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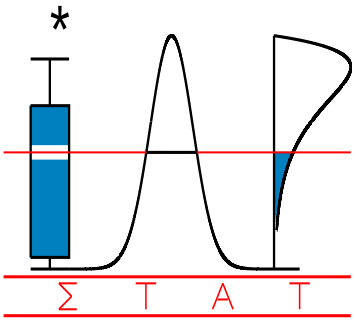

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SOLVENCY CAPITAL, INFLATION AND TIME HORIZON IN PENSION LIABILITIES

Pierre DEVOLDER, Habiba TASSA (*)

ABSTRACT

Risk measurement as applicable for insurers (Solvency 2) or banks (Basel 2) can also be considered for pension fund liabilities. The purpose of this paper is to present various stochastic models in continuous time in order to estimate solvency capital for two important risks faced by pension funds: market risk and inflation risk. We address the situation of a Defined Benefit Pension Scheme (DB) with liabilities linked to final salary. We try to develop in this context a methodology coherent with IAS norms based on the so called projected unit credit cost method but including a risk measure approach.
We also show that pension portability could be modeled using classical ruin theory.

Keywords: Inflation risk, market risk, solvency requirement, value at risk, probability of ruin, IAS norms

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1. Introduction:

Solvency requirements for financial institutions are clearly on the agenda in the context of the recent financial crisis. If regulations exist for banks (Basel 2) or are nearly ready for insurance companies (Solvency 2), nothing is done for the time being for pension funds. Of course pension liabilities are different from classical life insurance contracts and the juridical context of pension funds (sometimes quite different from one country to another) has its own specificities.
Nevertheless, in a context of risk management, it seems strange that the only international consistent valuation for pension liabilities is based only on a best estimate approach (IAS 19) without integrating any risk measure or solvency point of view.

The purpose of this paper is to try to develop a risk based approach for a defined benefit pension scheme. Following the IAS norms, we use as funding technique the so called projected unit credit cost method in order to compute contributions and actuarial liabilities. Two main risks are then considered: investment risk and inflation risk (with correlation). No longevity risk is taken into account here. The long term aspect of pension liability is taken systematically into account by analyzing the risks not only on a one year horizon (as in Solvency 2) but until maturity (retirement age). An ALM approach for the assets and liabilities of the scheme is proposed and various classical risk measures are applied to the surplus of the pension fund (probability of default, value at risk). The effect of the duration of the liability is clearly illustrated and is surely one of the key factors if we want to consider solvency measures for pension. We develop also a model based on probability of ruin instead of considering solvency only at maturity.

The paper is organized as follows. In section 2 we describe the general framework of the model in terms of asset and liability structure of the DB scheme. Section 3 is based on a static risk measurement and analyzes the influence of the time horizon on the probability of default at maturity. In section 4, we compute a solvency level using a classical value at risk approach. Finally in section 5, we consider a continuous time measurement, motivated by a pension portability argument and based on probability of ruin during a whole time interval as in the classical actuarial ruin theory.

2. ALM Model for a DB pension scheme:

We consider a Defined Benefit pension scheme based on final salary. At time t=0 an affiliate aged x is entering the scheme with an initial salary S(0).
At retirement age, at time t=T, a lump sum will be paid, expressed as a multiple of its last wage, for instance:

$$B = \frac{N}{40}\alpha.S \qquad (2.1)$$

where:

N = years of service credited by the scheme
S = final salary
B= benefit to pay (lump sum)
α= coefficient

In order to compute the contributions to the pension fund, we will use the IAS norms and in particular the projected unit credit cost method as funding technique. We need then the following assumptions:

1°) a fixed discount rate: r (risk free rate);
2°) a life table: we will assume here no mortality before retirement (no longevity risk is considered in this paper);
3°) a salary scale: the salary at time t denoted by S(t) will follow a stochastic evolution given by :

$$dS(t) = \mu S(t)dt + \eta S(t)dz(t) \qquad (2.2)$$

where: μ = average salary increase
η = volatility on salary evolution
z = standard Brownian motion

Using a best estimate approach, the contribution for the first year of service (or the normal cost) is then given by :

$$NC_0 = \frac{1}{40}.\alpha S(0).e^{(\mu - r)T} \qquad (2.3)$$

(present value at the risk free rate of the average projected benefit at retirement age).

At time t (t=1,2,..,T-1), the normal cost will have the same form :

$$NC_t = \frac{1}{40}.\alpha S(t).e^{(\mu - r)(T-t)} \qquad (2.4)$$

We can also introduce a loading factor β on the contribution ; the normal cost becomes then :

$$NC_t = \frac{1}{40}.\alpha S(t).(1+\beta).e^{(\mu-r)(T-t)} \qquad (2.5)$$

The actuarial liability AL at time t is then given by (t=0,1,..,T-1) :

$$AL_t = \frac{t+1}{40}.\alpha S(t).(1+\beta).e^{(\mu-r)(T-t)} \qquad (2.6)$$

On the asset side we assume each contribution is invested in a Geometric Brownian motion whose evolution is solution of :

$$dA(s) = \delta A(s)ds + \sigma A(s)dw(s)$$

where : δ = mean return of the investment fund
σ = volatility of the return
w = standard Brownian motion

The two sources of risk (inflation and market risks) are off course correlated :

$$corr(w(t), z(t)) = \rho t$$

Considering first the risk attached to the first year of contribution , we could take as initial condition : $A(0) = NC_0$

Then the corresponding final asset is given by (projection between t=0 and t=T):

$$A_0(T) = NC_0.e^{(\delta-\sigma^2/2)T+\sigma w(T)} = \frac{1}{40}\alpha S(0)(1+\beta).e^{(\mu+\delta-r-\sigma^2/2)T+\sigma w(T)} \qquad (2.7)$$

More generally we could consider the risk between time t and time T by computing the future evolution till maturity of the investment of the actuarial liability AL existing at time t in the reference asset A :

$$\begin{aligned} A_t(T) &= AL_t.e^{(\delta-\sigma^2/2)(T-t)+\sigma(w(T)-w(t))} \\ &= \frac{t+1}{40}\alpha S(t)(1+\beta).e^{(\mu+\delta-r-\sigma^2/2)(T-t)+\sigma(w(T)-w(t))} \end{aligned} \qquad (2.8)$$

the initial condition being then: $A_t(t) = AL_t$.

In a ALM approach, these asset values must be compared to their respective liability counterparts.
We obtain successively:

- for the final liability corresponding to the first year contribution:

$$L_0(T) = \frac{1}{40}\alpha S(0).e^{(\mu-\eta^2/2)T+\eta z(T)} \tag{2.9}$$

- for the final liability corresponding to the actuarial liability until time t:

$$L_t(T) = \frac{t+1}{40}\alpha S(t).e^{(\mu-\eta^2/2)(T-t)+\eta(z(T)-z(t))} \tag{2.10}$$

In the next sections we will make various risk measures based on this last value.

3. Probability of default

A first interesting question is to look at the probability of default at maturity without any extra resource (i.e. the risk to have not enough assets at maturity to pay the required pension benefit). In particular we can consider this probability as a function of the residual time T- t. This probability computed at time t (t=0,1,..,T-1) is given by:

$$\begin{aligned}\varphi(t,T) &= P(A_t(T) < L_t(T)) \\ &= P(Y(t,T) < M)\end{aligned} \tag{3.1}$$

where:

$$\begin{aligned}Y(t,T) &= \sigma(w(T)-w(t)) - \eta(z(T)-z(t)) = N(0,\overline{\sigma}^2(T-t)) \\ \overline{\sigma}^2 &= \sigma^2+\eta^2-2\rho\sigma\eta \\ M &= (r-\delta+\sigma^2/2-\eta^2/2)(T-t)-\ln(1+\beta)\end{aligned} \tag{3.2}$$

So finally the probability of default at maturity depends on the residual time and is given by:

$$\varphi(t,T) = \Phi(a(T-t)) \tag{3.3}$$

With :

$$\Phi = \text{distribution function } N(0,1)$$

$$a(s) = \frac{(r - \delta + \sigma^2/2 - \eta^2/2)\sqrt{s} - \ln(1+\beta)/\sqrt{s}}{\overline{\sigma}} \qquad (3.4)$$

Example 1 :

- risk free rate : r = 2%
- mean return of the fund : δ = 6%
- volatility of the fund : σ =10%
- average increase of salary : μ =5%
- volatility of the salary: η =5%
- correlation : ρ = 50%
- safety loading : β = 5%

Figure 1 shows the evolution of the probability of default as a function of the residual time T-t .

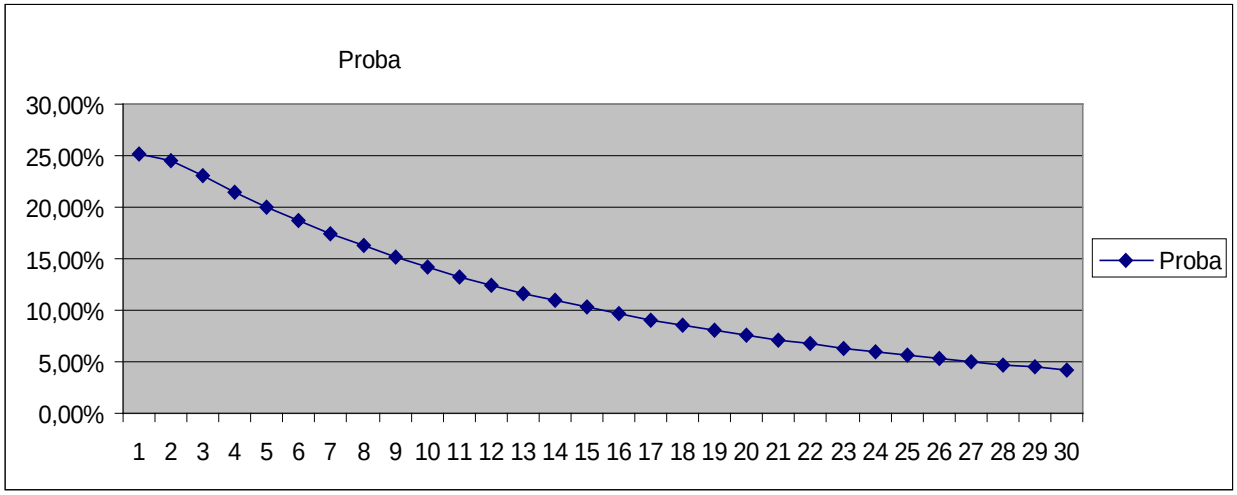

Figure 1 : Probability of default

We can see clearly a time effect: for short residual time to maturity this probability is quite high but it decreases rapidly for long residual time.

4. Value at risk approach :

In order to control this probability of default, we could as in Solvency 2 introduce a solvency level based on a value at risk approach.
We will use the following notations:

SC = solvency capital using a value at risk methodology
VaR = value at risk
α(N) = chosen safety level for a horizon of N years (for instance 99.5% on one year in Solvency 2) .

For this safety level we can choice the following value based on yearly independent default probabilities (probability of default of (1 – α) independently each year):

$$\alpha_N = (\alpha)^N \qquad (4.1)$$

We will assume that the solvency capital is invested in the reference investment fund ; so we can define this solvency capital SC(t,T) at time t (t=0,1,..,T-1) for the risk until time T as solution of:

$$P\{A_t(T) + SC(t,T).\frac{A_t(T)}{A_t(t)} < L_t(T)\} = 1 - \alpha_{T-t} \qquad (4.2)$$

Using (2.8) and (2.10), this condition becomes :

$$P((\frac{t+1}{40}\alpha S(t)(1+\beta) + SC(t,T)).e^{(\delta-\sigma^2/2)(T-t)+\sigma(w(T)-w(t))} < \frac{t+1}{40}\alpha S(t).e^{(\mu-\eta^2/2)(T-t)+\eta(z(T)-z(t))}) = 1-\alpha_{T-t}$$

After direct computation, we obtain the following value for the solvency capital:

$$SC(t,T) = \frac{t+1}{40}\alpha S(t).\{e^{(\mu-\delta)(T-t)+z_{\alpha_{T-t}}\bar{\sigma}\sqrt{T-t}+(\sigma^2-\eta^2)(T-t)/2} - e^{(\mu-r)(T-t)}(1+\beta)\} \qquad (4.3)$$

where

$$z_\beta = \beta \text{ quantile of the normal distribution such that : } \Phi(z_\beta) = \beta$$

We can express the solvency capital as a percentage of the actuarial liability AL given by (2.6) (solvency level in percent):

$$SC^{\%}(t,T) = \frac{SC(t,T)}{AL_t} = \frac{1}{1+\beta} e^{-(\delta-r)(T-t)+z_{\alpha_{T-t}}\bar{\sigma}\sqrt{T-t}+(\sigma^2-\eta^2)(T-t)/2} - 1 \qquad (4.4)$$

This relative level does not depend on the salary increase.
In particular if we look at a one year risk (as in Solvency 2), we get:

$$SC^{\%}(0,1) = \frac{SC(0,1)}{AL_0} = \frac{1}{1+\beta} e^{-(\delta-r)+z_{\alpha}\bar{\sigma}+(\sigma^2-\eta^2)/2} - 1$$

Example 2. :

- same assumptions as in example 1
- safety level on one year : $\alpha = 99.5\%$
- safety level on N years : $\alpha_N = (\alpha)^N$

Figure 2 shows then the evolution of the solvency level in percent as a function of the residual time T-t . Negative values for the SCR correspond to cases where no additional solvency is needed.

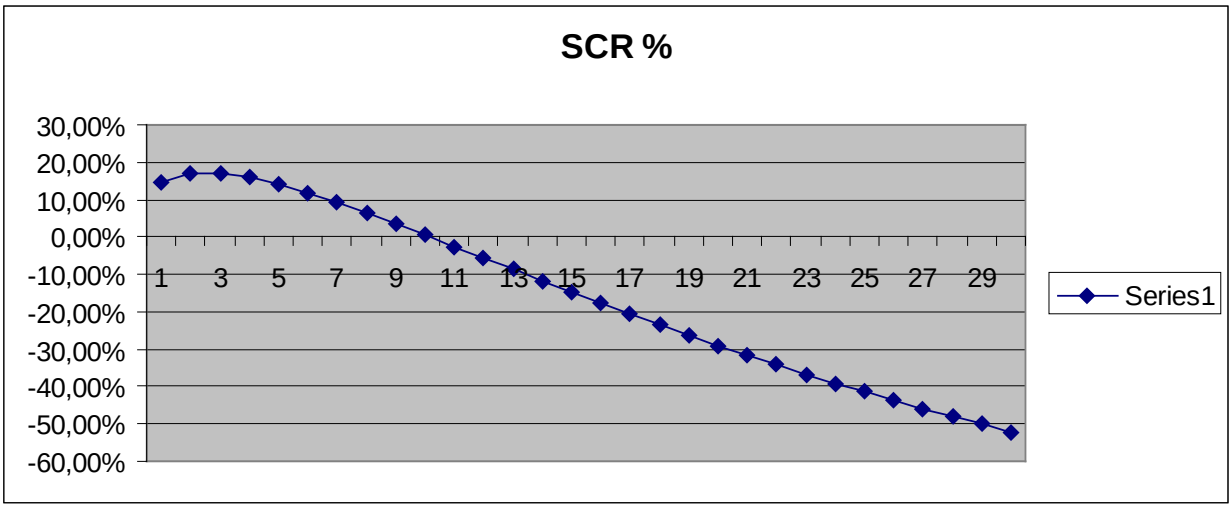

Figure 2 : Value at Risk

We can also observe as in Figure 1 a time effect.

5. Pension Portability and Ruin approach :

The risk considered until now is only related to the payment of the pension benefit at retirement age (time T). But sometimes, affiliates leave their pension scheme before retirement and a pension portability clause can be applied: the sponsor must then transfer the corresponding liability immediately (and generally without any possible penalty) to another scheme or another fund. This means that the solvency should be checked not only at maturity of the contract but at any time. The purpose of this section is to analyze this philosophy directly inspired by the actuarial classical ruin theory.

5.1. Probability of ruin without capital:

We first look at the probability of ruin without solvency capital between time t (t=0,1,…,T-1) and maturity, given by:

$$\Psi(t,T) = 1 - P\{A_t(s) \geq L_t(s),\ \forall s \in [t,T]\} \tag{5.1}$$

In order to compute this probability , we have to define the level of liability L corresponding to the years of service until time t , not only at maturity time T but also at any time s (between computation time t and maturity T). The asset process is simply defined by its market value generated by the investment at time t of the actuarial liability (cf. (2.8)):

$$\begin{aligned} A_t(s) &= AL_t.e^{(\delta-\sigma^2/2)(s-t)+\sigma(w(s)-w(t))} \\ &= \left(\frac{t+1}{40}\alpha S(t)(1+\beta).e^{(\mu-r)(T-t)}\right).e^{(\delta-\sigma^2/2)(s-t)+\sigma(w(s)-w(t))} \end{aligned} \tag{5.2}$$

The corresponding liability process can be defined as present value at time s of the final benefit at retirement T based on the first t+1 years of service and incorporating inflation till time s of departure (t < s < T):

$$L_t(s) = \frac{t+1}{40}\alpha S(t).e^{(\mu-\eta^2/2)(s-t)+\eta(z(s)-z(t))}.e^{-r(T-s)} \tag{5.3}$$

Then the probability (5.1) can be seen as a probability related to the minimum of a Geometric Brownian motion along an interval:

$$
\begin{aligned}
&P\{A_t(s) \geq L_t(s), \forall s \in [t,T]\} \\
&= P\{e^{(\delta-\mu-r+\eta^2/2-\sigma^2/2)(s-t)+\sigma(w(s)-w(t))-\eta(z(s)-z(t))} \geq \frac{e^{-\mu(T-t)}}{1+\beta}, \forall s \in [t,T]\} \\
&= 1 - P\{\min_{t\leq s\leq T}(e^{(\delta-\mu-r+\eta^2/2-\sigma^2/2)(s-t)+\sigma(w(s)-w(t))-\eta(z(s)-z(t))}) < \frac{e^{-\mu(T-t)}}{1+\beta}\}
\end{aligned}
\tag{5.4}
$$

We can then obtain the following result:

Proposition 5.1.
The probability of ruin (5.1) is explicitly given by:

$$
\begin{aligned}
\Psi(t,T) &= \Phi\left(\frac{-\ln(1+\beta)+(r-\delta+\sigma^2/2-\eta^2/2)(T-t)}{\bar{\sigma}\sqrt{T-t}}\right) \\
&+\left(\frac{e^{-\mu(T-t)}}{1+\beta}\right)^{\frac{2(\delta-r-\mu)+\eta^2-\sigma^2}{\bar{\sigma}^2}} \Phi\left(\frac{-\ln(1+\beta)-(r-\delta+2\mu+\sigma^2/2-\eta^2/2)(T-t)}{\bar{\sigma}\sqrt{T-t}}\right)
\end{aligned}
\tag{5.5}
$$

Proof :
This is a direct consequence of the law of the minimum of a Geometric Brownian motion (see for instance Back(2005)) :

If the process S is given by :

$$S(t) = e^{at+bw(t)}$$

and if :

$$z = \min_{0\leq s\leq t}\{S(s)\}$$

$$0 < L \leq 1$$

Then:

$$P(z \leq L) = \Phi(d_1) + L^{\frac{2a}{b^2}}.\Phi(d_2)$$

with:

$$d_{1,2} = \frac{\ln L \mp a.t}{\sigma\sqrt{t}}$$

Following (5.4), we can apply this result, using as parameters:

$$L = \frac{e^{-\mu(T-t)}}{1+\beta}$$

$$a = \delta - r - \mu + \eta^2/2 - \sigma^2/2$$

$$b^2 = \overline{\sigma}^2 = \sigma^2 + \eta^2 - 2\rho\sigma\eta$$

Remarks :

1°) The first term of (5.5) corresponds to the probability of default at maturity as given by (3.3) . Then the second part can be seen as the additional probability to cover the ruin strictly before maturity:

$$\Psi(t,T) = \varphi(t,T)$$
$$+\left(\frac{e^{-\mu(T-t)}}{1+\beta}\right)^{\frac{2(\delta-r-\mu)+\eta^2-\sigma^2}{\overline{\sigma}^2}} \Phi\left(\frac{-\ln(1+\beta) - (r-\delta+2\mu+\sigma^2/2-\eta^2/2)(T-t)}{\overline{\sigma}\sqrt{T-t}}\right)$$

2°) if β=0 (no loading on the best estimate) , then this probability becomes :

$$\Psi(t,T) = \Phi\left(\frac{(r-\delta+\sigma^2/2-\eta^2/2)(T-t)}{\overline{\sigma}\sqrt{T-t}}\right)$$
$$+\left(e^{-\mu(T-t)}\right)^{\frac{2(\delta-r-\mu)+\eta^2-\sigma^2}{\overline{\sigma}^2}} \Phi\left(\frac{-(r-\delta+2\mu+\sigma^2/2-\eta^2/2)(T-t)}{\overline{\sigma}\sqrt{T-t}}\right)$$

In particular for a one year horizon (cf. Solvency 2) , we get :

$$\Psi(t,t+1) = \Phi\left(\frac{(r-\delta+\sigma^2/2-\eta^2/2)}{\overline{\sigma}}\right)$$
$$+\left(e^{-\mu}\right)^{\frac{2(\delta-r-\mu)+\eta^2-\sigma^2}{\overline{\sigma}^2}} \Phi\left(\frac{-(r-\delta+2\mu+\sigma^2/2-\eta^2/2)}{\overline{\sigma}}\right)$$

3°) if β =0 and μ = 0 (no salary evolution) then the probability of ruin is equal to one.

5.2.. Probability of ruin with risky capital:

Assuming now that a solvency capital SC* is invested at time t (t=0,1,..,T-1) in

the reference investment fund A till maturity, we obtain for the probability of ruin :

$$\Psi(t,T,SC*(t,T)) = 1 - P\{A_t(s) + SC*(t,T).\frac{A_t(s)}{A_t(t)} \geq L_t(s),\ \forall s \in [t,T]\} \qquad (5.6)$$

As in (4.4) , we can express this solvency level in percentage of the actuarial liability existing at computation time t .

$$SC^*_{\%}(t,T) = \frac{SC*(t,T)}{AL_t} \qquad (5.7)$$

Then the probability of non ruin can be written as :

$$P\{A_t(s) \geq L_t(s), \forall s \in [t,T]\}$$

$$= P\{(1 + SC^*_{\%}(t,T))e^{(\delta-\mu-r+\eta^2/2-\sigma^2/2)(s-t)+\sigma(w(s)-w(t))-\eta(z(s)-z(t))} \geq \frac{e^{-\mu(T-t)}}{1+\beta}, \forall s \in [t,T]\}$$

$$= 1 - P\{\min_{t\leq s\leq T}(e^{(\delta-\mu-r+\eta^2/2-\sigma^2/2)(s-t)+\sigma(w(s)-w(t))-\eta(z(s)-z(t))}) < \frac{e^{-\mu(T-t)}}{(1+\beta)(1+SC^*_{\%}(t,T)}\}$$

We can obtain the following direct extension of proposition 5.1 using the same basic result :

Proposition 5.2:

With a risky solvency capital SC* (in percentage of the actuarial liability), the probability of ruin at time t becomes :

$$\Psi(t,T,SC^*) = \Phi(\frac{-\ln(1+\beta) - \ln(1+SC^*) + (r - \delta + \sigma^2/2 - \eta^2/2)(T-t)}{\bar{\sigma}\sqrt{T-t}})$$

$$+ \left(\frac{e^{-\mu(T-t)}}{(1+\beta)(1+SC^*)}\right)^{\frac{2(\delta-r-\mu)+\eta^2-\sigma^2}{\bar{\sigma}^2}} \Phi(\frac{-\ln(1+\beta) - \ln(1+SC^*) - (r - \delta + 2\mu + \sigma^2/2 - \eta^2/2)(T-t)}{\bar{\sigma}\sqrt{T-t}})$$

Remark :

This relation can be used to compute a well defined level of capital SC* corresponding to a fixed level of safety . Then SC* is solution of the implicit equation: $\Psi(t,T,SC^*) = 1-\alpha$

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