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Bernstein Estimator for Unbounded Density Copula

Taoufik Bouezmarni* Anouar El Gouch† Abderrahim Taamouti‡

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ABSTRACT

We study the asymptotic properties of the Bernstein estimator for unbounded density copula functions. We show that the estimator converges to infinity at the corner. We establish its relative convergence when the copula is unbounded and we provide the uniform strong consistency of the estimator on every compact in the interior region. We also check the finite simple performance of the estimator via an extensive simulation study and we compare it with other well known nonparametric methods. Finally, we consider an empirical application where the asymmetric dependence between international equity markets (US, Canada, UK, and France) is re-examined.

Key words: Unbounded copula, Nonparametric estimation; Bernstein polynomial; asymptotic properties; uniform strong consistency; relative convergence; boundary bias.

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1 Introduction

The copula function has the advantage to model completely the dependence among variables. In fact, any continuous joint distribution function can be controlled by the marginal distributions, which give the information on each component, and a *unique* copula that captures the dependence between components. This gives rise to a flexible two step modelling approach where in the first step one models the marginal distributions and in the second one characterizes the dependence using a copula function. In finance, for example, copulas are a powerful tool for modelling dependence between risky assets, and are applicable in multi-asset pricing, credit portfolio modelling, risk management, etc. The aim of the present paper is to investigate the properties of the nonparametric Bernstein estimator of the copula density. Although many common families of copula are unbounded (e.g. Clayton, Gumbel, Gaussian and Student), the properties of the Bernstein estimator have been studied only under the boundedness condition of the copula density at the corners. Hence, we examine the asymptotic properties of the Bernstein estimator for unbounded density copula functions.

Several approaches have been proposed to estimate the copula functions. First, a parametric approach that imposes a specific model for the copula that is known up to some parameters. The latter can be estimated using the maximum likelihood method. This approach is widely used in practice because of its simplicity; see Joe (1997) and Nelsen (2006) for textbook details. The second possibility is a semiparametric approach that assumes a parametric model for the copula and a nonparametric model for the marginal distributions. Liebscher (2005) proposes to estimate the density function based on semiparametric copulas and on the standard kernel estimator for the marginal densities, which solves the curse of dimensionality problem but not the boundary problem. Bouezmarni and Rombouts (2008) estimate the multivariate density function using semiparametric copula and asymmetric kernels for the marginal densities, which allows them to address the boundary and the curse of dimensionality problems simultaneously. In a recent paper, Kim et al. (2007) compare semiparametric and parametric methods for estimating copulas. The third way of estimating copulas is based on a fully nonparametric approach. The advantage of this approach is its flexibility to adapt to any kind of dependence structure. An important contribution is Deheuvels (1979) who suggests the multivariate empirical distribution to estimate the copula function. Gijbels and Mielniczuk (1990) estimate a bivariate copula using smoothing kernel methods. They also suggest the reflection method in order to solve the well known boundary bias problem of the kernel methods. Chen and Huang (2007) propose a bivariate estimator based on the local linear estimator, which is consistent everywhere in the support of the copula function, and Rödel (1987) uses the orthogonal series method.

Motivated by Weierstrass theorem, Bernstein polynomials are considered by Lorentz (1953) who proves that any continuous function can be approximated by Bernstein polynomials. For density functions, estimation using the Bernstein polynomial is suggested by Vitale (1975) and with a slight modification by Grawronski and Stadtmüller (1981). Tenbusch (1994) investigates the Bernstein estimator for bivariate density functions and Bouezmarni and Rolin (2007) prove the consistency of Bernstein estimator for unbounded probability density functions. Kakizawa (2004) and Kakizawa (2006) consider the Bernstein polynomial to estimate density and spectral density functions, respectively. Tenbusch (1997) and Brown and Chen (1999) propose estimators of the regression functions based on the Bernstein polynomial. In the Bayesian context, Bernstein polynomials are explored by Petrone (1999a), Petrone (1999b), Petrone and Wasserman (2002), and Ghosal (2001). The copula Bernstein estimator was first studied by Sancetta and Satchell (2004) for independent and identically distributed (i.i.d.) data and by Bouezmarni et al. (2010) for dependent data.

In this paper, we focus our attention on the behavior of the Bernstein copula estimator at the boundary regions. In finance, for example, having a good estimator of the copula density at the boundary region is essential for obtaining a valid risk evaluation (risk management). Without assuming any unnecessarily assumption like the existence of the first derivative, we are able to prove that the Bernstein estimator converges to infinity at the corner when the density copula is unbounded. To show the performance of the Bernstein density copula, we establish the relative weak and strong convergence at the boundary region. The uniform strong consistency on each compact in the interior region is also provided. Further, we ran a simulation study to investigate the finite sample properties of the Bernstein estimator for the copula density. The results show that this estimator has a quite good "performance" compared to many other well known estimators like Local linear estimator, Mirror-reflection estimator, Beta kernel estimator, and the Transformation estimator using multiplicative Epanechnikov kernel and Gaussian transformation. Since the Bernstein estimator depends on a bandwidth parameter, we also investigate the least square cross validation method to select the optimal bandwidth. Finally, we consider an empirical application where the asymmetric dependence between international equity markets (US, Canada, UK, and France) is re-examined. We find that the Bernstein estimator is a good estimator at the extremes. The results show that this estimator is able to capture the well known asymmetric dependence phenomena that is observed in the international equity markets.

This paper is organized as follows. The Bernstein copula estimator is introduced in Section 2. Section 3 provides the asymptotic properties of the Bernstein copula density estimator at the corners. In Section 4, we provide simulation results that show the performance of the Bernstein estimator compared with the other existing nonparametric estimators of copula density. In Section

5, we investigate the least square cross validation method to select the optimal bandwidth. Section 6 presents an empirical illustration using financial data and Section 7 concludes.

2 Bernstein copula estimator

Let $X \equiv (X_1, ..., X_d)^\top$ be a random vector in $\mathbb{R}^d$ with distribution function F and density function f from which an $i.i.d$ sample of length n, say $\left\{X_i \equiv (X_{i1}, ..., X_{id})^\top,\ i = 1, .., n\right\}$, is observed. According to Sklar (1959), the distribution function of X can be expressed via a copula:

$$F(x_1, ..., x_d) = C(F_1(x_1), ..., F_d(x_d)), \tag{1}$$

where F_j, for $j = 1, \ldots, d$, is the marginal distribution function of random variable X_j, and C is a copula function that captures the dependence structure in the vector X. If we derive (1) with respect to $(x_1, ..., x_d)$, we obtain the density function of X that can be expressed as follows:

$$f(x_1, ..., x_d) = \left(\prod_{j=1}^{d} f_j(x_j)\right) \times c\left(F_1(x_1), ..., F_d(x_d)\right),$$

where f_j, for $j = 1, \ldots, d$, is the marginal density of random variable X_j and c is the copula density. Hence, the estimation of the joint density function can be done by estimating the univariate marginal densities and the copula density function.

For simplicity of exposition we take $d = 2$. However, the results in this paper can be generalized for any other dimension d. Sancetta and Satchell (2004) proposed an empirical Bernstein estimator of copula function which is defined as follows:

$$\hat{C}(u, w) = \sum_{\upsilon_1=0}^{k} \sum_{\upsilon_2=0}^{k} C_n\left(\frac{\upsilon_1}{k}, \frac{\upsilon_2}{k}\right) p_{\upsilon_1,k}(u) p_{\upsilon_2,k}(w), \text{ for } (u, w) \in [0, 1]^2, \tag{2}$$

where $k \equiv k_n$ is an integer that depends on the sample size n and plays the role of a bandwidth parameter, C_n is the empirical copula function of the vector X, and $p_{\upsilon_j,k}(s_j)$, for $j = 1, 2$, is the binomial distribution function:

$$p_{\upsilon_j,k}(s_j) = \begin{pmatrix} k \\ \upsilon_j \end{pmatrix} s_j^{\upsilon_j} (1 - s_j)^{k-\upsilon_j}.$$

In this paper, our interest lies in the estimation of the copula density function using Bernstein polynomials. Indeed, if we derive (2) with respect to (u, w) we obtain the following Bernstein density copula:

$$\hat{c}(u, w) = \frac{1}{n} \sum_{i=1}^{n} K_{k,S_i}(u, w), \tag{3}$$

where $S_i = (F_{n1}(X_{i1}), F_{n2}(X_{i2}))$, with $F_{nj}(.)$ is the empirical distribution function of X_{ij},

$$K_{k,S_i}(u,w) = k^2 \sum_{\upsilon_1=0}^{k-1} \sum_{\upsilon_2=0}^{k-1} A_{S_i,\upsilon} p_{\upsilon_1,k-1}(u) p_{\upsilon_2,k-1}(w),$$

and

$$A_{S_i,\upsilon} = \mathbf{1}_{\{S_i \in B_\upsilon\}}, \quad \text{with} \quad B_\upsilon = \left] \frac{\upsilon_1}{k}, \frac{\upsilon_1+1}{k} \right] \times \left] \frac{\upsilon_2}{k}, \frac{\upsilon_2+1}{k} \right].$$

Hereafter, we will denote $p_{\upsilon_j,k-1}(s_j)$ by $p_{\upsilon_j}(s_j)$ and the double sums $\sum_{\upsilon_1=0}^{k-1} \sum_{\upsilon_2=0}^{k-1}$ by $\sum_\upsilon$. Finally, observe that the Bernstein estimator for the density copula function is simple to implement, non-negative, and integrates to one.

3 Main results

We study the asymptotic properties of the Bernstein estimator for unbounded copula densities. Recall that for i.i.d data and when the copula density has a finite first derivative everywhere on its support, Sancetta and Satchell (2004) derive upper bounds of the bias and variance of the Bernstein copula estimator and show the pointwise convergence of the estimator in term of mean squared error. Further, Bouezmarni et al. (2010) provide asymptotic properties of the Bernstein copula density for dependent data and derive the asymptotic bias, asymptotic variance, and show the uniform strong convergence of the Bernstein density copula when the underlying density is continuous on its support. They also establish the asymptotic normality of the estimator. However, although many common families of copula are unbounded at the corners (Clayton, Gumbel, Gaussian and Student copulas), the derivation of the previous results requires the boundedness of the density copula. Hence, in this section we show that the Bernstein estimator is still a consistent estimator even for unbounded copula densities. The following proposition establishes the uniform weak and strong consistency of the Bernstein copula density estimator on any compact I in the interior region without imposing boundedness condition of the copula at the corners.

Proposition 1. *Let $c(.)$ be a continuous copula density function on $(0,1)^2$. Let I be a compact set in $(0,1)^2$, and $\hat{c}(.)$ the Bernstein copula density estimator. We have,*
(i) *If $k_n \to \infty$ and $nk_n^{-4} \to \infty$, then*

$$\sup_{(u,w)\in I} |\hat{c}(u,w) - c(u,w)| \xrightarrow{P} 0, \quad as \quad n \to \infty,$$

(ii) *If $k \to \infty$ and $\frac{nk^{-4}}{\log(n)} \to \infty$, then*

$$\sup_{(u,w)\in I} |\hat{c}(u,w) - c(u,w)| \xrightarrow{a.s} 0, \quad as \quad n \to \infty.$$

■

Proof of Proposition 1 We start with part **(i)**. We can show that the bias term $\mathbb{E}(\hat{c}(u,w)) - c(u,w)$ converges uniformly to zero if the bandwidth parameter k tends to infinity. Thus, for $(u,w) \in I$, we have

$$\begin{aligned}
|\mathbb{E}(\hat{c}(u,w)) - c(u,w)| &= \left| k^2 \sum_v \int_{\frac{v_1}{k}}^{\frac{v_1+1}{k}} \int_{\frac{v_2}{k}}^{\frac{v_2+1}{k}} \{c(t_1,t_2) - c(u,w)\}\, dt_1 dt_2 p_{v_1,k}(u) p_{v_2,k}(w) \right| \\
&= \left| k^2 \mathbb{E}_{(\xi_1,\xi_2)} \left(\int_{\frac{\xi_1}{k}}^{\frac{\xi_1+1}{k}} \int_{\frac{\xi_2}{k}}^{\frac{\xi_2+1}{k}} \{c(t_1,t_2) - c(u,w)\} dt_1 dt_2 \right) \right| \\
&\leq k^2 \mathbb{E}_{(\xi_1,\xi_2)} \left(\int_{\frac{\xi_1}{k}}^{\frac{\xi_1+1}{k}} \int_{\frac{\xi_2}{k}}^{\frac{\xi_2+1}{k}} |c(t_1,t_2) - c(u,w)|\, dt_1 dt_2 \right),
\end{aligned}$$

where ξ_1 and ξ_2 are two independent Binomial random variables with corresponding parameters (k,u) and (k,w), respectively. Let δ be a positive real number such that $\delta < \min(u, 1-u, w, 1-w)$, and denote by $A \equiv \left\{ |\frac{\xi_1}{k} - u| \leq \delta, |\frac{\xi_2}{k} - w| \leq \delta \right\}$, then we have

$$\begin{aligned}
|\mathbb{E}(\hat{c}(u,w)) - c(u,w)| &\leq k^2 \mathbb{E}_{(\xi_1,\xi_2)} \left(\int_{\frac{\xi_1}{k}}^{\frac{\xi_1+1}{k}} \int_{\frac{\xi_2}{k}}^{\frac{\xi_2+1}{k}} |c(t_1,t_2) - c(u,w)|\, \mathbf{1}_A \, dt_1 dt_2 \right) \\
&+ \quad k^2 \mathbb{E}_{(\xi_1,\xi_2)} \left(\int_{\frac{\xi_1}{k}}^{\frac{\xi_1+1}{k}} \int_{\frac{\xi_2}{k}}^{\frac{\xi_2+1}{k}} |c(t_1,t_2) - c(u,w)|\, \mathbf{1}_{A^c} \, dt_1 dt_2 \right) \\
&= I_1 + I_2,
\end{aligned}$$

where A^c is the complementary event of A. Observe that A^c contains 8 events**,** where for each there exists at least $j \in \{1,2\}$ such that $\frac{\xi_j}{k} - u_j \geq \delta$ or $\frac{\xi_j}{k} - u_j \leq -\delta$, $(u_1,u_2) = (u,w)$. Thus, using Lemma 2.1 of Bouezmarni and Rolin (2007), we obtain

$$I_2 \leq 8(k^2 + ||c||_I) \exp(-2k\delta^2).$$

Consequently, because of the uniform continuity of the copula density on I, it is straightforward to show that $I_1 = o(1)$ uniformly. Hence, $\sup_I |E(\hat{c}(u,w)) - c(u,w)| \to 0$, when $k \to \infty$.

Next, we show that $\sup_I |\hat{c}(u,w) - \mathbb{E}(\hat{c}(u,w))|$ converges to zero. Indeed, we have

$$\begin{aligned}
|\hat{c}(u,w) - \mathbb{E}(\hat{c}(u,w))| &= \left| k^2 \sum_v \{C_n(B_v) - C(B_v)\}\, p_{v_1,k}(u) p_{v_2,k}(w) \right| \\
&\leq 4k^2 \sup_{(t_1,t_2)\in[0,1]^2} |C_n(t_1,t_2) - C(t_1,t_2)|,
\end{aligned}$$

where

$$C(B_v) = C\left(\frac{v_1+1}{k}, \frac{v_2+1}{k}\right) - C\left(\frac{v_1}{k}, \frac{v_2+1}{k}\right) - C\left(\frac{v_1+1}{k}, \frac{v_2}{k}\right) + C\left(\frac{v_1}{k}, \frac{v_2}{k}\right)$$

is the copula function which is evaluated at the set B_v and $C_n(B_v)$ is its empirical version. Thus, from Kiefer (1961) and for $\epsilon > 0$, we obtain

$$
\begin{aligned}
P\left(\sup_{(u,w)\in I} |\hat{c}(u,w) - \mathbb{E}(\hat{c}(u,w))| > \epsilon\right) &\leq P\left(\sup_{t_1,t_2} \sqrt{n}\,|C_n(t_1,t_2) - C(t_1,t_2)| \geq \frac{1}{4}\epsilon n^{1/2}k^{-2}\right) \\
&\leq C\exp\left(-(2-\epsilon^*)\frac{1}{16}\epsilon^2 nk^{-4}\right).
\end{aligned}
$$

Hence, for any $\epsilon^* > 0$ we conclude the proof of part **(i)**.
Now, let's prove part **(ii)** of Proposition 1. Observe that from part **(i)**, we have

$$\sup_{I} |\mathbb{E}(\hat{c}(u,w)) - c(u,w)| \to 0, \text{ when } k \to \infty.$$

Further, from part **(i)**, we can see that

$$\sum_{n\geq 1} P\left(\sup_{(u,w)\in I} |\hat{c}(u,w) - \mathbb{E}(\hat{c}(u,w))| > \epsilon\right) \leq C\sum_{n\geq 1}\exp\left(-(2-\epsilon^*)\frac{1}{16}\epsilon^2 nk^{-4}\right).$$

Hence, we conclude the proof of Proposition 1 by considering $\epsilon = \epsilon_n = \alpha k^2 \left(\frac{\log(n)}{n}\right)^{1/2}$, for $\alpha < \frac{4}{\sqrt{2-\epsilon^*}}$, and using Borel-Cantelli lemma.

■

The next proposition shows that the Bernstein copula density estimator converges to infinity when the density is unbounded at the corners. It also provides the relative convergence of the estimator at the corners. Without loss of generality, the following results are derived when the density is unbounded at $(0,0)$.

Proposition 2. *Let $c(.)$ be a copula density function that is unbounded at $(0,0)$. Let $\hat{c}(.)$ be the Bernstein copula density estimator. Then, under the conditions of part **(ii)** of Proposition 1, we have*

$$\hat{c}(0,0) \xrightarrow{a.s} \infty, \quad \text{as} \quad n \to \infty.$$

Further, we have

$$\frac{|\hat{c}(u,w) - c(u,w)|}{c(u,w)} \xrightarrow{a.s.} 0, \quad \text{as} \quad n \to \infty, \quad \text{and} \quad (u,w) \to (0,0).$$

■

Proof of Proposition 2 We first show that $\hat{c}(0,0)$ converges to infinity. From the proof of Proposition 1, we have $|\hat{c}(0,0) - \mathbb{E}(\hat{c}(0,0))| \xrightarrow{a.s.} 0$. Thus, it remains to show that $\mathbb{E}(\hat{c}(0,0))$ converges to infinity when the bandwidth parameter k tends to infinity. Observe that for $C > 0$, there exists

$\delta_1 > 0$ and $\delta_2 > 0$ such that $c(u,v) > C$, for $u < \delta_1$ and $v < \delta_2$. For n sufficiently large and for k_n tends to infinity, we have $min(\delta_1, \delta_2) > \frac{1}{k}$. Hence,

$$\mathbb{E}(\hat{c}(0,0)) = k^2 \int_0^{\frac{1}{k}} \int_0^{\frac{1}{k}} c(t_1,t_2) dt_1 dt_2 > C, \text{ for } n \text{ sufficiently large.}$$

We now show the relative convergence of the Bernstein estimator in the boundary region of unbounded density copula functions. We follow similar arguments to the those in the proof of Proposition 1 and we obtain

$$\begin{aligned}\frac{\mathbb{E}(\hat{c}(u,w)) - c(u,w)|}{c(u,v)} &\leq k^2 \mathbb{E}_{(\xi_1,\xi_2)} \left(\int_{\frac{\xi_1}{k}}^{\frac{\xi_1+1}{k}} \int_{\frac{\xi_2}{k}}^{\frac{\xi_2+1}{k}} \frac{|c(t_1,t_2) - c(u,w)|}{c(u,w)} \mathbf{1}_A \, dt_1 dt_2 \right) \\ &\quad + k^2 \mathbb{E}_{(\xi_1,\xi_2)} \left(\int_{\frac{\xi_1}{k}}^{\frac{\xi_1+1}{k}} \int_{\frac{\xi_2}{k}}^{\frac{\xi_2+1}{k}} \frac{|c(t_1,t_2) - c(u,w)|}{c(u,w)} \mathbf{1}_{A^c} \, dt_1 dt_2 \right) \\ &= I_1 + I_2.\end{aligned}$$

Using Lemma 2.1 of Bouezmarni and Rolin (2007), we have

$$I_2 \leq 8 \left(\frac{k^2}{c(u,w)} + 1 \right) \exp(-2k\delta^2).$$

Further, because of the continuity of the copula density on $(0,1)^2$, it is straightforward to show that $I_1 = o(1)$ uniformly. Hence, $\frac{|E(\hat{c}(u,w)) - c(u,w)|}{c(u,w)} \to 0$, when $k \to \infty$. We also have that

$$\begin{aligned}\frac{|\hat{c}(u,w) - \mathbb{E}(\hat{c}(u,w))|}{c(u,w)} &\leq \frac{k^2}{c(u,w)} \sum_v |C_n(B_v) - C(B_v)| \, p_{v_1,k}(u) p_{v_2,k}(w) \\ &\leq 4 \frac{k^2}{c(u,w)} \sup_{(t_1,t_2) \in [0,1]^2} |C_n(t_1,t_2) - C(t_1,t_2)| \, .\end{aligned}$$

Hence, from Kiefer (1961) and for $\epsilon > 0$, we obtain

$$\begin{aligned}P\left(\frac{|\hat{c}(u,w) - \mathbb{E}(\hat{c}(u,w))|}{c(u,w)} > \epsilon \right) &\leq P\left(\sup_{(t_1,t_2)} \sqrt{n} \, |C_n(t_1,t_2) - c(t_1,t_2)| \geq \frac{1}{4} \epsilon n^{1/2} k^{-2} c(u,w) \right) \\ &\leq C \exp\left(-(2 - \epsilon^*) \frac{1}{16} \epsilon^2 n k^{-4} c^2(u,w) \right).\end{aligned}$$

This concludes the proof of Proposition 2.

■

4 Monte Carlo Simulations

In this section, we run Monte Carlo simulations to evaluate the performance of Bernstein estimator of copula density in the interior region and at the corners. We compare the finite sample properties of the Bernstein estimator [hereafter **BR**] with those of: **(1)** Local linear estimator with multiplicative Epanechnikov kernel [hereafter **LL**]; **(2)** Mirror-reflection estimator with multiplicative Epanechnikov kernel [hereafter **MR**]; **(3)** Beta kernel estimator [hereafter **BT**]; and **(4)** Transformation estimator using multiplicative Epanechnikov kernel and Gaussian transformation [hereafter **TR**]. We choose the estimators that are known to have somewhat a quite good behavior at the borders. For more details about the LL estimator and the MR, BT and TR estimators, the reader can consult Chen and Huang (2007) and Charpentier et al. (2006), respectively.

We consider several data generating processes (DGPs) for the copula density $c\,(u,w)$. We choose our DGPs to illustrate performance in different contexts that one can encounter in practice. Thus, we simulate our bivariate data $\left\{(X_{i1},X_{i2})^{\top}\right\}_{i=1}^{n}$ using a uniform distribution $Unif[0,1]^2$ and under one of the following copula densities: **(1)** Normal copula [hereafter $c_{(n)}$]; **(2)** Student copula [hereafter $c_{(t)}$]; **(3)** Clayton copula [hereafter $c_{(cl)}$]; **(4)** Gumbel copula [hereafter $c_{(g)}$]; and **(5)** Frank copula [hereafter $c_{(f)}$]**.** Those copulas are extremely useful to model the dependence between stock and bond returns. They all, except Frank copula, become unbounded at $(0,0)$ or/and at $(1,1)$ with strong dependent data. Here we consider two different degrees of dependence that are measured by the Kendall rank correlation coefficient: $\tau=0.25$ (weak dependence) and 0.75 (strong dependence).

Like the BR estimator, BT, LL, TR and MR estimators depend on a bandwidth parameter h. In the simulations and for each of the previous estimators, we use an optimal fixed bandwidth obtained by a grid search over the set $\{0.01,0.04,\ldots,0.97\}$ for h and over the set $\{3,6,\ldots,54,65,75,\ldots,205\}$ for k the bandwidth parameter needed to calculate the BR estimator. This allows us to compare all the estimators under their quasi optimal settings. In the next section, we discuss the problem of bandwidth selection for the Bernstein estimator. Finally, to keep the computing time reasonable, we consider small and moderate sample sizes: $n=50$ and $n=150$ and we perform $N=1000$ simulations to evaluate the performance of the estimators using two distance measures. The first one is the Integrated Mean Squared Error (IMSE) given by:

$$\begin{aligned} IMSE &= \mathbb{IB}ias^2+\mathbb{IV}ar \\ &\approx \frac{1}{I}\sum_{i=1}^{I}\left(\left(\frac{1}{N}\sum_{j=1}^{N}\hat{c}_j(\mathbf{s}_i)-c(\mathbf{s}_i)\right)^2\right) \\ &\quad+\frac{1}{I}\sum_{i=1}^{I}\left(\frac{1}{N}\sum_{j=1}^{N}\left(\hat{c}_j(\mathbf{s}_i)-\frac{1}{N}\sum_{j}\hat{c}_j(\mathbf{s}_i)\right)^2\right)^2,\ I=2500, \end{aligned}$$

where $\mathbf{s}_i=(u_i,w_i)\in\{(0.01,0.01),(0.01,0.03),\ldots,(0.99,0.99)\}$, and $\hat{c}_j(\boldsymbol{s})$, for $j=1,\ldots,N$, is the estimator of the copula density that corresponds to the j-th replication. Observe that $IMSE\approx\frac{1}{N}\sum_j ISE(\hat{c}_j)$, where $ISE(\hat{c})=\frac{1}{I}\sum_i(\hat{c}(\boldsymbol{s}_i)-c(\boldsymbol{s}_i))^2$ is the approximated Integrated Squared Error (ISE). To assess the uncertainty in the $IMSE$ measure, we also report the Square root of the Integrated Squared Error (SISE), which is defined by the square root of $\frac{1}{N}\sum_j \left(ISE(\hat{c}_j) - \frac{1}{N}\sum_j ISE(\hat{c}_j)\right)^2$. The second measure that we consider is given by the Integrated Median Absolute Relative Error (IMdAE) approximated by:

$$IMdAE(\mathbf{s}_i) \approx \frac{1}{I}\sum_i MdAE(\mathbf{s}_i),$$

where $MdAE(\mathbf{s})$ is the empirical median of the sequence $\left\{\left|\frac{\hat{c}_j(\mathbf{s})-c(\mathbf{s})}{c(\mathbf{s})}\right|\right\}_{j=1}^{N}$.

Table 1 reports the IMSE for the BR estimator using an optimal bandwidth k_{opt} which is obtained by a grid search over one of the following sets: **(1)** $\{(u_i, w_i) = (0.01, 0.01), (0.01, 0.03), \ldots, (0.99, 0.99)\}$ [hereafter set **ALL**]; **(2)** elements of the set ALL that satisfy $\sqrt{u_i^2 + w_i^2} < 0.56$, i.e. the 25% extreme left points [hereafter set $\mathcal{V}_0$]; **(3)** elements of the set ALL that satisfy $\sqrt{u_i^2 + w_i^2} > 0.98$, i.e. the 25% extreme right points [hereafter set $\mathcal{V}_1$]; and **(4)** the interior of the set ALL, i.e. $ALL \setminus \{\mathcal{V}_0, \mathcal{V}_1\}$ [hereafter **INT**]. As expected, we see that the IMSE decreases with the sample size n and increases with the Kendall's rank correlation τ. In other words, we find that the strength of the dependence between X_1 and X_2 makes its estimation difficult. We get similar results when we use the LL, BT, MR, and TR estimators. These results are available from the author upon request. However, except for the Clayton copula density and for $\tau = 0.75$, we obtain relatively small integrated mean squared error. We also find that the performance of the BR estimator depends on the target region. In the interior region the estimator behaves clearly better than at the borders. Further, generally the optimal bandwidth k_{opt} increases with n and τ, as predicted by the theory. Interestingly, we see that k_{opt} also depends on the target region. This indicates that in general we should use a larger value of k near the extreme points where the value of the copula function become very large. Thus, in practice an adaptive bandwidth should be used in order to get a better approximation. One such an adaptation could be the method of "shrinking" of the bandwidth at the borders. This will be investigated in a future research project.

Tables 2 and 3 compare the performance of the previous estimators of copula densities over the set ALL and for $n = 50$ and $n = 150$, respectively. To facilitate such a comparison, we provide measures of relative performance of the Bernstein estimator with respect to each of the other estimators, $E = LL$, MR, BT, TR. These measures are given by the ratios $RE_{IMSE} = \frac{IMSE(E)}{IMSE(BR)}$ and $RE_{IMdAE} = \frac{IMdAE(E)}{IMdAE(BR)}$. We also compute and report the ratio of the integrated variance (IVAR) to IMSE ($\frac{IVAR}{IMSE}$) and compute the standard deviation of the integrated squared error for each estimator. Note that the values reported in tables 2 and 3 are expressed as percentages. From these, we see that for weakly dependent data ($\tau = 0.25$), the BR and LL estimators are quite comparable and give the best results in terms of IMSE and IMdAE. The BR estimator performs better than the rest of the estimators for three of the five copulas that we studied here (Clayton, Gumbel and Student copulas). Looking at the ratio $\frac{IVAR}{IMSE}$ and at the standard deviation of the integrated squared error (SISE), we see clearly that the BR estimator has the advantage to be less variable than the LL estimator. We also see that the Transformation estimator (TR) gives the worse result with a very high variance and a large IMSE. However, the opposite happens for strongly dependent data ($\tau = 0.75$). In the latter case the TR method become the best since it behaves better for three of the five copulas considered here (Clayton, Gumbel and Student copulas). The performance measure IMdAE indicates that the BR estimator still behaves reasonably well. The only case where the BR estimator is beaten by the TR estimator is when we consider the Clayton copula with $\tau = 0.75$. This can be attributed to the fact that the used bandwidth is too large for the points in the interior domain [see Table 1].

5 Bandwidth Selection

Here, we investigate the performance of the Bernstein estimator when an automatic data-driven bandwidth is used. We use the least-squared cross-validation (LSCV) method which selects a bandwidth $\hat{k}$ that minimizes the following function:

$$LSCV(k) = \int_0^1 \hat{c}^2(u,w)dudw - 2n^{-1}\sum_{i=1}^{n} \hat{c}^{(-i)}(X_{1,i}, X_{2,i}),$$

where $\hat{c}^{(-i)}(\cdot,\cdot)$, for $i = 1,\ldots,n$, is the Bernstein copula estimator calculated using a bandwidth k and all the data except the observation $(X_{1,i}, X_{2,i})$. Observe that:

$$\mathbb{E}\left(LSCV(k)\right) = \mathbb{E}\left(\int_0^1 (\hat{c}(u,w) - c(u,w))^2 dudw\right) - \int_0^1 c^2(u,w)dudw.$$

Thus, the bandwidth $\hat{k}$ minimizes an unbiased estimator of the expected integrated squared error. We can also show that

$$\int_0^1 \hat{c}^2(u,w)dudw = n^{-2}\sum_i\sum_j \tilde{B}(\lfloor X_{1,i}k\rfloor, \lfloor X_{1,j}k\rfloor)\tilde{B}(\lfloor X_{2,i}k\rfloor, \lfloor X_{2,j}k\rfloor),$$

where $\lfloor\cdot\rfloor$ denotes the floor function and

$$\tilde{B}(a,b) = \frac{B(a+b+1, 2k-a-b-1,)}{B(a+1,k-a)B(b+1,k-b)},$$

where $B(a,b)$ is the usual Beta function, i.e. $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$. The above formula facilitates the calculation of the LSCV function, which reduces the simulation running time.

We now repeat the simulation study described in section 4 using the data-driven bandwidth $\hat{k}$ instead of the optimal bandwidth k_{opt}. Figure 1 shows a box-plot of 1000 observations of the ratio $\frac{ISE(\hat{c}_{j,\hat{k}_j})-IMSE(k_{opt})}{IMSE(k_{opt})}$. The latter should fluctuates around zero if the data-driven bandwidth $\hat{k}$ and the optimal bandwidth k_{opt} lead approximately to the same integrated squared error (ISE).

Although our results show that this is not the case for all considered scenarios, the integrated squared error (ISE) obtained using $\hat{k}$ remains reasonably small and typically does not exceed $2 * IMSE(k_{opt})$. The results for $n = 50$ are not very satisfactory, probably because the sample size is too small. However, these results improve when the sample size n increases, which seems to indicate the consistency of the bandwidth selection method.

For weak dependent data ($\tau = 0.25$), we see that the integrate mean squared error changes a lot across the simulations, especially for Normal and Frank copulas. Moreover, for the latter two cases, the bandwidth $\hat{k}$ seems to lead to some bias approximations. Surprisingly, when the dependence between X_1 and X_2 is strong ($\tau = 0.75$), we find much better results in term of ISE, even if the boundary problems are more severe in this case. An explanation can be obtained by comparing $\hat{k}$ to k_{opt}. Thus, Figure 2 shows a box-plot of 1000 observations of the ratio $\frac{\hat{k}_j - k_{opt}}{k_{opt}}$, where $\hat{k}_j$ is the bandwidth that corresponds to the j-th replication selected using the least-squared cross-validation (LSCV) method. This figure clearly shows that the LSCV method tend to choose a large bandwidth k when τ is small, which leads to an over-smoothing. However, the opposite happens when τ is large, but the under-smoothing is much less severe except for the Clayton copula density. Consequently, we recommend to correct the LSCV bandwidth by taking into account the degree of dependency in the data. This will be investigated and studied in a future work.

6 Empirical illustration

In this section, we re-examine the asymmetric dependence between international equity markets using two nonparametric estimators of copula densities. Recent research have suggested an increase in the correlation between international equity markets during volatile periods. This increase is especially observed during market downturns. Ang and Bekaert (2002) use a two-regime switching model and find evidence of one state with low returns and high correlation and volatilities, and a second state with high returns and low correlation and volatilities. Longin and Solnik (2001), use extreme value theory and develop a new concept named exceedance correlation, and find a high correlation between large negative returns and zero correlation between large positive returns.

Rather than to use correlation coefficients, in this section we use copula densities which represent a natural way to model the dependence between equity market returns. We focus on four equity markets (US, Canada, UK, and France) and we use weekly observations that spans 19 years.

6.1 Data description

Our data consists of weekly observations on MSCI Equity Indices series for the US, Canada, the UK and France. The sample runs from October 16th 1984 to December 21th 2004 for a total of 1054 observations. The returns are computed using the standard continuous compounding formula. All returns are derived on a weekly basis from daily prices expressed in US dollars. Summary statistics for the US, Canada, the UK and France equity returns are presented in Table 4. These weekly returns are displayed in Figure 3. The unconditional distributions of the US, Canada, the UK and France equity weekly returns show the expected excess kurtosis and negative skewness. The sample kurtosis is greater than the normal distribution value of three. The values of Jarque-Bera test statistic show that these equity returns are not normally distributed. The time series plots of returns show the familiar volatility clustering effect, along with some occasional large absolute returns.

6.2 Results

To estimate the dependence between US, Canada, UK and France equity markets, we use the two best estimators of the copula density that we have selected on the basis of simulation results of Section (4). These estimators are the Bernstein estimator (BR) and the Local linear estimator with multiplicative Epanechnikov kernel (LL). To assess the sensitivity of our estimation results, we consider various values for the bandwidth parameter k. These values are $k = 25$, 50, 100 for the BR estimator and $k = 0.035$, 0.1 and 0.5 for the LL estimator. The bandwidth of the BR estimator plays the inverse role of the bandwidth of the LL estimator, that is a large value of BR's bandwidth reduces the bias but increases the variance.

The empirical results for the copula density estimation for the pairs US-Canada, US-UK and US-France are presented in Figures 4 and 5. From these, we see that using a small bandwidth it over-smooths the BR estimator, whereas a large bandwidth under-smooths the estimator. The opposite happens with LL estimator: we over-smooth the estimation of the copula density when we choose a large value of the bandwidth and we under-smooth the estimator when a small bandwidth is chosen. Intermediate values like $k = 50$ for the BR estimator and $k = 0.1$ for the LL estimator, produce more reasonable results. As expected, we find that the dependence between US and Canada, UK, France equity markets is *asymmetric*. That is, the international equity market returns are more dependent during the bear market than during the bull market. The latter result is confirmed by comparing the values that takes the estimator of the density copula at the extremes (0,0) and (1,1); with (0,0) corresponds to the bear market and (1,1) to the bull market. We find that at (0,0) the estimator of the copula density takes a larger value than the one it takes at (1,1). The result is quite stable when we use BR estimator, and it is more striking in the US-France case. Since a large bandwidth tends to increase the bias in the LL estimator and to decrease its variance, we find that using a large bandwidth for the LL estimator decreases the asymmetry in the estimated dependence [see Figure 4]. Thus, the dependence between US and Canada, UK, France equity markets look more symmetric. This is due to the high value of the bandwidth, so to the high-biased LL estimator.

We also considered many other values for the bandwidth parameter k. The results are more or less similar to the those presented in Figures 4 and 5. We also analyzed the dependence between other pairs of countries (Canada-UK, Canada-France, UK-France) and the conclusions are also similar to those obtained before. Thus, it seems that the Bernstein estimator is a good estimator at the extremes. The empirical results show that this estimator is able to capture the well known asymmetric dependence phenomena that is observed in the international equity markets.

7 Conclusion

In this paper, we examined the asymptotic properties of the Bernstein estimator (BR) for unbounded density copula functions. We showed that the BR estimator converges to infinity at the corners. We established its relative convergence when the copula is unbounded. We also provided its uniform strong consistency on every compact in the interior region. Furthermore, we studied the finite simple performance of the estimator via an extensive simulation study and we compared with other well known nonparametric estimators. Finally, we considered an empirical application where we re-examined the asymmetric dependence between international equity markets US, Canada, UK, and France. We compared the empirical results using the Bernstein estimator and the Local linear estimator with multiplicative Epanechnikov kernel. We found that the Bernstein estimator is a good estimator at the extremes. Our results showed that this estimator is able to capture the well known asymmetric dependence phenomena observed in the international equity markets.

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Table 1: Integrated Mean Squared Error (IMSE) of the BR estimator in the unit square (ALL), near 0 ($\mathcal{V}_0$), near 1 ($\mathcal{V}_1$) and in the interior set (INT), under different families of copula and using optimal bandwidth.

			$\mathcal{V}_0$		$\mathcal{V}_1$		$INT.$		ALL	
n	τ	Cop.	k_{opt}	$IMSE$	k_{opt}	$IMSE$	k_{opt}	$IMSE$	k_{opt}	$IMSE$
50	0,25	$c_{(cl)}$	9	0.597	3	0.071	6	0.056	6	0.216
		$c_{(f)}$	3	0.098	3	0.121	6	0.050	3	0.080
		$c_{(g)}$	3	0.084	6	0.534	3	0.043	6	0.185
		$c_{(n)}$	3	0.133	6	0.163	6	0.047	6	0.097
		$c_{(t)}$	9	0.656	9	0.725	6	0.072	9	0.385
	0,75	$c_{(cl)}$	195	33.348	9	0.453	27	0.201	105	9.402
		$c_{(f)}$	15	1.052	15	0.893	36	0.247	21	0.635
		$c_{(g)}$	21	1.353	75	11.249	27	0.190	45	3.431
		$c_{(n)}$	30	2.277	33	2.251	24	0.158	30	1.197
		$c_{(t)}$	75	8.567	75	8.455	30	0.239	75	4.364
150	0,25	$c_{(cl)}$	21	0.365	9	0.057	9	0.027	12	0.139
		$c_{(f)}$	6	0.054	6	0.066	6	0.025	6	0.042
		$c_{(g)}$	9	0.063	21	0.343	9	0.025	9	0.128
		$c_{(n)}$	6	0.082	6	0.091	6	0.024	6	0.054
		$c_{(t)}$	24	0.430	24	0.470	9	0.040	21	0.256
	0,75	$c_{(cl)}$	195	28.236	15	0.240	48	0.108	195	7.542
		$c_{(f)}$	30	0.584	27	0.492	65	0.133	36	0.351
		$c_{(g)}$	39	0.762	95	8.669	45	0.105	95	2.480
		$c_{(n)}$	65	1.343	65	1.292	39	0.084	54	0.698
		$c_{(t)}$	75	6.812	75	6.789	54	0.133	75	3.415

Table 2: The ratios R_{IMSE} (%), R_{IMdAE} (%) and $IVAR/IMSE$ (%), and the $SISE$ (%) of the estimators BR, LL, MR, BT, and TR in the set ALL and for $n = 50$.

		$\tau = 0.25$				$\tau = 0.75$			
Copula	Method	RE_{IMSE}	RE_{IMdAE}	$\frac{IVAR}{IMSE}$	$SISE$	RE_{IMSE}	RE_{IMdAE}	$\frac{IVAR}{IMSE}$	$SISE$
$c_{(cl)}$	BR	100.0	100.0	29.6	5.5	100.0	100.0	20.2	244.8
	LL	101.3	112.3	32.7	6.1	103.2	87.4	14.2	167.5
	BT	104.5	110.6	29.9	5.6	103.6	2087.0	16.1	206.8
	MR	114.8	120.1	20.0	5.6	116.8	87.4	8.1	74.1
	TR	168.7	113.8	87.5	19.8	***89.4***	***72.9***	24.3	326.3
$c_{(f)}$	BR	100.0	100.0	40.1	3.9	100.0	100.0	52.7	17.3
	LL	***95.2***	100.0	91.0	6.1	112.3	107.3	63.3	34.4
	BT	108.3	109.9	46.1	3.8	94.8	113.5	53.3	15.0
	MR	110.1	111.8	44.2	4.9	***92.8***	106.6	47.0	18.0
	TR	399.8	149.5	88.1	16.5	161.3	110.4	48.1	38.8
$c_{(g)}$	BR	100.0	100.0	33.1	4.7	100.0	100.0	24.5	79.9
	LL	100.0	108.4	36.1	5.5	107.0	119.2	27.8	75.0
	BT	105.2	108.1	27.4	4.6	111.1	101.4	41.5	96.4
	MR	114.1	124.4	22.4	4.8	116.1	120.1	13.6	31.0
	TR	196.2	133.1	88.0	18.9	***92.5***	***89.9***	28.4	99.5
$c_{(n)}$	BR	100.0	100.0	62.6	4.3	100.0	100.0	45.5	31.0
	LL	***89.9***	100.0	73.6	5.3	116.8	121.3	43.8	46.2
	BT	106.8	116.8	36.7	3.7	108.4	105.9	24.4	23.5
	MR	114.0	125.4	32.5	4.5	124.4	122.8	24.7	19.7
	TR	338.5	133.2	89.8	17.3	107.0	99.2	46.5	41.3
$c_{(t)}$	BR	100.0	100.0	29.7	7.5	100.0	100.0	31.9	107.6
	LL	111.7	112.5	33.1	10.4	108.2	120.4	31.7	88.1
	BT	103.0	103.5	28.5	7.4	107.0	105.4	34.3	107.5
	MR	115.8	124.3	22.2	7.1	122.3	123.8	16.5	44.6
	TR	127.9	104.0	80.2	22.0	***91.1***	100.9	32.0	117.9

Table 3: The ratios R_{IMSE} (%), R_{IMdAE} (%) and $IVAR/IMSE$ (%), and the $SISE$ (%) of the estimators BR, LL, MR, BT, and TR in the set ALL and for $n = 150$.

		$\tau = 0.25$				$\tau = 0.75$			
Copula	Method	RE_{IMSE}	RE_{IMdAE}	$\frac{IVAR}{IMSE}$	$SISE$	RE_{IMSE}	RE_{IMdAE}	$\frac{IVAR}{IMSE}$	$SISE$
$c_{(cl)}$	BR	100,0	100.0	41.2	3.8	100.0	100.0	15.6	213.2
	LL	104.8	103.9	39.5	4.0	109.3	80.0	9.5	147.0
	BT	114.0	113.4	35.0	3.7	114.7	2865.0	6.0	128.4
	MR	135.8	126.7	29.5	3.6	126.6	80.0	17.2	127.1
	TR	108.4	115.6	69.7	6.3	***79.0***	***60.5***	31.9	317.2
$c_{(f)}$	BR	100.0	100.0	48.0	1.9	100.0	100.0	58.0	8.5
	LL	***65.8***	***86.6***	75.5	1.7	***98.2***	105.3	62.3	13.2
	BT	116.7	112.1	55.9	2.0	110.2	105.8	27.0	6.5
	MR	127.0	116.6	43.3	2.5	***89.8***	105.9	56.8	8.8
	TR	306.7	147.7	70.8	5.6	166.2	110.8	51.4	18.5
$c_{(g)}$	BR	100.0	100.0	38.2	3.0	100.0	100.0	24.1	66.3
	LL	104.5	105.7	36.0	3.2	107.9	123.2	30.7	61.4
	BT	111.8	111.4	27.0	2.9	108.0	108.5	19.6	58.6
	MR	126.6	130.7	29.3	2.9	129.4	125.8	17.8	31.0
	TR	117.7	130.1	70.5	5.7	***84.1***	***93.4***	39.4	91.7
$c_{(n)}$	BR	100.0	100.0	56.1	2.1	100.0	100.0	53.0	20.3
	LL	***81.7***	***92.0***	47.9	1.8	121.4	116.4	58.7	28.0
	BT	122.3	122.3	41.7	2.2	118.8	108.6	63.4	22.0
	MR	140.5	134.6	35.4	2.6	149.8	123.7	28.0	12.6
	TR	243.9	134.1	74.4	5.9	104.6	***94.4***	53.6	23.9
$c_{(t)}$	BR	100.0	100.0	43.5	5.8	100.0	100.0	13.3	65.4
	LL	111.7	108.4	38.8	7.1	100.0	119.3	25.0	63.5
	BT	110.5	110.4	37.8	5.5	104.0	102.2	15.0	64.4
	MR	131.7	132.3	27.9	4.4	127.3	124.3	38.1	63.0
	TR	***88.7***	110.7	58.3	7.6	***77.2***	100.9	40.2	104.8

Table 4: Summary statistics for weekly stock returns

Returns	Mean	Median	Max	Min	Std.Dev.	Skewness	Kurtosis	JB
r_{US}	0.001889	0.0037	0.1261	-0.2805	0.023989	-1.683636	22.51981	17231.21
r_{Canada}	0.001459	0.00325	0.1012	-0.2453	0.024768	-1.383057	13.79779	5456.367
r_{UK}	0.001873	0.0023	0.1002	-0.2542	0.025856	-1.101227	12.62567	4282.064
r_{France}	0.002434	0.0032	0.1403	-0.1509	0.029687	-0.348288	5.739768	350.9621

Correlations

	US	Canada	UK
Canada	0.7133		
UK	0.5520	0.5406	
France	0.5315	0.5041	0.6363

Note: This table summarizes the weekly returns characteristics for the US, Canada, the UK and France equity indices and the correlation between them. In this table JB represents the Jarque-Bera test statistic. The sample covers the period from October 16th 1984 to December 21th 2004 for a total of 1054 observations.

Figure 1: The ratio $(ISE(\hat{k}) - IMSE(k_{opt}))/IMSE(k_{opt})$ of the BR estimator under weak and strong dependence and different families of the copula, $n = 150$

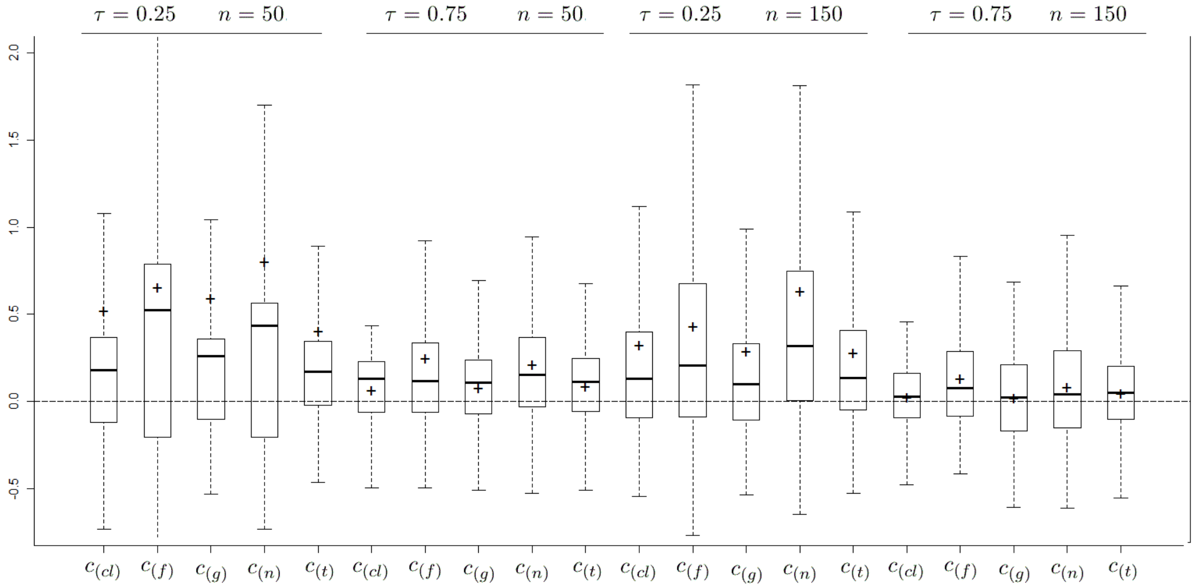
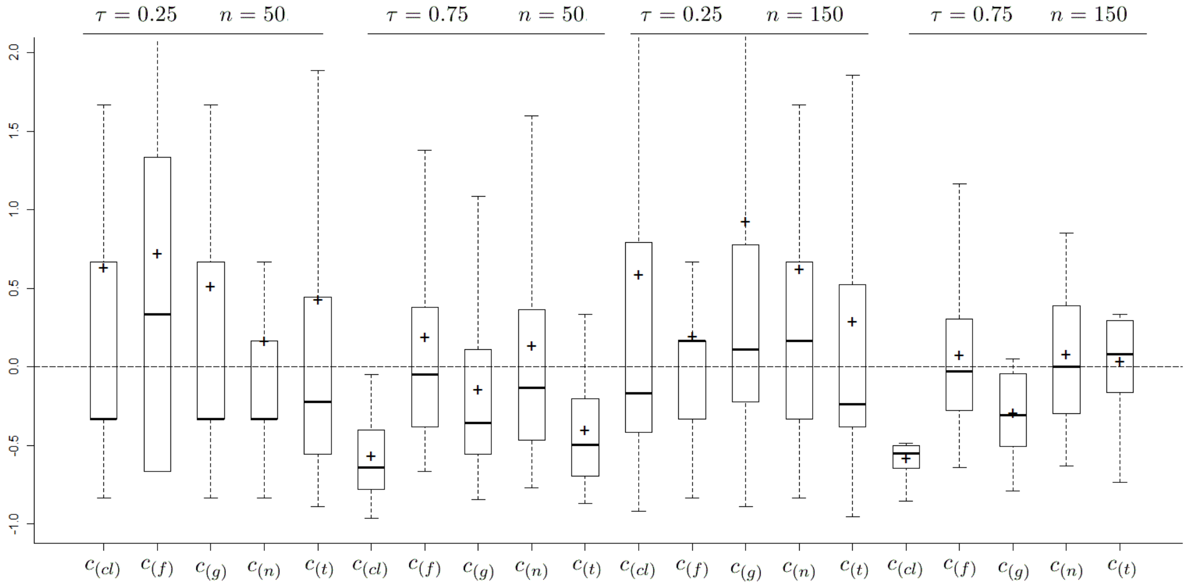
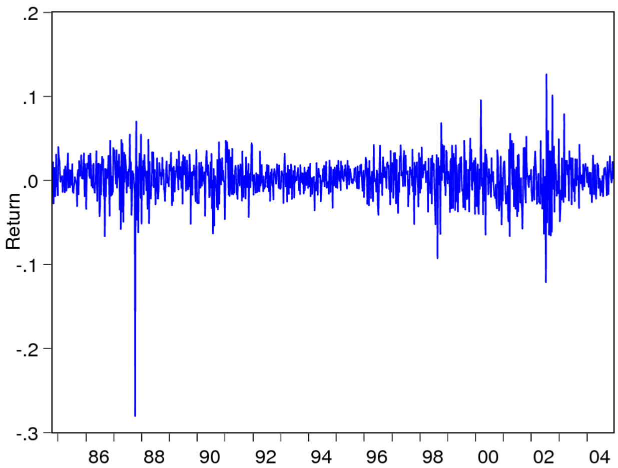
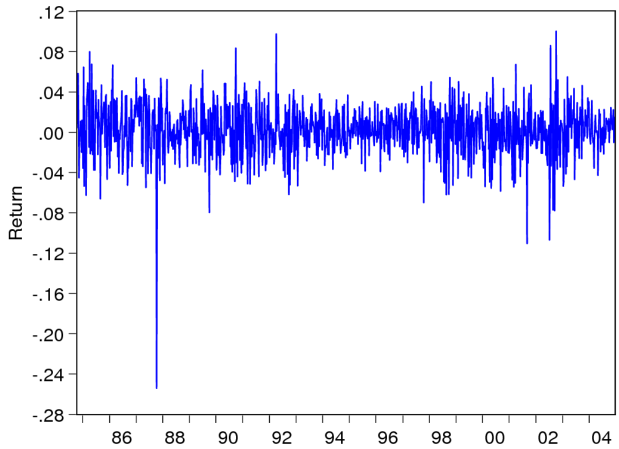


Figure 2: The ratio $(\hat{k} - k_{opt})/k_{opt}$ of the BR estimator under weak and strong dependence and different families of the copula, $n = 150$

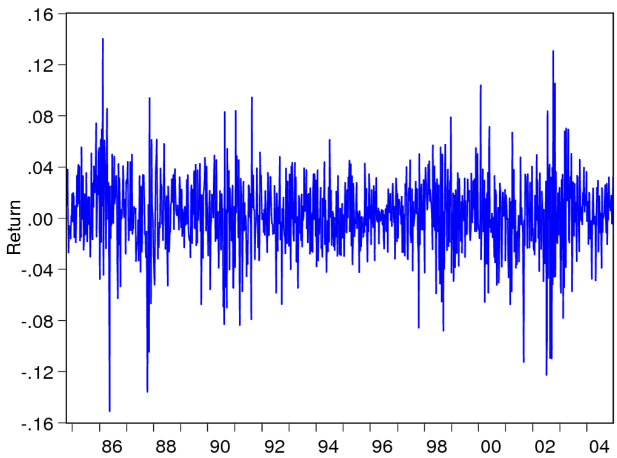




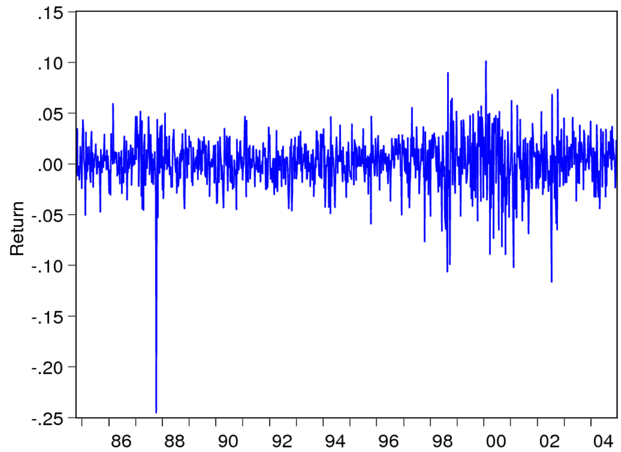
(a) US



(b) UK

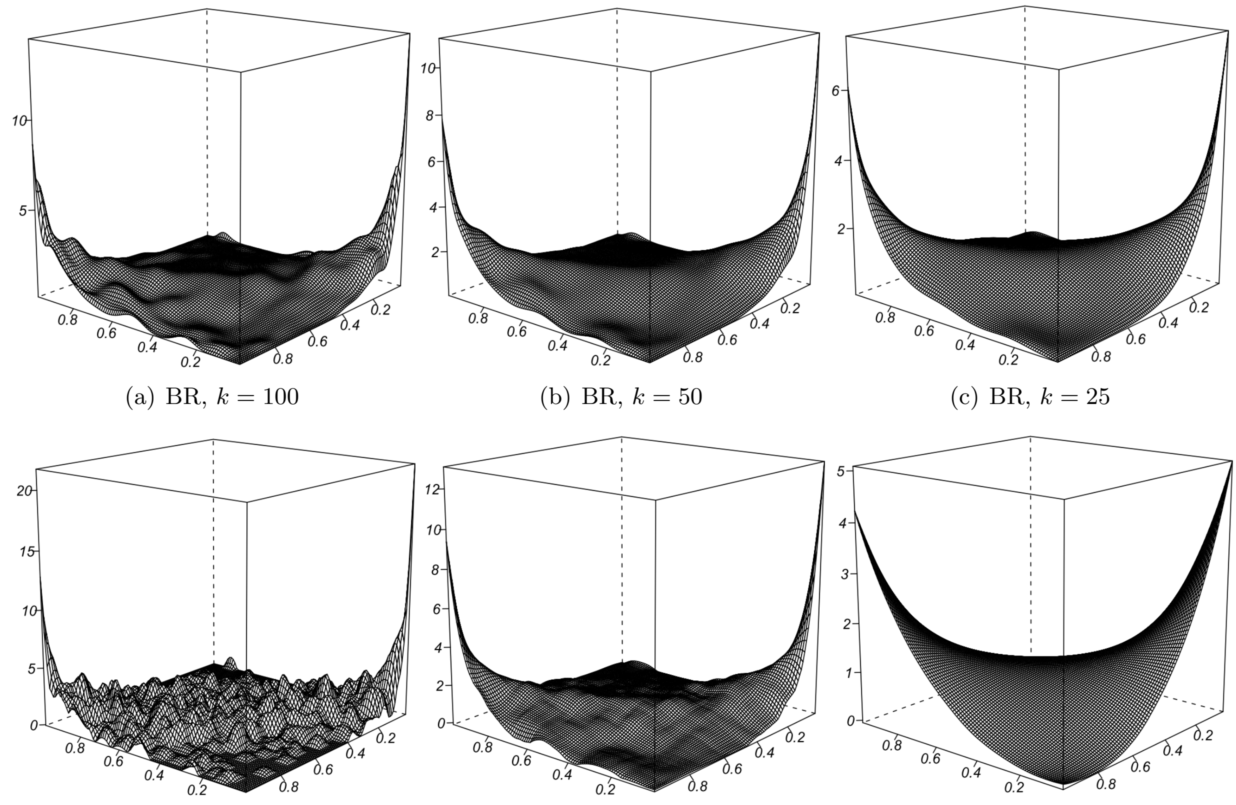


(c) FRANCE



(d) CANADA

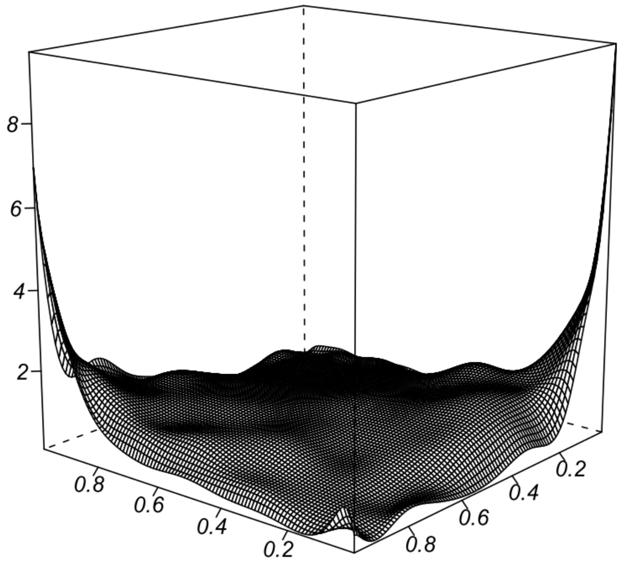
Figure 3: These graphs represent weekly equity returns in the US, Canada, the UK, and France. The sample covers the period from October 16th 1984 to December 21th 2004 for a total of 1054 observations.



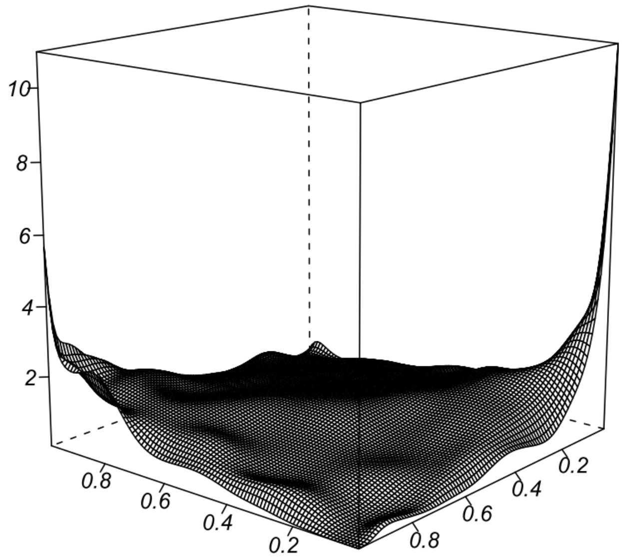
(a) BR, $k = 100$ (b) BR, $k = 50$ (c) BR, $k = 25$

(d) LL, $h = 0.035$ (e) LL, $h = 0.1$ (f) LL, $h = 0.5$

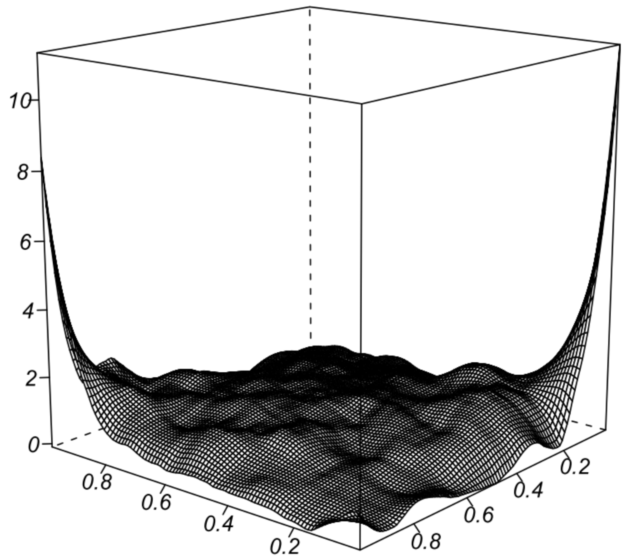
Figure 4: Bernstein (BR) and Local linear (LL) estimators of the copula density for the pair US-Canada, using weekly equity returns and different bandwidth parameters.



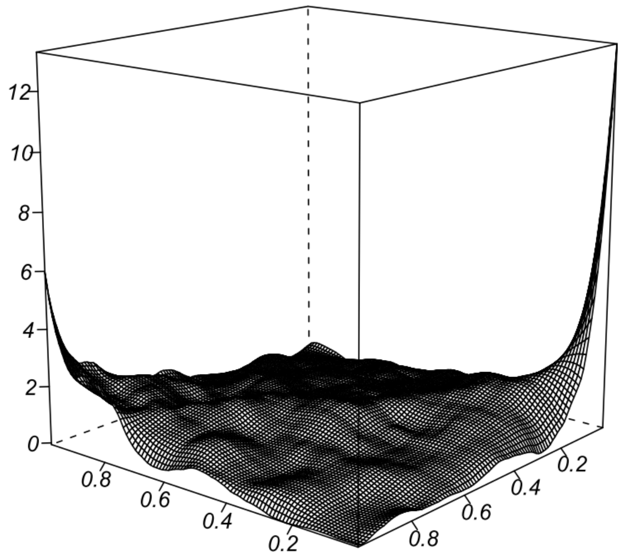
(a) BR, USA-UK, $k = 50$



(b) BR, USA-FR, $k = 50$



(c) LL, USA-UK, $h = 0.1$



(d) LL, USA-FR, $h = 0.1$

Figure 5: Bernstein (BR) and Local linear (LL) estimators of the copula density for the pairs US-UK and US-France, using weekly equity returns for.