Y}_i.$$

(vi) *Let X_{ij}, $i, j = 1, 2, 3, \ldots$, be independent random variables, identically distributed for a given j. Then, given two non-negative integer-valued random vectors $\mathbf{N}$ and $\mathbf{M}$, independent of the X_{ij}'s,*

$$\mathbf{N} \preceq_{Lt} \mathbf{M} \Rightarrow \left(\sum_{i=1}^{N_1} X_{i1}, \sum_{i=1}^{N_2} X_{i2}, \ldots, \sum_{i=1}^{N_n} X_{in} \right) \preceq_{Lt} \left(\sum_{i=1}^{M_1} X_{i1}, \sum_{i=1}^{M_2} X_{i2}, \ldots, \sum_{i=1}^{M_n} X_{in} \right).$$

(vii) *Closure under concatenation: let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ (resp. $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n$) be non-negative independent random vectors such that $\mathbf{X}_i \preceq_{Lt} \mathbf{Y}_i$ holds for $i = 1, 2, \ldots, n$; then,*

$$\mathbf{X}_c = (\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n) \preceq_{Lt} \mathbf{Y}_c = (\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n).$$

Proof. (i) is obvious. (ii) is a straight corollary of Property 2.3(ii) together with Proposition 3.1. (iii) follows from

$$\begin{aligned}
\mathbf{X}_i \preceq_{Lt} \mathbf{Y}_i \;&\Leftrightarrow\; \mathbf{t}'\mathbf{X}_i \preceq_{Lt} \mathbf{t}'\mathbf{Y}_i \text{ for all } \mathbf{t} \geq \mathbf{0} \text{ by Proposition 3.1} \\
&\Rightarrow\; \mathbf{t}' \sum_{i=1}^{n} \mathbf{X}_i \preceq_{Lt} \mathbf{t}' \sum_{i=1}^{n} \mathbf{Y}_i \text{ for all } \mathbf{t} \geq \mathbf{0} \text{ by Property 2.3(iii)} \\
&\Leftrightarrow\; \sum_{i=1}^{n} \mathbf{X}_i \preceq_{Lt} \sum_{i=1}^{n} \mathbf{Y}_i \text{ by Proposition 3.1.}
\end{aligned}$$

(iv) is obvious. The proof of (v) is as follows. Let us first prove that, for any $\mathbf{k} \in \mathbb{N}^n$, the stochastic inequality

$$\left(\sum_{i=1}^{k_1} X_{i1}, \sum_{i=1}^{k_2} X_{i2}, \ldots, \sum_{i=1}^{k_n} X_{in} \right) \preceq_{Lt} \left(\sum_{i=1}^{k_1} Y_{i1}, \sum_{i=1}^{k_2} Y_{i2}, \ldots, \sum_{i=1}^{k_n} Y_{in} \right). \tag{3.3}$$

holds. Let $k_{(1)}, k_{(2)}, \ldots, k_{(n)}$ denote the k_i's arranged in ascending order and let $k_{(i)} = k_{j_i}$. Then, it suffices to invoke (iv) to get

$$\begin{array}{rcl}
(X_{11}, X_{12}, \ldots, X_{1n}) & \preceq_{Lt} & (Y_{11}, Y_{12}, \ldots, Y_{1n}) \\
(X_{21}, X_{22}, \ldots, X_{2n}) & \preceq_{Lt} & (Y_{21}, Y_{22}, \ldots, Y_{2n}) \\
\vdots & \vdots & \vdots \\
(X_{k_{(1)}1}, X_{k_{(1)}2}, \ldots, X_{k_{(1)}n}) & \preceq_{Lt} & (Y_{k_{(1)}1}, Y_{k_{(1)}2}, \ldots, Y_{k_{(1)}n})
\end{array}$$

$$(X_{k_{(1)}+1;1}, X_{k_{(1)}+1;2}, \ldots, X_{k_{(1)}+1;j_1-1}, 0, X_{k_{(1)}+1;j_1+1}, \ldots, X_{k_{(1)}+1;n})$$
$$\preceq_{Lt} (Y_{k_{(1)}+1;1}, Y_{k_{(1)}+1;2}, \ldots, Y_{k_{(1)}+1;j_1-1}, 0, Y_{k_{(1)}+1;j_1+1}, \ldots, Y_{k_{(1)}+1;n})$$

$$\begin{array}{rcl}
\vdots & \vdots & \vdots \\
(0, 0, \ldots, 0, X_{k_{(n)};j_n}, 0, \ldots) & \preceq_{Lt} & (0, 0, \ldots, 0, Y_{k_{(n)};j_n}, 0, \ldots).
\end{array}$$

Considering (iii), we get (3.3) by summing up all these stochastic inequalities. Now, in virtue of (i) with $\mathbf{\Theta} = \mathbf{N}$, we obtain the announced result. (vi) follows from

$$E \exp\left(-\sum_{j=1}^{n} t_j \sum_{i=1}^{N_j} X_{ij}\right) = E \prod_{i=1}^{n} \phi_j(N_j)$$

where

$$n \mapsto \phi_j(n) = E \exp\left(-t_j \sum_{i=1}^{n} X_{ij}\right) = \exp\left\{n \ln E \exp(-t_j X_{1j})\right\} \in \mathcal{U}_{Lt}^{I\!N}.$$

Finally, let us prove (vii). From Proposition 3.1, we know that $\mathbf{t}_i'\mathbf{X}_i \preceq_{Lt} \mathbf{t}_i'\mathbf{Y}_i$ holds for $i = 1, 2, \ldots, n$ and for any $\mathbf{t} \geq \mathbf{0}$. Now, from Property 2.3(iii), we get that

$$\mathbf{t}_i'\mathbf{X}_i \preceq_{Lt} \mathbf{t}_i'\mathbf{Y}_i \text{ for } i = 1, 2, \ldots, n \Rightarrow \sum_{i=1}^{n} \mathbf{t}_i'\mathbf{X}_i \preceq_{Lt} \sum_{i=1}^{n} \mathbf{t}_i'\mathbf{Y}_i,$$

which is equivalent to $\mathbf{t}'\mathbf{X}_c \preceq_{Lt} \mathbf{t}'\mathbf{Y}_c$ for all $\mathbf{t} \geq 0$. Using Proposition 3.1, we finally deduce that $\mathbf{X}_c \preceq_{Lt} \mathbf{Y}_c$ holds, as announced. $\diamond$

4 Applications

4.1 Multivariate counting distribution

Let us examine the general model for multivariate counting distributions recently considered by Ambagaspitiya (1998). This author investigates the family of the n-dimensional discrete random vectors $\mathbf{N}$ that can be represented as

$$\mathbf{N} = \mathbf{A}\mathbf{Z}^{\perp}, \tag{4.1}$$

where $\mathbf{A} = (a_{ij})$ is a $n \times k$ non-negative integer element matrix and $\mathbf{Z}^{\perp}$ is a k-dimensional random vector valued in $I\!N^k$ with independent components. The independence of the components in $\mathbf{Z}^{\perp}$ yields

$$E \exp(-\mathbf{t}'\mathbf{N}) = E \exp(-\mathbf{Z}^{\perp\prime}\mathbf{A}'\mathbf{t}) = \prod_{i=1}^{k} E \exp\left(-\mathbf{A}_i'\mathbf{t} Z_i^{\perp}\right),$$

with $\mathbf{A}_i$ the ith column of $\mathbf{A}$. Now, let $\widetilde{\mathbf{N}}$ be of the form $\widetilde{\mathbf{A}}\mathbf{Z}^{\perp}$; we thus have that

$$\mathbf{A}_i'\mathbf{t} \geq \widetilde{\mathbf{A}}_i'\mathbf{t} \text{ for } i = 1, 2, \ldots, k \text{ and all } \mathbf{t} \geq \mathbf{0} \Rightarrow \mathbf{N} \preceq_{Lt} \widetilde{\mathbf{N}}.$$

Similarly, if $\widetilde{\mathbf{N}}$ is of the form $\mathbf{A}\widetilde{\mathbf{Z}}^{\perp}$, we have that

$$Z_i^{\perp} \preceq_{Lt} \widetilde{Z}_i^{\perp} \text{ for } i = 1, 2, \ldots, n \Rightarrow \mathbf{N} \preceq_{Lt} \widetilde{\mathbf{N}}.$$

Let us now consider an insurance company which sells a product including several types of coverages. Think for instance of a travel insurance contract, including medical costs, repatriation costs, a sum in case of death, a sum in case of disablement (proportional to the degree of disability), and so on. It is natural to assume that the losses for each type of coverage can be expressed in terms of compound correlated random variables (at the portfolio level). Let us suppose that there are n different kinds of coverage and denote as S_{N_k} the aggregate loss for the kth type of coverage. Then the losses for the whole portfolio are given by

$$\mathbf{S} = (S_{N_1}, S_{N_2}, \ldots, S_{N_n})'; \text{ with } S_{N_k} = \sum_{i=1}^{N_k} X_{ik}, \ \ k = 1, 2, \ldots, n,$$

where N_k is the number of claims and X_{ik}, $i = 1, 2, \ldots$, are the claims sizes. The total aggregate claims is then given by $T = \mathbf{S'1}$. We make the following assumptions: the N_k's are correlated and distributed as in (4.1), the X_{ik}'s are independent for all i and k (and identically distributed for a given k); claims sizes are independent of claim numbers. Thus the dependence in the model arises from the dependence among the components of $\mathbf{N}$, only. Considering two portfolios differing only in their respective claim frequency vectors $\mathbf{N}$ and $\widetilde{\mathbf{N}}$, we get from Proposition 3.4 (vi) that

$$\mathbf{N} \preceq_{Lt} \widetilde{\mathbf{N}} \Rightarrow \mathbf{S} \preceq_{Lt} \widetilde{\mathbf{S}}.$$

It then follows from Proposition 3.1 that $T \preceq_{Lt} \widetilde{T}$.

4.2 Ordering Markov chains

Let $\mathcal{X} = \{X_t, \ t \in \mathcal{T}\}$ and $\mathcal{Y} = \{Y_t, \ t \in \mathcal{T}\}$ be two stochastic processes with state space $\mathcal{S} \subseteq I\!R$ and time parameter space $\mathcal{T}$ (usually $\mathcal{T} = I\!R^+$ or $I\!N$). Suppose that for all choices of an integer n and $t_1 < t_2 < \ldots < t_n \in \mathcal{T}$, it holds that

$$(X_{t_1}, X_{t_2}, \ldots, X_{t_n}) \preceq_{Lt} (Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n});$$

then, $\mathcal{X}$ is said to be smaller that $\mathcal{Y}$ in the $\preceq_{Lt}$-sense, written as $\mathcal{X} \preceq_{Lt} \mathcal{Y}$.

As an example, consider the following process defined by Alzaid, Kim and Proschan (1991). Let $I\!N_0$ be the set of the positive integers. Let $\{X_n, \ n \in I\!N_0\}$ be a sequence of non-negative independent random variables describing random influences on the state of a system at time n. For $n = 1, 2, \ldots$, the system at time $n+1$ is described by a random variable $Z_{n+1} = \Phi_n(Z_n, X_n)$, where Φ_n is a non-negative function having a completely monotone derivative with respect to the first variable. The process $\{Z_n, \ n \in I\!N_0\}$ is a Markov chain, and if $\Phi_n \equiv \Phi$ it also has stationary transition probabilities.

Alzaid *et al.* (1991, Theorem 3.9) obtained the following result: Let $\{Z_n, \ n \in I\!N_0\}$ be defined as above with a sequence $\{X_n, \ n \in I\!N_0\}$ of independent and identically distributed random variables and $\Phi_n \equiv \Phi$. Then,

$$Z_1 \preceq_{Lt} Z_2 \Rightarrow Z_n \preceq_{Lt} Z_{n+1} \text{ for all } n.$$

Similarly, it is easy to prove that

$$Z_2 \preceq_{Lt} Z_1 \Rightarrow Z_{n+1} \preceq_{Lt} Z_n \text{ for all } n.$$

Indeed, it suffices to proceed by induction. Assume it holds for n and let us prove it for $n+1$. For any $\phi \in \mathcal{U}_{cm}$, we have that

$$\begin{aligned} E\phi(Z_{n+1}) = E\phi(\Phi(Z_n, X_n)) &= E\left\{E[\phi(\Phi(Z_n, X_n))|X_n]\right\} \\ &\leq E\left\{E[\phi(\Phi(Z_{n-1}, X_n))|X_n]\right\} \\ &= E\phi(\Phi(Z_{n-1}, X_n)) \\ &= E\phi(\Phi(Z_{n-1}, X_{n-1})), \end{aligned}$$

where the inequality follows from Property 2.3(i) and (ii). It is worth mentioning that the conditions $Z_1 \preceq_{Lt} Z_2$ or $Z_2 \preceq_{Lt} Z_1$ are in fact conditions on the transition function Φ.

They also proved that if $\{Z_n^{(1)},\ n \in \mathbb{N}_0\}$ and $\{Z_n^{(2)},\ n \in \mathbb{N}_0\}$ are two Markov chains defined with the same Φ_n's and different sequences $\{X_n^{(1)},\ n \in \mathbb{N}_0\}$ and $\{X_n^{(2)},\ n \in \mathbb{N}_0\}$,

$$X_n^{(1)} \preceq_{Lt} X_n^{(2)} \text{ for all } n \Leftrightarrow \{X_n^{(1)},\ n \in \mathbb{N}_0\} \preceq_{Lt} \{X_n^{(2)},\ n \in \mathbb{N}_0\} \Rightarrow Z_n^{(1)} \preceq_{Lt} Z_n^{(2)} \text{ for all } n. \tag{4.2}$$

Let us now apply these results in credibility theory. Let X_j be the claim amount of a given insurance policy for the jth year it is in force, $j \in \mathbb{N}_0$. The pure premium of this policy is μ. Assume that the claim amounts for different years are independent. After n years, we have observed $X_1, X_2, \ldots, X_n$. The insurance company may want to charge for the $(n+1)$-th year an amount of

$$Z_{n+1} = \alpha_n \sum_{i=1}^{n} X_i + (1-\alpha_n)n\mu,$$

where $\{\alpha_n,\ n \in \mathbb{N}\}$ is a non-decreasing sequence of numbers in $[0,1]$. In such a case,

$$\Phi_n(z,x) = \frac{\alpha_n}{\alpha_{n-1}} z + \alpha_n x + R(\alpha_n, \alpha_{n-1}, \mu, n),$$

where R takes all the remaining terms. It is easily seen that $\Phi_n(z,x)$ has a completely monotone derivative with respect to z so that (4.2) holds. Let us now prove that in this case we even have

$$\{X_n^{(1)},\ n \in \mathbb{N}_0\} \preceq_{Lt} \{X_n^{(2)},\ n \in \mathbb{N}_0\} \Rightarrow \{Z_n^{(1)},\ n \in \mathbb{N}_0\} \preceq_{Lt} \{Z_n^{(2)},\ n \in \mathbb{N}_0\}.$$

For this purpose, it suffices to show that $\mathbf{S}_1 \preceq_{Lt} \mathbf{S}_2$ holds, where

$$\mathbf{S}_i = \left(\alpha_{k_1} \sum_{j=1}^{k_1} X_j^{(i)}, \alpha_{k_2} \sum_{j=1}^{k_2} X_j^{(i)}, \ldots, \alpha_{k_n} \sum_{j=1}^{k_n} X_j^{(i)}\right),\ \ i = 1, 2,$$

with $k_1 < k_2 < \ldots < k_n \in \mathbb{N}_0$. This stochastic inequality simply follows from

$$\begin{aligned} E\exp(-\mathbf{t}'\mathbf{S}_i) &= E\exp\left(-\sum_{\ell=1}^{n} t_\ell \alpha_{k_\ell} \sum_{j=1}^{k_\ell} X_j^{(i)}\right) \\ &= \prod_{m=1}^{n} E\exp\left(-\left(\sum_{\ell=m}^{n} t_\ell \alpha_{k_\ell}\right) \sum_{j=k_{m-1}+1}^{k_m} X_j^{(i)}\right), \end{aligned}$$

with the convention that $k_0 = 0$. We thus have proven that if the annual amounts of claims are ordered in the $\preceq_{Lt}$-sense, then the sequence of the annual premiums are ordered in the same way.

4.3 Portfolios with dependent risks

Let us consider the model recently defined by Marceau *et al.* (1999). This model allows for dependence between the risks of an insurance portfolio in the individual risk model. Consider a portfolio consisting of n policies with claim amounts $X_1, X_2, \ldots, X_n$. Let S be the aggregate claim amount for the insurance portfolio, i.e. $S = \sum_{i=1}^n X_i$. Let X_i, $i = 1, 2, \ldots, n$, be of the form $X_i = I_i B_i$ where the I_i's are Bernoulli random variables such that $P[I_i = 1] = p_i$ and $P[I_i = 0] = q_i$, $p_i + q_i = 1$, and the B_i's are independent and identically distributed positive random variables. Assume further that the random vectors $(I_1, I_2, \ldots, I_n)$ and $(B_1, B_2, \ldots, B_n)$ are mutually independent. Now, suppose that the I_i's satisfy

$$I_i = \min\{J_i + J_0, 1\}, \;\; i = 1, 2, \ldots, n,$$

where the J_i's, $i = 0, 1, \ldots, n$, are independent Bernoulli random variables with $P[J_i = 1] = p_{ii}$ and $P[J_i = 0] = q_{ii}$, $p_{ii} + q_{ii} = 1$. The random vector $(I_1, I_2, \ldots, I_n)$ has clearly dependent components, so that $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ also has. Nevertheless, the I_i's remain Bernoulli distributed with parameters $q_i = q_{00}q_{ii}$ and $p_i = 1 - q_{00}q_{ii}$. Obviously, if $q_{00} = 1$, we find the independent case.

Now, let us consider two portfolio's differing only in their respective parameters $p_{00}^{(1)}$ and $p_{00}^{(2)}$. Then,

$$\begin{aligned}
E\exp(-\mathbf{t}'\mathbf{X}^{(1)}) &= E\exp\left(-\sum_{i=1}^n t_i \min\{J_0^{(1)} + J_i, 1\} B_i\right) \\
&= p_{00}^{(1)} E\exp\left(-\sum_{i=1}^n t_i B_i\right) + q_{00}^{(1)} E\exp\left(-\sum_{i=1}^n t_i J_i B_i\right) \\
&= p_{00}^{(1)} \left(E\exp\left(-\sum_{i=1}^n t_i B_i\right) - E\exp\left(-\sum_{i=1}^n t_i J_i B_i\right)\right) + E\exp\left(-\sum_{i=1}^n t_i J_i B_i\right).
\end{aligned}$$

Since we obviously have that

$$E\exp\left(-\sum_{i=1}^n t_i B_i\right) \leq E\exp\left(-\sum_{i=1}^n t_i J_i B_i\right),$$

we get

$$p_{00}^{(1)} \leq p_{00}^{(2)} \Rightarrow \mathbf{X}^{(2)} \preceq_{Lt} \mathbf{X}^{(1)}.$$

In particular, the corresponding aggregate claims satisfy $S^{(2)} \preceq_{Lt} S^{(1)}$. The latter stochastic inequality is very natural: a higher value of p_{00} obviously makes the portfolio more dangerous. In fact, the simple structure of the model allows us to order the $S^{(i)}$'s according to stronger orderings than $\preceq_{Lt}$ when $p_{00}^{(1)} \leq p_{00}^{(2)}$. Let $\phi : I\!R^+ \to I\!R$ be a non-decreasing function; then

$$E\phi(S^{(1)}) = p_{00}^{(1)} \left(E\phi\left(\sum_{i=1}^n B_i\right) - E\phi\left(\sum_{i=1}^n J_i B_i\right)\right) + E\phi\left(\sum_{i=1}^n J_i B_i\right).$$

so that

$$p_{00}^{(1)} \leq p_{00}^{(2)} \Rightarrow S^{(1)} \preceq_{st} S^{(2)}.$$

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