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Solvency requirement for long term guarantee: risk measure versus probability of ruin

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SOLVENCY REQUIREMENT FOR LONG TERM GUARANTEE: RISK MEASURE VERSUS PROBABILITY OF RUIN

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ABSTRACT

Solvency requirements are based on the idea that risk can be accepted if enough capital is present. The determination of this minimum level of capital depends on the way to consider and measure the underlying risk. Apart from the kind of risk measure used, an important factor is the way to integrate time in the process. This topic is particularly important for long term liabilities such as life insurance or pension benefits.

In this paper we study the market risk of a life insurer offering a fixed guaranteed rate on a certain time horizon and investing the premium in a risky fund.

We develop and compare various risk measurements based either on a single point analysis or on a continuous time test. Dynamic risk measures are also considered.

Keywords: Solvency capital, risk measure, probability of ruin, dynamic risk measure

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1. Introduction:

Solvency requirements for insurance companies are more than ever of first importance in the context of the recent financial crisis and the achievement of Solvency 2 rules. One of the most important aspects of Solvency 2 is to recognize that risks affecting insurance companies are of different natures and that in particular, financial risks are at the heart of the question, especially for life insurance companies. Amongst the various financial drivers of risk, equity risk plays a very special role. In deed, even if traditionally insurance companies invest the biggest part in risk free instruments (bonds, cash,…), a significant proportion of premiums are invested in equity in order to boost the returns. Pension funds have also important positions in equity. At the same time, investing in equity is riskier and will require more capital. A good equilibrium has to be found between the safety of the liability proposed to the clients and the performance of the investment strategy. A pertinent measurement of this market risk is therefore crucial. In particular the influence of the time horizon is a classical question in finance (see for instance Bodi(1995),Campbell/Viceira(2002),Lee(1990),Samuelson(1994) , Thorley(1995)).

Solvency 2 rules are only based on an annual measure (annual value at risk at 99.5%). The purpose of this paper is to compare various instruments of solvency measure of the market risk, integrating the time dimension and the horizon of the liabilities.

In order to integrate time in the process, we have considered 3 different ways to check the solvency of the contract. First, we can consider classical risk measures such as value at risk or tail value at risk but applied to the long term risk.
The idea is then to analyze the form of the needed capital in relation with the maturity of the liability. A second approach is based on a continuous check of the solvency throughout the duration and not only at maturity or at the end of the year. The classical actuarial concept of probability of ruin has been used and comparisons can be done with “static” risk measures. Finally the technique of dynamic risk measure can be proposed to obtain time consistent measurement.

The paper is organized as follows. In section 2 we describe the general framework of the model in terms of asset and liability structure. Section 3 is based on a static risk measurement and analyzes the influence of the time horizon on the probability of default at maturity and on the solvency capital using a value at risk or a tail value at risk approach. In section 4, we consider a continuous time measurement based on probability of ruin during a whole time interval (and no more at a single point) using a similar philosophy as in classical actuarial ruin theory. In section 5 we move to dynamic risk measures and we compute solvency requirements based on iterated values at risk. Section 6 concludes the paper.

2. General framework:

In order to capture the market risk of a life insurance contract with guarantee, we consider a unit amount of money paid at time t=0 (single premium) assumed to be invested on the asset side in a fund driven by a classical geometric Brownian motion . On this amount paid initially by the policyholder we assume the insurer will guarantee a fixed return at a fixed maturity t = N (liability side). Being only interested in the market risk, we will consider in this paper only pure financial products without any mortality effect. The methodology can be easily adapted introducing a deterministic life table.
For N=1, we have a yearly guarantee (horizon of one year as in Solvency 2). The guarantee can be nominal (0% guarantee corresponding to the guarantee to pay back at maturity the initial premium) or based on a positive return denoted by r_G. In general we assume: $r_G \geq 0$.

So the liability at maturity is given by:

$$L(N) = e^{r_G N} \qquad (2.1)$$

On the asset side, we denote by A(t) the value of the risky asset at time t, given by a geometric Brownian motion:

$$A(t) = \exp((\delta - \sigma^2/2)t + \sigma w(t)) \quad (0 \leq t \leq N) \qquad (2.2)$$

where: δ = mean return of the fund
σ = volatility of the fund
w= standard Brownian motion.

We can easily introduce an asset allocation in order to measure the influence of the investment policy on the risk. We consider in this context a continuous rebalancing between the risky asset modeled by (2.2) and a riskless asset, assuming a deterministic constant risk free rate. Denoting by:

$$r = \text{risk free rate}$$

$$\beta = \text{proportion in risky asset } (0 < \beta \leq 1)$$

then the new mixed asset becomes:

$$A_\beta(t) = \exp((\beta\delta + (1-\beta)r - \beta^2\sigma^2/2)t + \beta\sigma w(t)) \qquad (2.3)$$

3. Static risk measurement

We start our solvency analysis of this product by applying static risk measures based on the eventual deficit at maturity of the contract (at time N).

3.1. Probability of default:

A first interesting question is to look at the probability of default at maturity without any extra capital (i.e. the risk to have not enough assets at maturity to pay the required liability). In particular we can consider this probability as a function of the time horizon N. This probability can be easily computed:

$$
\begin{aligned}
\varphi(N) = P(A(N) < L(N)) &= P(\exp((\delta - \sigma^2/2)N + \sigma w(N)) < e^{r_G N}) \\
&= P((\delta - \sigma^2/2)N + \sigma w(N) < N r_G) \\
&= P(w(N) < \frac{N(r_G - (\delta - \sigma^2/2))}{\sigma}) \\
&= \Phi(\frac{N(r_G - (\delta - \sigma^2/2))}{\sigma\sqrt{N}}) \\
&= \Phi(a\sqrt{N}) \qquad (3.1)
\end{aligned}
$$

With :

$$
a = \frac{r_G - (\delta - \sigma^2/2)}{\sigma}
$$

$$
\Phi(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{-t^2/2} dt \qquad (3.2)
$$

(The number a representing the spread between the guaranteed rate and the mean return normalized by the volatility).

For a coherent level of guarantee (compared to the mean return of the asset), this spread is negative:

$$
r_G < \delta - \sigma^2/2
$$

Then the probability of ruin decreases with the time horizon.
We can conclude here that investing in risky assets such as equities for the long run gives better chance to achieve a fixed guaranteed return. This could lead to more aggressive strategies for long duration (Bodie Z. (1995), Campbell /Viceira (2002)).

Example 3.1 :

We will work with the central following scenario:
- guaranteed rate: $r_G = 2\%$
- risk free rate : $r = 4\%$
- mean return of the equity fund : $\delta = 7\%$
- volatility of the equity fund : $\sigma = 16\%$

Figure 1 shows the evolution of the probability of default as a function of the time horizon N . For this scenario, the probability of default on one year is equal to 40.81% but on 30 years this probability decreases to 10.14%.

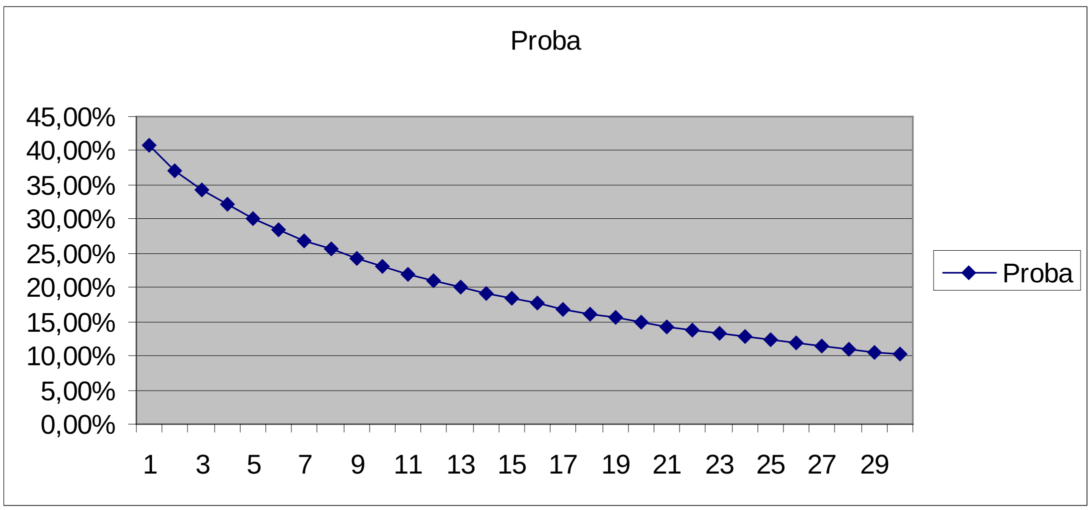

Figure 1 : Probability of default

Remark 3.1. :

We can also look at the influence of the asset allocation on this probability.
Using a mixed investment strategy, the probability of default, considered now as a joined function of the time horizon and the asset allocation becomes:

$$\varphi(N,\beta) = P(A_\beta(N) < L(N)) = \Phi(a(\beta).\sqrt{N})$$

with :

$$a(\beta) = \frac{r_G - (\beta\delta + (1-\beta)r - \beta^2\sigma^2/2)}{\beta\sigma}$$

3.2. Value at risk :

The Solvency 2 framework is based on a quantile measurement. So we will use in this section the value at risk methodology. We introduce the following notations:

SC = initial solvency capital using a value at risk methodology
VaR = value at risk
r = risk free rate
α(N) = chosen safety level for a horizon of N years (99.5% on one year in Solvency 2) .
For this safety level we can for instance choice the following value based on yearly independent default probabilities (probability of default of (1 – α) independently each year):

$$\alpha_N = (\alpha)^N \qquad (3.3)$$

Assuming first that the solvency capital is invested in the riskless asset, we can define the solvency capital SC as the solution of:

$$P\{A(N) + SC.e^{rN} < L(N)\} = 1 - \alpha_N$$

This probability is given by:

$$P\{e^{(\delta-\sigma^2/2)N+\sigma w(N)} + SC.e^{rN} < e^{r_G N}\} = P\{w(N) < \frac{\ln\{-SC.e^{rN}\} + r_G N - (\delta - \sigma^2/2)N}{\sigma}\}$$

$$= \Phi(\frac{\ln\{-SC.e^{rN}\} + r_G N - (\delta - \sigma^2/2)N}{\sigma\sqrt{N}})$$

So we obtain in this case the following value for the solvency capital:

$$SC = \frac{e^{r_G N} - e^{(\delta-\sigma^2/2)N+\sigma\sqrt{N}\, z_{1-\alpha_N}}}{e^{rN}} \qquad (3.4)$$

where

$$z_\beta = \beta \text{ quantile of the normal distribution such that} : \Phi(z_\beta) = \beta$$

We can no more conclude here in absolute terms that the risk measure is systematically a decreasing function of the time horizon even if a decreasing form appears on long term as showed by the following example:

<u>Example 3.2. :</u>

- safety level on one year : α=99.5%
- safety level on N years : $\alpha_N = (\alpha)^N$

Figure 2 shows then the evolution of the value at risk as a function of the time horizon N with same assumptions as in example 3.1.

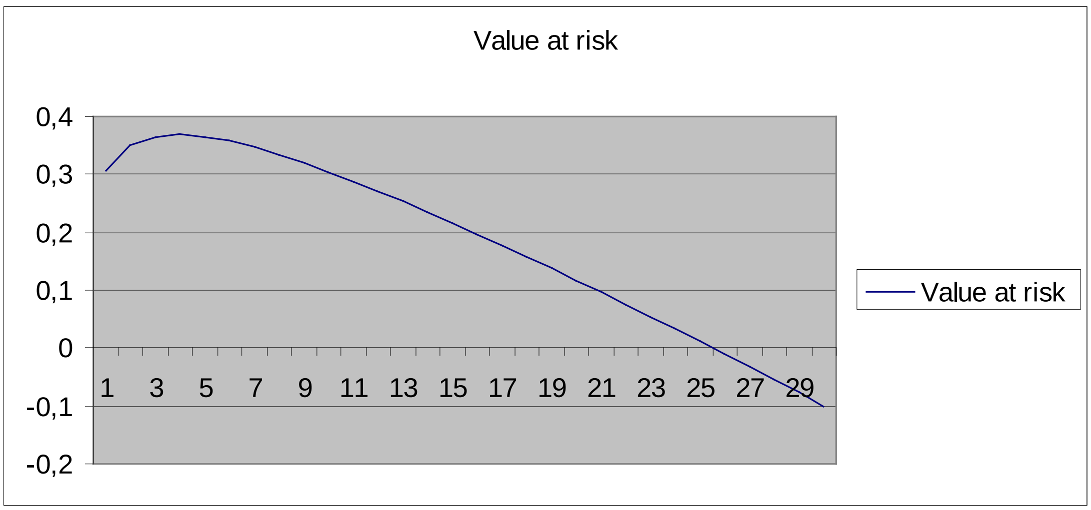

Figure 2 : Value at Risk

<u>Remark 3.2.</u> : We could also assume that the initial solvency capital is invested in the risky asset (cf. Briys /De Varenne (1997)). Then the solvency condition becomes:

$$P\{A(N) + SC.A(N) < L(N)\} = 1 - \alpha_N$$

Similar direct computations lead to the following alternative value for the solvency capital SC:

$$SC = e^{N(r_G - (\delta - \sigma^2/2)) - \sigma\sqrt{N} z_{1-\alpha_N}} - 1 \tag{3.5}$$

<u>Remark 3.3</u> : as in section 3.1., we could introduce a mixed investment strategy for the premium and/or the solvency capital.

3.3. Tail Value at risk:

Value at risk is a classical tool in risk management and is the reference for Basel 2 and Solvency 2. But from a theoretical point of view it is well known that value at risk presents some important problems (Artzner et all (1997)).

The methodology presented above can be easily extended using tail value at risk instead of value at risk. Assuming that the solvency capital is invested in the riskless asset and denoting by:

SC = initial solvency capital using a tail value at risk methodology

then we obtain:

$$SC = e^{-rN}(E\{L(N) - A(N) \mid L(N) - A(N) > V_{\alpha_N} \})$$

with

$$V_{\alpha_N} = e^{r_G N} - e^{(\delta - \sigma^2/2)N + \sigma\sqrt{N}.z_{1-\alpha_N}}$$

Then using the properties of the log normal distribution we get:

$$\begin{aligned} SC &= e^{-rN}(e^{r_G N} - E\{A(N) \mid A(N) < e^{(\delta-\sigma^2/2)N + \sigma\sqrt{N} z_{1-\alpha_N}} \}) \\ &= e^{-rN}(e^{r_G N} - e^{\delta N} \frac{1}{1-\alpha_N} \Phi(z_{1-\alpha_N} - \sigma\sqrt{N})) \end{aligned}$$

Finally:

$$SC = e^{(r_G - r)N} - e^{(\delta - r)N} \frac{1}{1-\alpha_N} \Phi(z_{1-\alpha_N} - \sigma\sqrt{N})) \qquad (3.6)$$

<u>Example 3.3. :</u>

Figure 3 shows then the evolution of the tail value at risk as a function of the time horizon N with same assumptions as in examples 3.1. and 3.2. The form is similar to the value at risk (first increasing then decreasing on a long term basis).

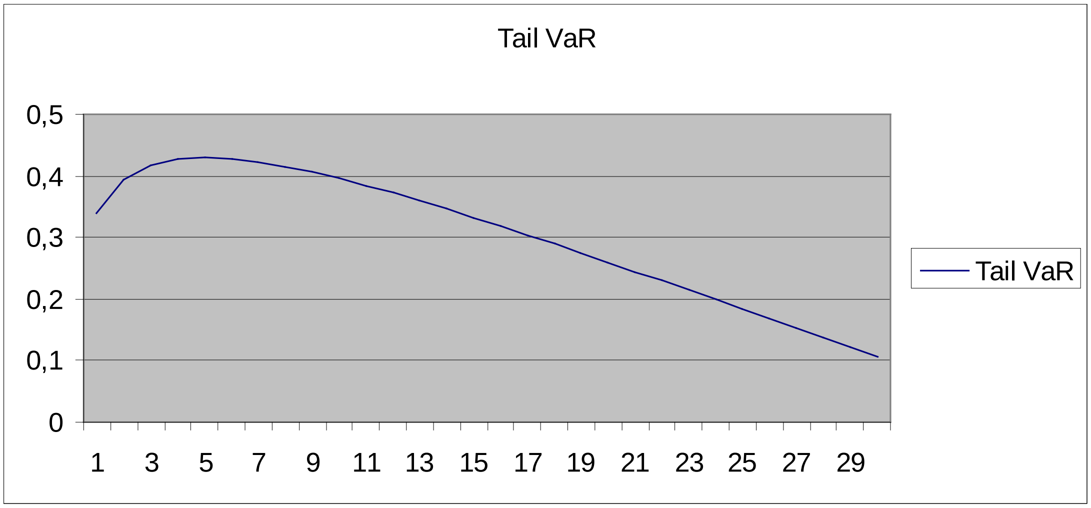

Figure 3 : Tail value at risk

4. Continuous time risk measurement :

The different approaches presented in the preceding section show undoubtedly that a horizon effect exists in terms of measurement of equity risk when based on solvency tools. We used for this a static risk measurement consisting on a point measure only at maturity. Another classical actuarial approach should be to check the solvency not only at maturity of the contract but at any time. The purpose of this section is to analyze this philosophy inspired by the actuarial classical ruin theory.

4.1. Probability of ruin without capital:

We first look at the probability of ruin without solvency capital given by:

$$\Psi(N) = 1 - P\{A(t) \geq L(t),\ \forall t \in [0, N]\} \qquad (4.1)$$

In order to compute this probability , we have to define the level of liability L(t) not only at maturity but also at any time , the asset process being still defined by its market value (2.2).

4.1.1. Accounting view :

We will first use an accounting view (mathematical reserve approach) :

$$L(t) = e^{r_G t} \qquad (4.2)$$

Then the probability (4.1) can be seen as a probability related to the minimum of a Geometric Brownian motion along an interval :

$$\begin{aligned}\Psi(N) &= P(\min_{0\le s\le N} e^{(\delta-\sigma^2/2)s+\sigma w(s)} < e^{r_G s}) \\ &= P(\min_{0\le s\le N} e^{(\delta-r_G-\sigma^2/2)s+\sigma w(s)} < 1)\end{aligned} \tag{4.3}$$

We can then obtain the following result :

Proposition 4.1.

Without solvency capital and using the accounting value (4.2) for the liabilities, the ruin is sure whatever the mean return of the risky asset is.

Proof:
This is a direct consequence of the law of the minimum of a Geometric Brownian motion (see for instance Back(2005)) :

If the process S is given by :

$$S(t) = e^{(\mu-\sigma^2/2)t+\sigma w(t)}$$

and if :

$$z = \min_{0\le s\le t}\{S(s)\}$$

$$0 < L \le 1$$

Then: $$P(z \le L) = \Phi(d_1) + L^{\frac{2\mu}{\sigma^2}-1}.\Phi(d_2) \tag{4.4}$$

with: $$d_{1,2} = \frac{\ln L \mp (\mu-\sigma^2/2)t}{\sigma\sqrt{t}}$$

Taking L=1 in (4.4) , we obtain for the probability of ruin (4.3) :

$$\Psi(N) = \Phi(\frac{-(\delta-r_G-\sigma^2/2)N}{\sigma\sqrt{N}}) + \Phi(\frac{(\delta-r_G-\sigma^2/2)N}{\sigma\sqrt{N}}) = 1$$

4.1.2. Surrender view :

Instead of using for the liability valuation the mathematical reserve, we could introduce a classical surrender penalization (see for instance Grosen / Jorgensen (2002)) , and alternatively substitute to (4.2) the following form for the liability :

$$L(t) = \rho(t).e^{r_G t}$$

where the function ρ is a non decreasing function such that :

$$\rho(t) \leq 1 \ \text{ for } \ 0 \leq t \leq N$$
$$\rho(N) = 1$$

(1- ρ (t)) representing the surrender charge in case of surrender at time t.
As example we will use here an exponential form for this penalty (used in some life insurance products):

$$\rho(t) = e^{-\lambda(N-t)} \qquad (\lambda > 0)$$

leading to the following liability function:

$$\begin{aligned} L(t) &= e^{-\lambda(N-t)}.e^{r_G t} \\ &= e^{-\lambda N}.e^{(r_G+\lambda)t} \end{aligned} \tag{4.5}$$

Then the probability of ruin (4.3) becomes:

$$\Psi(N,\lambda) = P(\min_{0\leq s\leq N} e^{(\delta - r_G - \lambda - \sigma^2/2)s + \sigma w(s)} < e^{-\lambda N})$$

Using the general formula (4.4) we obtain the following result:

Proposition 4.2:

Without solvency capital and using a negative exponential surrender penalty of the form (4.5), the probability of ruin is given by :

$$\begin{aligned} \Psi(N,\lambda) &= \Phi\left(\frac{-(\delta - r_G - \sigma^2/2)N}{\sigma\sqrt{N}}\right) + (e^{-\lambda N})^{\frac{2(\delta - r_G - \lambda)}{\sigma^2} - 1}\Phi\left(\frac{(\delta - r_G - 2\lambda - \sigma^2/2)N}{\sigma\sqrt{N}}\right) \\ &= \Phi(a\sqrt{N}) + (e^{-\lambda N})^{\frac{2(\delta - r_G - \lambda)}{\sigma^2} - 1}\Phi((-a - 2\lambda/\sigma)\sqrt{N}) \end{aligned} \tag{4.6}$$

<u>Remark 4.1:</u> the first part of this probability corresponds to the probability of default (3.1) only at maturity. The second term reflects the risk of insolvency before maturity.

4.1.3. Fair valuation of liability:

Another possibility for the liability function is to introduce a fair valuation concept. Then the liability at time t can be seen as the present value of the final liability discounted at the risk free rate:

$$L(t) = e^{-r(N-t)}.e^{r_G N}$$

or:

$$L(t) = e^{-(r-r_G)(N-t)}.e^{r_G t}$$

So this expression can be seen as a special case of section 4.1.2 with:

$$\lambda = r - r_G$$

This parameter is strictly positive as prescribed in section 4.1.2. if the risk free rate is strictly higher than the guaranteed rate of the product. We can then obtain the following corollary of proposition 4.2 :

Corollary 4.1 :

If the liability is measured at its fair value and if the guaranteed rate is less than the risk free rate , the probability of ruin without solvency capital is given by :

$$\Psi(N) = \Phi(a\sqrt{N}) + (e^{-(r-r_G)N})^{\frac{2(\delta - r\ )}{\sigma^2} - 1} \Phi((-a - 2(r - r_G)/\sigma)\sqrt{N}) \qquad (4.7)$$

4.2. Probability of ruin with riskless capital:

We assume now that an initial solvency capital SC is injected in the business at time t=0 and invested in the riskless asset till maturity.
Then the probability of ruin (4.1) becomes:

$$\Psi(N, SC) = 1 - P\{A(t) + SC.e^{rt} \geq L(t),\ \forall t \in [0, N]\}$$

Using as liability valuation the surrender view of section 4.1.2, this probability can be written as:

$$\Psi(N, SC, \lambda) = 1 - P\{A(t) + SC.e^{rt} \geq e^{-\lambda(N-t)}.e^{r_G t},\ \forall t \in [0, N]\}$$

This probability can be once again expressed as a minimum but unfortunately not as the minimum of a Geometric Brownian motion:

$$\Psi(N,SC,\lambda) = P(\min_{0\le s\le N}(e^{(\delta-\sigma^2/2)s+\sigma w(s)} / (e^{-\lambda N} e^{(r_G+\lambda)t} - SC.e^{rt})) < 1) \quad (4.8)$$

Nevertheless we can obtain an explicit form of this probability if we use a fair valuation approach for the liabilities :

Proposition 4.3.

If the liability is measured by its fair value and if the guaranteed rate is less than the risk free rate, the probability of ruin with a riskless solvency capital SC strictly less than $e^{-(r-r_G)N}$ *is given by:*

$$\Psi(N,SC) = \Phi\left(\frac{\ln(e^{-(r-r_G)N} - SC) - (\delta - r - \sigma^2/2)N}{\sigma\sqrt{N}}\right)$$
$$+ (e^{-(r-r_G)N} - SC)^{\frac{2(\delta-r)}{\sigma^2}-1}\Phi\left(\frac{\ln(e^{-(r-r_G)N} - SC) + (\delta - r - \sigma^2/2)N}{\sigma\sqrt{N}}\right) \quad (4.9)$$

Proof :

The fair valuation approach corresponding to the case $\lambda = r - r_G$ then probability (4.8) becomes :

$$P(\min_{0\le s\le N}(e^{(\delta-\sigma^2/2)s+\sigma w(s)} / (e^{-\lambda N} e^{(r_G+\lambda)t} - SC.e^{rt})) < 1)$$
$$= P(\min_{0\le s\le N}(e^{(\delta-r-\sigma^2/2)s+\sigma w(s)}) < e^{-(r-r_G)N} - SC)$$

We can then use directly formula (4.4) with :

$$L = e^{-(r-r_G)N} - SC$$

which satisfies the condition $0 < L \le 1$ taking into account the assumptions of the proposition.

Remark 4.2. :

The condition on the solvency capital just means that the initial capital is less than the present value of the total liability at maturity, which seems trivial.

4.3. Probability of ruin with risky capital:

Assuming finally that the initial solvency capital SC is invested in the risky asset A given by (2.2) till maturity, we obtain for the probability of ruin :

$$\Psi(N, SC, \lambda) = 1 - P\{A(t) + SC.\, A(t) \geq e^{-\lambda(N-t)}.e^{r_G t},\ \forall t \in [0, N]\}$$

In this case it is possible to obtain an explicit value of this probability whatever the value of λ is:

Proposition 4.4:

With a risky solvency capital and using a negative exponential surrender penalty of the form (4.5), the probability of ruin is given by :

$$\Psi(N, \lambda, SC) = \Phi(a\sqrt{N} - \frac{\ln(1+SC)}{\sigma\sqrt{N}})$$
$$+ (\frac{e^{-\lambda N}}{1+SC})^{\frac{2(\delta - r_G - \lambda)}{\sigma^2} - 1} \Phi(-a\sqrt{N} - \frac{\ln(1+SC) + 2\lambda N}{\sigma\sqrt{N}}) \qquad (4.10)$$

Proof :

The probability of ruin can be written in this case :

$$\Psi(N, SC, \lambda) = 1 - P\{A(t) + SC.\, A(t) \geq e^{-\lambda(N-t)}.e^{r_G t},\ \forall t \in [0, N]\}$$
$$= P(\min_{0 \leq s \leq N}((A(s) + SC.A(s)) / e^{(r_G + \lambda)s}) < e^{-\lambda N})$$
$$= P(\min_{0 \leq s \leq N}(e^{(\delta - r_G - \lambda - \sigma^2/2)s + \sigma w(s)}) < e^{-\lambda N} / (1 + SC))$$

We can then apply directly formula (4.4) to obtain the result.

Remarks 4.3 :

1°) For $\lambda = 0$ we obtain the accounting view (section 4.1.1) and for $\lambda = r - r_G$, the fair valuation (section 4.1.3).

2°) Formula (4.10) shows the influence of the two possible tools available for the insurer to decrease its probability of ruin: putting a surrender penalty on the product and/or injecting an initial solvency capital.

4.4. Computation of the solvency capital :

For a fixed level of solvency capital SC, the formulas developed in sections 4.2 and 4.3 permit us to compute the corresponding probability of ruin. We could also use a reverse methodology and try to determine the level of capital corresponding to a maximum accepted probability of ruin (equivalent to section 3.2 in static measurement). Taking into account the form of these probabilities of ruin (see for instance (4.9) or (4.10)), no explicit solution of the capital can be hoped. Numerical procedures must then be applied.
For instance, in the case of a risky solvency capital, using (4.10), the solvency capital SC corresponding to a fixed level of safety α_N is the solution of the following implicit equation:

$$\Phi(a\sqrt{N}-\frac{\ln(1+SC)}{\sigma\sqrt{N}})+(\frac{e^{-\lambda N}}{1+SC})^{\frac{2(\delta-r_G-\lambda)}{\sigma^2}-1}\Phi(-a\sqrt{N}-\frac{\ln(1+SC)+2\lambda N}{\sigma\sqrt{N}})=1-\alpha_N$$

This equation can also be written as :

$$SC=e^{-\lambda N}\left(\frac{\Phi(d^*(SC))}{1-\alpha_N-\Phi(d(SC))}\right)^{1/K}-1$$

with :

$$K=\frac{2(\delta-r_G-\lambda)}{\sigma^2}-1$$

$$d(SC)=a\sqrt{N}-\frac{\ln(1+SC)}{\sigma\sqrt{N}}\quad \text{and}\quad d^*(SC)=-a\sqrt{N}-\frac{\ln(1+SC)+2\lambda N}{\sigma\sqrt{N}}$$

suggesting the following recursive algorithm for the computation of SC:

$$SC_n=e^{-\lambda N}\left(\frac{\Phi(d^*(SC_{n-1}))}{1-\alpha_N-\Phi(d(SC_{n-1}))}\right)^{1/K}-1 \qquad (4.11)$$

5. Iterated risk measurement :

Instead of using only a point approach as in section 3 or a continuous risk measurement through a probability of ruin as in section 4, we could introduce an intermediate requirement by asking an initial solvency capital such as to remain solvent only at some fixed discrete times (for instance at the end of each year). Dynamic risk measures must be then considered; in particular we can work with the concept of iterated risk measures (see for instance Pflug / Romisch(2007) or Hardy/Wirch (2004)).

5.1. General principle of iterated measurement:

We will use the following notations in this section:

$$X = \text{risk} = \text{stochastic cash flow to be paid at maturity } t = N$$

$$SC_{t,N}(X) = \text{solvency capital at time t to cover the risk X at time N}$$

$$\rho = \text{static risk measure (for instance value at risk)}$$

$$\{\Im_t, 0 \le t \le N\} = \text{filtration (progressive arrival of information on X)}$$

$$r = \text{riskfree rate}$$

Then the initial needed capital at time t=0 using a classical static risk measurement is given by the present value of the risk measure (cf. section 3):

$$SC_{0,N}(X) = e^{-rN}.\rho(X|\Im_0) \qquad (5.1)$$

With dynamic risk measurement we are now interested to compute the needed capital not only at time 0 but for instance at the end of each year.
Two approaches can be immediately introduced:

- the *accumulated approach*: successive solvency capitals are just the accumulation with interest of the initial capital (assumed to be invested in a riskless asset):

$$SC_{t,N}(X) = e^{rt}.SC_{0,N}(X) = e^{-r(N-t)}.\rho(X|\Im_0) \qquad (5.2)$$

As suggested by formula (5.2) no additional information is integrated in the succesive capitals.

- The *recalculated approach*: the static risk measurement is applied at each time t and gives then:

$$SC_{t,N}(X) = e^{-r(N-t)}.\rho(X|\Im_t) \tag{5.3}$$

This approach seems to be more time consistent, taking into account the progressive arrival of new information on the risk X.

But it could generate during the process unexpected variations of the solvency capital.

Then a third approach called the *iterated risk measurement*, tries to anticipate these possible future capital requirements; in this vision, the successive risks are no more the future cash flow X but the needed capital at the next period of time. In this recursive method we first compute the needed capital one period before maturity (we will assume here annual units of time following the accounting rules but off course formula can be developed using other time units).

At time t=N-1 we will use the recalculated formula (5.3) :

$$SC_{N-1,N}(X) = e^{-r}.\rho(X|\Im_{N-1})$$

Then at time t=N-2, the solvency capital becomes :

$$SC_{N-2,N}(X) = e^{-r}.\rho(SC_{N-1,N}|\Im_{N-2})$$

In general , at time t :

$$SC_{t,N}(X) = e^{-r}.\rho(SC_{t+1,N}|\Im_t) \tag{5.4}$$

5.2. Application to solvency requirement:

Let us adapt here the value at risk computations of section 3.2 in this iterated approach. More precisely we would like to generalize, in a dynamic iterated environment, formula (3.4) written as:

$$SC_{0,N} = e^{(r_G-r)N} - e^{(\delta-r-\sigma^2/2)N+\sigma\sqrt{N}z_{1-\alpha_N}}$$

Proposition 5.1.

The iterated solvency capital at time t (t=0,1,…, N-1) using a value at risk approach is given by :

$$SC_{t,N} = e^{r_G N}.e^{-r(N-t)} - A(t)e^{(\delta - r - \sigma^2/2)(N-t) + \sigma(N-t) z_{1-\alpha_1}} \qquad (5.5)$$

where A(t) is the asset value at time t .

In particular the initial solvency capital at time t=0 is given by:

$$SC_{0,N} = e^{(r_G - r)N} - e^{(\delta - r - \sigma^2/2)N + \sigma N z_{1-\alpha_1}} \qquad (5.6)$$

These values can be compared with a simple recalculated approach at each time:

$$SC_{t,N} = e^{r_G N}.e^{-r(N-t)} - A(t)e^{(\delta - r - \sigma^2/2)(N-t) + \sigma\sqrt{N-t}\, z_{1-\alpha_{N-t}}} \qquad (5.7)$$

<u>Proof :</u>

We start the recursive calculation with the capital at time t=N-1.
This first computation gives by definition a same value as in the recalculated approach:

$$SC_{N-1,N} = e^{r_G N}.e^{-r} - A(N-1).e^{(\delta - r - \sigma^2/2) + \sigma z_{1-\alpha_1}}$$

Then at time t=N-2, this capital is stochastic because of the presence of the future value of the asset. This random variable, seen from time t=N-2 , can be written:

$$A(t-1) = A(t-2).e^{(\delta - \sigma^2/2) + \sigma(w(t-1) - w(t-2))}$$

So the needed solvency capital at time t=N-1 seen from time t=N-2 is a random variable X given by :

$$X = e^{r_G N}.e^{-r} - A(N-2).e^{2(\delta - \sigma^2/2) - r + \sigma(w(N-1) - w(N-2)) + \sigma z_{1-\alpha_1}}$$

The needed capital at time t=N-2 is then the amount $SC_{N-2,N}$ such that :

$$P(SC_{N-2}.e^{r} \geq X) = \alpha_1$$

Or :

$$P(SC_{N-2,N}.e^{r} \geq (e^{r_G N}.e^{-r} - A(N-2).e^{2(\delta - \sigma^2/2) - r + \sigma(w(N-1) - w(N-2)) + \sigma z_{1-\alpha_1}})) = \alpha_1$$

Or:

$$P((w(t-1)-w(t-2)) \geq \frac{1}{\sigma}(\ln(e^{r_G N-r}.-SC_{N-2,N}.e^{r})-\ln(A(N-2).e^{2(\delta-\sigma^2/2)-r+\sigma z_{1-\alpha_1}})) = \alpha_1$$

So that:

$$\ln(e^{r_G N-r}.-SC_{N-2,N}.e^{r})-\ln(A(N-2).e^{2(\delta-\sigma^2/2)-r+\sigma z_{1-\alpha_1}}) = \sigma.z_{1-\alpha_1}$$

And finally:

$$SC_{N-2,N} = e^{r_G N}.e^{-2r} - A(N-2)e^{2(\delta-r-\sigma^2/2)+2\sigma z_{1-\alpha_1}}$$

The same procedure can be used recursively to generate all the values (5.5).

Corollary 5.1 ***:***

1°) If we use a same safety level for any time horizon and higher than 50% :

$$\alpha_s = \alpha > 1/2$$

then the iterated approach (5.5) gives higher capital requirements than the classical approach (5.7).

2°) It is possible to have a same capital level in the 2 approaches if we adjust the safety level according to the equilibrium relation:

$$z_{1-\alpha_S} = \sqrt{s}.z_{1-\alpha_1}$$

6. Conclusion :

Starting from the classical problem of solvency linked to a guaranteed interest rate in life insurance and assuming the investment in some risky asset driven by a geometric Brownian motion, we have developed various methodologies to measure the risk and determine the solvency capital requirement. More precisely we have compared 3 different approaches:

- a static approach based on a risk measure on the final position of the insurer;
- a continuous approach based on the probability of ruin during all the life of the contract;
- a dynamic approach based on iterated risk measures on annual successive risks.

These different results illustrate the importance of considering in the capital requirement computation, not only the kind of risk measure used but also the way to integrate the time horizon.

References

Artzner P., Delbaen F.,Eber J.M.,and Heath D. (1999): Coherent Measures of Risk, *Mathematical Finance* **9,** pp. 203-228

Back K.(2005): *A Course in Derivatives Securities,* Springer

Bodie,Z.(1995): On the Risks of Stocks in the Long Run. *Financial Analysts Journal* **51(3),**pp.68-76

Bodie, Z , Merton and Samuelson (1992): Labor Supply Flexibility and Portfolio Choice in a Lifecycle Model , *Journal of Economic Dynamics and Control* **16(3),** pp.427-449

Briys E.,de Varenne F. (1997): On the Risk of Life Insurance Liabilities : Debunking Some Common Pitfalls, *Journal of Risk and Insurance* **64(4)**, pp. 673-694

Campbell J., Viceira L. (2002): *Strategic Asset Allocation – Portfolio Choice for Long term Investors,* Oxford University Press

Grosen A. , Jorgensen P. (2002) : Life Insurance Liabilities at Market Value : An Analysis of Insolvency Risk, Bonus Policy and Regulatory Intervention Rules in a Barrier Option framework , *Journal of Risk and Insurance* , **69,1,**pp. 63-91

Hardy M., Wirch J.L. (2004) : The Iterated CTE : a Dynamic Risk Measure, *North American Actuarial Journal*, **8**, pp. 62-75

Lee,W. (1990): Diversification and Time ,Do Investment Horizons matter?, *Journal of Portfolio Management* **4***,* pp. 60-69

McNeil A.,Frey R. ,Embrechts P. (2005) : *Quantitative Risk Management*, Princeton

Merton R. (1992)*: Continuous Time Finance,* Wiley

Pflug G. , Romisch W.(2007): *Modeling, Measuring and Managing Risk*, World Scientific

Samuelson ,P.(1994): The Long Term Case for Equities. *Journal of Portfolio Management* **21(1),** pp 15-24

Samuelson P. (1963) : Risk and uncertainty : A fallacy of Large Numbers, *Scientia*, **98***,* pp. 108-113

Sandstrom A. (2006) : *Solvency* , Chapman & Hall

Thorley S. (1995): The Time Diversification Controversy *, Financial Analyst Journal* **51(3),** pp.*18-22*

Wirch J.L. ,Hardy M.R. (1999) : A synthesis of risk measures for capital adequacy , *Insurance : Mathematics and Economics*, **25**, pp. 337-347