T E C H N I C A L
R E P O R T

# 11012

# Semi-Markov regime switching interest rate models and minimal entropy measure

HUNT, J. and P. DEVOLDER

I A P S T A T I S T I C S
N E T W O R K

INTERUNIVERSITY ATTRACTION POLE

http://www.stat.ucl.ac.be/IAP

# Semi Markov regime switching interest rate models and minimal entropy measure

Julien Hunt* Pierre Devolder †

February 16, 2011

**Abstract**

In this paper, we present a discrete time regime switching binomial-like model of the term structure where the regime switches are governed by a discrete time semi-Markov process. We model the evolution of the prices of zero-coupon when given an initial term structure as in the model by Ho and Lee that we aim to extend. We discuss and derive conditions for the model to be arbitrage free and relate this to the notion of martingale measure. We explicitely show that due to the extra source of uncertainty coming from the underlying semi-Markov process, there are an infinite number of equivalent martingale measures. The notion of path independence is also studied in some detail, especially in the presence of regime switches. We deal with the market incompleteness by giving an explicit characterization of the minimal entropy martingale measure. We give an application to the pricing of a European bond option both in a Markov and semi-Markov framework. Finally, we conclude.

## 1 Introduction

Interest rate structures have undoubtedly become one of the most active topics of research in modern quantitative finance and in life insurance. Various methodologies have been developed in order to model the uncertainty related to the future evolution of interest rates and its consequences in terms of risk management. Classical models include discrete time approaches (such as binomial models [22]) as well as continuous time frameworks (such as the famous Vasicek, CIR or Hull and White models (see [32], [6] and [24])).

Amongst these approaches, regime switching techniques are surprisingly not so frequent. Regime switching models in finance are based on the simple and natural idea that the economic environment is not stable but subject to regular changes at some non-predictable stopping times. For instance, we can think of successive periods of expansion and recession in the economy. In modelling terms, these changes should induce a sudden modification of the underlying parameters of the basic structure.

Regime switching models have been developed in the financial literature in the early seventies with the paper of Goldfeld *et al.*([18]). This was later followed by

*Université Catholique de Louvain, ISBA, Voie du roman pays 20, B-1348 Louvain-La-Neuve. E-Mail: julien.hunt@uclouvain.be. J.Hunt acknowledges financial support from IAP research network P6/03 of the Belgian Government(Belgian Science Policy).

†Université Catholique de Louvain, ISBA, Voie du roman pays 20, B-1348 Louvain-La-Neuve. E-Mail: pierre.devolder@uclouvain.be

the paper by Hamilton ([19]) which triggered an enormous amount of research into the field of regime switching time series. A large part of this litterature focuses on Markov switching models. Applications of regime switching models to financial derivatives, interest rates and portfolio optimization have mainly been explored in a continuous time Markov switching framework. There is a long list of papers that deal with this subject. Among others, let us cite [11], [15], [16] and [29]. As for regime switching interest rate models let us cite [26] and [33] (to mention only a few). There is also one paper that briefly presents a discrete-time Markov switching model for option pricing ([25]).

Although popular, we feel there are many drawbacks with the simple homogeneous Markov switching models that can be easily dealt with by using semi-Markov switching models. The first major drawback of these models is the memoryless property of Markov processes which, as argued in [30], seems inadequate in real-world data. Clearly, semi-Markov processes offer an interesting alternative in that respect. Second, as discussed in [7], constant transition probabilities for the switching process are somewhat unrealistic for interest rates data. Again, semi-Markov processes provide a simple and flexible alternative. Third, the idea of switching processes is to deal with the impact of a changing environment (i.e. the business cycle) on the parameters of our model, the underlying process being a model for this changing environment. It has been documented (see [10]) that the business cycle exhibits duration dependence. This feature is directly captured by semi-Markov processes through their dependence on backward recurrence time i.e. the time elapsed since the last jump of the process (or to put it otherwise: the duration). Moreover, since Markov process are a subclass of semi-Markov processes (see [21] for more details), semi-Markov switching models should always perform at least as well as Markov switching models and at little or no cost in terms of complexity. This is an extremely powerful argument to motivate the use of semi-Markov switching models. Finally, some authors have shown that interest rates data often rejects the Markov property (see [23]). Again, this could point in the direction of semi-Markov switching models. All these reasons push us to consider semi-Markov switching models as an easy to understand, flexible and coherent alternative to Markov switching processes.

This has lead some authors to study the idea of **semi-Markov** regime switching models in the field of financial derivatives. A nice feature of semi-Markov models is that they are intuitively coherent and easy to understand whilst remaining parcimonious. We mention some papers published about semi-Markov switching models (or that can be adapted easily to encompass for semi-Markov switching) (see [17], [26], [30], [31]). Most of these papers are about option pricing in a continuous time semi-Markov regime switching framework.

Very few papers specifically deal with semi-Markov regime switching interest rates. Our paper studies this subject in the context of a generalisation of the Ho and Lee model to a semi-Markov regime switching framework. The paper is organised as follows. The first part is a brief reminder on discrete time semi-Markov processes. We then introduce a market model that allows for the regime switching extension. We move on to discuss the notion of absence of arbitrage and its relation to martingale measures in this framework. We discuss the notion of path independence in the presence of regime switching. Furthermore, we deal with the market incompleteness by giving an explicit characterization of the minimal entropy martingale measure. We also give an application of our model and of our measure to the pricing of a European bond option in both a Markov and semi-Markov switching framework. Finally, we conclude by giving some general comments on our model.

## 2 A framework for binomial regime switching term structure modelling

Let us consider a discrete time financial market built on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We suppose that financial transactions can only take place at fixed times 0,1,2,...,$T^*$. We note $\mathbb{T} = \{0, 1, ..., T^*\}$.

We define the set $E = \{1, ..., m\}$ for $m$ finite and $\mathcal{E}$ as the sigma-algebra on $E$. We suppose our probability space carries a pair of processes $(X_n, T_n)$ taking values in $E \times \mathbb{N}$. We suppose $(X_n, T_n)$ is a homogeneous Markov renewal process of semi-Markov kernel $Q$ i.e. that for all $n$, $X_n$, $j$ and $t$

$$\mathbb{P}(X_{n+1} = j; T_{n+1} - T_n \leq t | X_0, T_0, ..., X_n, T_n) = \mathbb{P}(X_{n+1} = j; T_{n+1} - T_n \leq t | X_n) := Q_{X_n j}(t)$$

We define $\nu_t$ as $\nu_t = \max_n \{T_n \leq t\}$. Then, the semi-Markov process $Y$ of kernel $Q$ is defined by

$$Y_t = X_{\nu_t}$$

where we suppose that $Y_0$ is known and non-random. The process $Y$ will control the "regime" or "state" of the economy.

Similarly, the process $K_t$ is defined as

$$K_t = t - T_{\nu_t}$$

and represents the time elapsed since the last jump. We suppose $K_0$ is known and non-random. Process $K_t$ is known as the backward recurrence time process. The introduction of the backward recurrence time process is justified by the fact that the pair $(Y_t, K_t)$ satisfies the Markov property (see [27] for more details).

Finally, we define $\mathcal{F}_t = \sigma(\zeta_s, Y_s, K_s, 0 \leq s \leq t)$.

There are two main approaches to discrete time term structure modelling, one is to model interest rates directly, the other is to model the evolution of zero-coupon bonds. We will focus on the second approach.

**Definition 2.1.** *A zero-coupon bond of maturity $T \leq T^*$ is a financial product that yields one unit of currency at maturity $T$. We will denote the time to maturity of a zero-coupon bond at time $t$ by $\tau$ (so $\tau = T - t$). We write $P_t(\tau)$ for the price at time $t$ of a zero-coupon whose time to maturity is $\tau$.*

We suppose as given an initial term structure i.e. a set of values $P_0(\tau)$ for all $\tau$.

For every $\tau$ and for every state $i \in E$, we assume the existence of two real values $u_i(\tau)$ and $d_i(\tau)$. We will further assume that $u_i(\tau)$ and $d_i(\tau)$ are strictly positive and $u_i(\tau) > d_i(\tau)$. We note by $u_i$ (resp. $d_i$) the vector whose entries are given by the $u_i(\tau)$'s (resp. $d_i(\tau)$'s) i.e. $u_i = (u_i(1), ..., u_i(T^*))$ (resp. $d_i = (d_i(1), ..., d_i(T^*))$).

We introduce an adapted vector stochastic processes $\zeta_t$ of size $T^*$. Conditional on the past, this process behaves like a "binomial" process. We impose that

$$\mathbb{P}(\zeta_{t+1} = u_{Y_t}|\mathcal{F}_t) = 1 - \mathbb{P}(\zeta_{t+1} = d_{Y_t}|\mathcal{F}_t)$$

This process describes the evolution of the term structure from time $t$ to $t+1$. We will say that the term structure goes up (resp. down) if $\zeta_{t+1} = u_{Y_t}$ (resp. $\zeta_{t+1} = d_{Y_t}$).

We suppose that $\zeta_{t+1}$ and $Y_{t+1}$ are conditionally independent given $\mathcal{F}_t$ that is, $\forall t \geq 1$:

$$\mathbb{P}[\zeta_{t+1} = u_{Y_t}, Y_{t+1} = j|\mathcal{F}_t] = \mathbb{P}[\zeta_{t+1} = u_{Y_t}|\mathcal{F}_t]\mathbb{P}[Y_{t+1} = j|\mathcal{F}_t]$$

We note $P[Y_{t+1} = j|\mathcal{F}_t] = \upsilon_t^j$. Because of the structure of process $Y$, it follows that $\upsilon_t^j = P[Y_{t+1} = j|Y_t, K_t]$. Furthermore, using the properties of semi-Markov processes, we can write (see [4]):

$$\upsilon_t^j = \begin{cases} \frac{Q_{Y_t j}(K_t+1) - Q_{Y_t j}(K_t)}{1 - \sum_{j=1}^m Q_{Y_t j}(K_t)}, \text{if } \ j \neq Y_t \\ \\ \frac{1 - \sum_{j=1}^m Q_{Y_t j}(K_t+1)}{1 - \sum_{j=1}^m Q_{Y_t j}(K_t)}, \text{if } \ j = Y_t \end{cases}$$

*Remark* 2.2. This probability ($\upsilon_t^j$) is a special case of the so-called "transition probability with initial backward". More details and some further applications can be found in [8], [9] and references cited therein.

Let us introduce some notation. We define $\mathbb{P}[\zeta_{t+1} = u_{Y_t}|\mathcal{F}_t] := z_t$. It is clear that

$$\pi_t^j := \mathbb{P}[\zeta_{t+1} = u_{Y_t}, Y_{t+1} = j|\mathcal{F}_t] = z_t \upsilon_t^j$$

Furthermore, we define

$$\kappa_t^j := (1 - z_t)\upsilon_t^j = \mathbb{P}[\zeta_{t+1} = d_{Y_t}, Y_{t+1} = j|\mathcal{F}_t]$$

It is then quite clear that whatever value $t$:

$$\sum_{j=1}^m [\pi_t^j + \kappa_t^j] = 1$$

When given $\mathcal{F}_t$, the pair of processes $Y$ and $K$ can take $m$ different values at time $t+1$. Indeed, suppose $Y_t = i$ and $K_t = k$, then either $(Y_{t+1}, K_{t+1}) = (i, k+1)$, either $(Y_{t+1}, K_{t+1}) = (j, 0)$ for every $j \neq i$. However, the system composed of process $\zeta$, $Y$ and $K$ can take $2m$ different values. These values are all determined by the following sets of events (that serve as a definition for the events $A_{t+1}^{j,u}$ and $A_{t+1}^{j,d}$ ):

$$\begin{aligned} A_{t+1}^{j,u} &:= \{\omega \in \Omega : \ Y_{t+1} = j, \zeta_{t+1} = u_{Y_t}\} \\ A_{t+1}^{j,d} &:= \{\omega \in \Omega : \ Y_{t+1} = j, \zeta_{t+1} = d_{Y_t}\} \end{aligned}$$

## 2.1 An extension of Ho and Lee

The classical model of Ho and Lee deals with the dynamics of zero-coupon bonds. Our framework allows to extend the model of Ho and Lee to include shifts in the parameter values of the model. We model the evolution of the term-structure as follows (given $\mathcal{F}_t$):

- on $A_{t+1}^{j,u}$, we have for all $\tau$:

$$P_{t+1}(\tau) = u_{Y_t}(\tau)\frac{P_t(\tau+1)}{P_t(1)}$$

- on $A_{t+1}^{j,d}$, we have for all $\tau$:

$$P_{t+1}(\tau) = d_{Y_t}(\tau)\frac{P_t(\tau+1)}{P_t(1)}$$

This can be written as:

$$P_{t+1}(\tau) = u_{Y_t}(\tau)\frac{P_t(\tau+1)}{P_t(1)}(\sum_{j=1}^{m}\mathbb{1}_{A_{t+1}^{j,u}}) + d_{Y_t}(\tau)\frac{P_t(\tau+1)}{P_t(1)}(\sum_{j=1}^{m}\mathbb{1}_{A_{t+1}^{j,d}})$$

Using our previous notation, we get

$$\begin{aligned}
\mathbb{P}[P_{t+1}(\tau) = u_{Y_t}(\tau)\frac{P_t(\tau+1)}{P_t(1)}|\mathcal{F}_t] &= z_t \\
\mathbb{P}[P_{t+1}(\tau) = d_{Y_t}(\tau)\frac{P_t(\tau+1)}{P_t(1)}|\mathcal{F}_t] &= 1 - z_t
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathbb{P}[P_{t+1}(\tau) = u_{Y_t}(\tau)\frac{P_t(\tau+1)}{P_t(1)}; Y_{t+1} = j|\mathcal{F}_t] &= \pi_t^j \\
\mathbb{P}[P_{t+1}(\tau) = d_{Y_t}(\tau)\frac{P_t(\tau+1)}{P_t(1)}; Y_{t+1} = j|\mathcal{F}_t] &= \kappa_t^j
\end{aligned}$$

We suppose that all these probabilities are strictly positive.

*Remark* 2.3. Let us note that the values obtained for the prices of zero-coupon bonds are the same when events $A_{i,u}$ and $A_{j,u}$ (or $A_{i,d}$ and $A_{j,d}$) happen i.e., when the term structure goes up (resp. down), the price of the zero-coupon bond is $P_{t+1}(\tau) = u_{Y_t}(\tau)\frac{P_t(\tau+1)}{P_t(1)}$ (resp. $P_{t+1}(\tau) = d_{Y_t}(\tau)\frac{P_t(\tau+1)}{P_t(1)}$) in the next period whether there is a regime change or not (see figure 1). The effect of the regime switch will only be felt in the next time period.

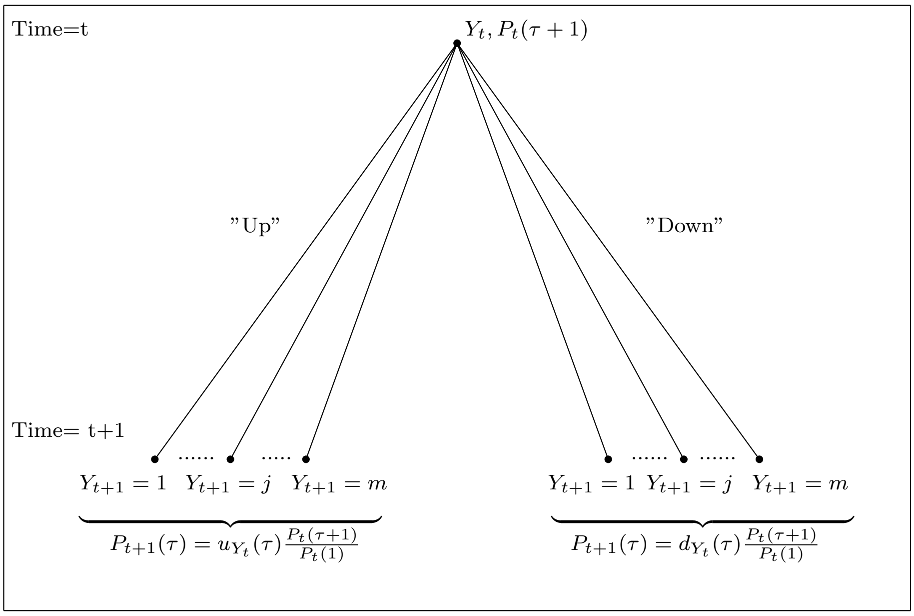


Figure 1: Regime switching semi-Markov binomial tree

# 3 Absence of arbitrage

**Definition 3.1.** *An arbitrage strategy is a strategy that allows to make a sure profit with no risk and no initial investment. The absence of arbitrage strategies on the market is known as the no-arbitrage condition and will henceforth be noted as NA.*

We are going to show that the no-arbitrage condition implies some conditions on the parameters of our model.

**Proposition 3.2.** *NA implies that $u_i(\tau) > 1 > d_i(\tau)$ for every $\tau$ and $i \in E$.*

*Proof.* Given $Y_t = i$, $K_t = k$, let us show that if $u_i(\tau) > d_i(\tau) > 1$, then we can exhibit an arbitrage strategy. The strategy is the following: at time $t$, buy a zero-coupon bond of maturity $T$ (of time to maturity $\tau$) that is paid for by borrowing the necessary sum at the risk-free rate. At time $t+1$, sell the zero-coupon bond. This will bring in $P_{t+1}(\tau-1)$. Repay the loan, this will cost $\frac{P_t(\tau)}{P_t(1)}$. Because $u_i(\tau) > d_i(\tau) > 1$, you have made a sure profit of $P_{t+1}(\tau) - \frac{P_t(\tau+1)}{P_t(1)} > 0$ whatever event ($A^{j,u}_{t+1}$ or $A^{j,d}_{t+1}$ (for every $j \in \{1, ..., m\}$)) happens. So there exists an arbitrage strategy. All the other cases are treated in a similar fashion. □

We have identified a necessary condition for absence of arbitrage. But this is not the only one. We are going to show that in order to avoid arbitrage, we need to have a relation linking $u_i$ to $d_i$. We will show that this condition is somehow related to the notion of martingale measure.

**Theorem 3.3.** *NA implies the existence of a process $p_t$ such that for each $t$: $0 < p_t < 1$ and for all $\tau$:*

$$p_t u_{Y_t}(\tau) + (1 - p_t) d_{Y_t}(\tau) = 1$$

*Proof.* Let us consider a one-period model. Assume we are given (exogeneously) a structure of the prices of zero-coupon bonds $P_t(.)$.

Let us construct a portfolio containing one zero-coupon bond of maturity $T+1$(time to maturity $\tau+1$) and $H$ zero-coupon bonds of maturity $T^{"}+1(\neq T+1)$(time to maturity $\tau^{"}+1$). At time $t$, this portfolio is worth

$$W(t) = P_t(\tau+1) + HP_t(\tau^{"}+1)$$

At time $t+1$, given our model, the portfolio can take two (and only two) values. On the set $\bigcup_{j\in E} A^{j,u}_{t+1}$, we get

$$W(t+1) = u_{Y_t}(\tau)\frac{P_t(\tau+1)}{P_t(1)} + Hu_{Y_t}(\tau^{"})\frac{P_t(\tau^{"}+1)}{P_t(1)}$$

Whereas on the set $\bigcup_{j\in E} A^{j,d}_{t+1}$ we get

$$W(t+1) = d_{Y_t}(\tau)\frac{P_t(\tau+1)}{P_t(1)} + Hd_{Y_t}(\tau^{"})\frac{P_t(\tau^{"}+1)}{P_t(1)}$$

Now, lets choose $H$ such that the value of the portfolio is the same whatever event happens. This implies that the portolio is in fact a risk free asset over that time period. So it should have the same return as a zero-coupon bond maturing at $t+1$. This leads to the two conditions

$$u_{Y_t}(\tau)P_t(\tau+1) + Hu_{Y_t}(\tau^{"})P_t(\tau^{"}+1) = d_{Y_t}(\tau)P_t(\tau+1) + Hd_{Y_t}(\tau^{"})P_t(\tau^{"}+1) \quad (1)$$

and

$$W(t) = P_t(1)W(t+1)$$

This last equation can be rewritten as

$$P_t(\tau+1) + HP_t(\tau^{"}+1) = d_{Y_t}(\tau)P_t(\tau+1) + Hd_{Y_t}(\tau^{"})P_t(\tau^{"}+1) \quad (2)$$

From 1 and 2, equate the common value $H$

$$\frac{(d_{Y_t}(\tau) - u_{Y_t}(\tau))P_t(\tau+1)}{(u_{Y_t}(\tau^{"}) - d_{Y_t}(\tau^{"}))P_t(\tau^{"}+1)} = \frac{(d_{Y_t}(\tau) - 1)P_t(\tau+1)}{(1 - d_{Y_t}(\tau^{"}))P_t(\tau^{"}+1)}$$

This gives

$$\frac{1 - d_{Y_t}(\tau^{"})}{u_{Y_t}(\tau^{"}) - d_{Y_t}(\tau^{"})} = \frac{1 - d_{Y_t}(\tau)}{u_{Y_t}(\tau) - d_{Y_t}(\tau)}$$

Because $\tau$ and $\tau^{"}$ are chosen arbitrarly, both sides are independent of the time to maturity and we can write

$$\frac{1 - d_{Y_t}(\tau^{"})}{u_{Y_t}(\tau^{"}) - d_{Y_t}(\tau^{"})} = p_{Y_t} := p_t$$

It is trivial to verify that $0 < p_t < 1$. Finally, this last relation yields

$$p_t u_{Y_t}(\tau^{"}) + (1-p_t)d_{Y_t}(\tau^{"}) = 1 \quad (3)$$

□

*Remark* 3.4. The last result is similar to what happens in the Ho and Lee model except here, because the value of the parameters change with time, we also have to allow for the probability to change with time and become a process $p_t$. This process can take value $\frac{1-d_{Y_t}(\tau)}{u_{Y_t}(\tau)-d_{Y_t}(\tau)}$ dependent on the value of $Y_t$ but independent of $\tau$.

*Remark* 3.5. Another way of looking at $p_t$ is the following. Let $p_i = \frac{1-d_i(\tau)}{u_i(\tau)-d_i(\tau)}$ for $i \in E$. Then, $p_t$ is simply a process that takes value in $(p_1, ..., p_m)$ according to the value of $Y_t$, i.e. $p_t = p_{Y_t}$. In view of this, this means that for each possible value of $Y_t$, for each possible state, we have an equality of the type of theorem 3.3, i.e. $p_{Y_t} u_{Y_t}(\tau) + (1 - p_{Y_t}) d_{Y_t}(\tau) = 1$ for each $Y_t \in E$.

**Corollary 3.6.**

$$P_t(\tau^" + 1) = P_t(1)(p_t u_{Y_t}(\tau^")\frac{P_t(\tau^" + 1)}{P_t(1)} + (1 - p_t) d_{Y_t}(\tau^")\frac{P_t(\tau^" + 1)}{P_t(1)}) \tag{4}$$

*Proof.* Multiplying both sides of (3) by $P_t(\tau^" + 1)$ yields the desired result. □

*Remark* 3.7. The important aspect of corollary 3.6 is that it tells us that the price of the zero-coupon bond at time $t$ is simply a sort of "average" of the discounted possible values at time $t + 1$. We will further develop this idea and the link with martingale measures in the next section.

# 4 Martingale measures

For each $j \in E$ and $t \in \mathbb{T}$, let us define a series of strictly positive parameters $(p_t^j, q_t^j)_{j\in E;t\in\mathbb{T}}$ such that for every $t \in \mathbb{T}$, $\sum_{j=1}^{m}(p_t^j + q_t^j) = 1$. Let $D_t$ be defined as

$$D_t = \prod_{s=0}^{t-1}\Big(\sum_{j=1}^{m}\Big[\frac{p_s^j}{\pi_s^j}\mathbb{1}_{A_{s+1}^{j,u}} + \frac{q_s^j}{\kappa_s^j}\mathbb{1}_{A_{s+1}^{j,d}}\Big]\Big) \tag{5}$$

**Lemma 4.1.** *Let $D_t$ be defined by (5), then $D_t > 0$ for all $t$, $\mathbb{E}[D_t] = 1$ and*

$$\mathbb{E}[D_{t+1}|\mathcal{F}_t] = D_t$$

*Proof.* The first statement is obvious from the definition of $D_t$ and of the quantities involved in this definition. The second statement is clear from the definition of $A_{s+1}^{j,u}$, $A_{s+1}^{j,d}$ and of $p_s^j$ and $q_s^j$. The third statement is immediate since

$$D_t = D_{t-1}\Big(\sum_{j=1}^{m}\Big[\frac{p_{t-1}^j}{\pi_{t-1}^j}\mathbb{1}_{A_t^{j,u}} + \frac{q_{t-1}^j}{\kappa_{t-1}^j}\mathbb{1}_{A_t^{j,d}}\Big]\Big)$$

□

This result allows us to think of $D_t$ as a density process. This process will be used to introduce equivalent measures.

**Definition 4.2.** *Define $\mathbb{P}^*$ as the equivalent measure with density $D_{T^*}$ with respect to $\mathbb{P}$.*

We will show that under one condition, the measure $\mathbb{P}^*$ is an equivalent martingale measure. This means that under this measure, every bond (properly discounted) should behave as a martingale.

**Theorem 4.3.** *Let us suppose that for all $t$, $\sum_{j=1}^{m} p_t^j = p_t$ (with $p_t$ defined by (3)). Then, the measure $\mathbb{P}^*$ is an equivalent martingale measure.*

*Proof.* By definition, $\mathbb{P}^*$ is an equivalent measure. It remains to be shown that under this measure, every bond behaves as a martingale. We have

$$\begin{aligned}
\mathbb{E}^*[P_t(1)P_{t+1}(\tau)|\mathcal{F}_t] &= \frac{1}{D_t}\mathbb{E}[P_t(1)P_{t+1}(\tau)D_{t+1}|\mathcal{F}_t] \\
&= P_t(\tau+1)\mathbb{E}\left[\sum_{j=1}^{m}\left[\frac{u_{Y_t}(\tau)p_t^j}{\pi_t^j}\mathbb{1}_{A_{t+1}^{j,u}} + \frac{d_{Y_t}(\tau)q_t^j}{\kappa_t^j}\mathbb{1}_{A_{t+1}^{j,d}}\right]|\mathcal{F}_t\right] \\
&= P_t(\tau+1)(u_{Y_t}(\tau)\sum_{j=1}^{m} p_t^j + d_{Y_t}(\tau)\sum_{j=1}^{m} q_t^j)
\end{aligned}$$

Let us now use the fact that $\sum_{j=1}^{m} p_t^j = p_t$. Then, by theorem 3.3, we get

$$\mathbb{E}^*[P_t(1)P_{t+1}(\tau)|\mathcal{F}_t] = P_t(\tau+1)$$

□

*Remark* 4.4. For every $t$, there are an infinite number of sets $(p_t^j, q_t^j)_{j\in E, t\in\mathbb{T}}$ that satisfy these conditions and so we have an infinite number of martingale measures. This seems coherent with intuition in the sense that we have added an extra source of uncertainty (the process $Y_t$) and this should lead to an incomplete market and so to an infinite number of martingale measures.

The next result will help us get a better understanding of $p_t$ in our framework and will shed light on the link between NA and martingale measures through the process $p_t$.

**Proposition 4.5.** *The process $p_t$ gives the probability that the term structure "goes up" under the equivalent martingale measure $\mathbb{P}^*$ at time $t$ i.e. $p_t = \mathbb{P}^*[\zeta_{t+1} = u_{Y_t}|\mathcal{F}_t]$*

*Proof.* Let us calculate $\mathbb{P}^*[A_{t+1}^{j,u}|\mathcal{F}_t]$.

$$\begin{aligned}
\mathbb{P}^*[A_{t+1}^{j,u}|\mathcal{F}_t] = \mathbb{P}^*[Y_{t+1}=j, \zeta_{t+1}=u_{Y_t}|\mathcal{F}_t] &= \mathbb{E}^*\left[\mathbb{1}_{A_{t+1}^{j,u}}|\mathcal{F}_t\right] \\
&= \frac{1}{D_t}\mathbb{E}\left[\mathbb{1}_{A_{t+1}^{j,u}}D_{t+1}|\mathcal{F}_t\right] \\
&= \mathbb{E}\left[\frac{p_t^j}{\pi_t^j}\mathbb{1}_{A_{t+1}^{j,u}}|\mathcal{F}_t\right] \\
&= p_t^j
\end{aligned}$$

Remember that $\sum_{j=1}^{m} p_t^j = p_t$. This means that $p_t = \mathbb{P}^*[\zeta_{t+1} = u_{Y_t}|\mathcal{F}_t]$. □

*Remark* 4.6. A classical idea of martingale pricing theory is that the price of an asset should be the expectation under the martingale measure of its future value (properly discounted). In view of proposition 4.5, equation (4) is a direct application of this principle. Indeed, although the real number of outcomes of the system at time $t+1$ is $2m$, the price of a zero-coupon bond can only take two values at time $t+1$ and the probability of these values is given by $p_t$ and $1-p_t$. Nevertheless, in a certain sense, equation (4) also takes into account the real number of outcomes through the fact that $p_t$ should be thought of as a sum of $m$ terms where each term gives the probability of one possible outcome. In this sense, equation (4) is a direct application of classical martingale pricing theory.

# 5 Model implications

In the classic approach by Ho and Lee, imposing the condition that an up movement followed by a down movement should be equivalent to a down movement followed by an up movement (path independence) lead to an explicit expression for the up and down parameters. The aim of this section is to discuss the idea of path independence in the context of regime switching models.

**Theorem 5.1.** *Let $\mathcal{F}_t$ be given and suppose as given a zero-coupon bond of price $P_t(\tau+2)$. Suppose the two following paths be given: the first is a realisation of the events $A_{t+1}^{Y_t,u}$ followed by $A_{t+2}^{Y_t,d}$ and the second a realisation of $A_{t+1}^{Y_t,d}$ followed by $A_{t+2}^{Y_t,u}$. If we impose that the price of the zero-coupon is the same whichever path is followed then there exist a constant $\delta_{Y_t}$ such that:*

$$\begin{aligned} u_{Y_t}(\tau) &= \frac{1}{p_{Y_t}+(1-p_{Y_t})\delta_{Y_t}^{\tau}} \\ d_{Y_t}(\tau) &= \delta_{Y_t}^{\tau} u_{Y_t}(\tau) \end{aligned}$$

*Proof.* With the first path we obtain

$$P_{t+2}(\tau) = \frac{d_{Y_t}(\tau)u_{Y_t}(\tau+1)P_t(\tau+2)}{u_{Y_t}(1)P_t(2)}$$

With the second we get

$$P_{t+2}(\tau) = \frac{u_{Y_t}(\tau)d_{Y_t}(\tau+1)P_t(\tau+2)}{d_{Y_t}(1)P_t(2)}$$

Imposing equality yields

$$\frac{d_{Y_t}(\tau)u_{Y_t}(\tau+1)}{u_{Y_t}(1)} = \frac{u_{Y_t}(\tau)d_{Y_t}(\tau+1)}{d_{Y_t}(1)} \tag{6}$$

We can then use equation (3) to eliminate $d_{Y_t}$ from this equation. Indeed

$$d_{Y_t}(\tau) = \frac{1-p_{Y_t}u_{Y_t}(\tau)}{1-p_{Y_t}}$$

Using this relation and the condition that $u_{Y_t}(0)=1$, we obtain a difference equation exactly similar to that of the classical Ho and Lee model. This is easily solved. Let us define

$$\delta_{Y_t} = \frac{1 - p_{Y_t} u_{Y_t}(1)}{u_{Y_t}(1) - p_{Y_t} u_{Y_t}(1)} \tag{7}$$

Then, we obtain the following solution for $u_{Y_t}$ and $d_{Y_t}$

$$\begin{aligned} u_{Y_t}(\tau) &= \frac{1}{p_{Y_t} + (1 - p_{Y_t})\delta_{Y_t}^{\tau}} \\ d_{Y_t}(\tau) &= \delta_{Y_t}^{\tau} u_{Y_t}(\tau) \end{aligned}$$

$\square$

*Remark* 5.2. From the last result, we see that the whole term structure is completely defined when we specify the $\delta_{Y_t}$'s (a constant for every state) and the $p_{Y_t}$'s (a probability of going up or down when the state is $Y_t$) for every possible state $Y_t$. This is the same as what happens in the Ho and Lee model except this has to be chosen for each possible state.

*Remark* 5.3. It follows from equation (7) that specifying $\delta_{Y_t}$ is in fact equivalent to specifying $u_{Y_t}(1)$.

Following our intuitive condition we get an explicit relation for $u_{Y_t}(\tau)$ and $d_{Y_t}(\tau)$. This was obtained when we applied path independence while staying in the same state. The next result will develop the indea of path independence when a change of state occurs. Loosely speaking we are going to build two paths with the same number of up's and down's in every state. We will then draw conclusions about path independence when change of states are allowed.

**Lemma 5.4.** *To simplify notation, assume $Y_t = i$. Let us suppose the existence of a zero-coupon bond of price $P_t(\tau+3)$. Assume the two following paths: $A^{j,d}_{t+1}$, $A^{i,d}_{t+2}$, $A^{i,u}_{t+3}$ and $A^{i,u}_{t+1}$, $A^{j,d}_{t+2}$ and $A^{i,d}_{t+3}$. Applying path independence along these two paths leads to a system of simultaneous equations (one for each $\tau$) of the type*

$$f(p_j) = 0$$

*With*

$$\begin{aligned} f(p_j) =& p_j^2(1-\delta_j)(1-\delta_j^{\tau})(p_i + (1-p_i)\delta_i^{\tau+1}) + p_j(\delta_j^{\tau+1}(\delta_i^{\tau+1}(p_i - 1)(p_i + 1) - \\ & \delta_i(1-p_i)p_i(\delta_i^{\tau-1} + 1) + p_i(p_i - 2)) + \delta_j^{\tau}(p_i + \delta_i^{\tau+1}(1-p_i)) + \delta_j(p_i + \delta_i^{\tau+1}(1-p_i)) - \\ & (p_i + \delta_i^{\tau}(1-p_i))(p_i + \delta_i(1-p_i))) + \delta_j^{\tau+1} p_i(1-p_i)(1-\delta_i)(1-\delta_i^{\tau}) \end{aligned} \tag{8}$$

*Proof.* Applying our model on the first path leads to

$$P_{t+3}(\tau) = \frac{u_i(\tau) d_j(\tau+1) d_i(\tau+2) P_t(\tau+3)}{d_j(1) d_i(2) P_t(3)} \tag{9}$$

Proceeding in an equivalent way, the second path yields

$$P_{t+3}(\tau) = \frac{d_j(\tau) d_i(\tau+1) u_i(\tau+2) P_t(\tau+3)}{d_i(1) u_i(2) P_t(3)} \tag{10}$$

Equating (9) and (10) gives

$$\frac{u_i(\tau)d_j(\tau+1)d_i(\tau+2)}{d_j(1)d_i(2)} = \frac{d_j(\tau)d_i(\tau+1)u_i(\tau+2)}{d_i(1)u_i(2)}$$

Using the fact that $d_i(\tau) = \delta_i^\tau u_i(\tau)$(of course this is also true for $u_j$ and $d_j$) and isolating the "$i$ terms" and the "$j$ terms" yields

$$\frac{u_i(\tau)u_i(1)}{u_i(\tau+1)} = \frac{d_j(\tau)d_j(1)}{d_j(\tau+1)} \tag{11}$$

Using the explicit expressions for $u_i$ and $d_i$, we obtain the following equality which should hold for every $\tau$

$$\begin{aligned}(p_j^2 + p_j(1-p_j)\delta_j + p_j(1-p_j)\delta_j^\tau + (1-p_j)^2\delta_j^{\tau+1})(p_i + (1-p_i)\delta_i^{\tau+1}) = \\ (p_j + (1-p_j)\delta_j^{\tau+1})(p_i^2 + p_i(1-p_i)\delta_i + p_i(1-p_i)\delta_i^\tau + (1-p_i)^2\delta_i^{\tau+1})\end{aligned} \tag{12}$$

Reordering the terms yields the desired result. □

*Remark* 5.5. Both paths used in the last proof have one up movement in the state $i$ and two down movements: one in state $j$ and one in state $i$. In this sense, the paths are "equivalent".

If we fix $p_i$, $\delta_i$ and $\delta_j$, equations (8) yield a system of second order equations for $p_j$ (one for every $\tau$). Given the coefficients of these equations are not equal because of their dependence on $\tau$, it is unclear that a solution $p_j \in ]0;1[$ can solve all these equations simultaneously. As a matter fact, counterexamples exist where there is no such solution (set $p_i = 0.6$, $\delta_i = 0.97$, $\delta_j = 0.98$).

An interesting question at this stage is whether one can find conditions under which a solution $p_j$ exists in the set $]0,1[$. A partial answer is provided by the next theorem.

**Theorem 5.6.** *Let us suppose that $\delta_i = \delta_j$, then given $p_i$ there exists at least one solution $p_j$ to the system of equations (8).*

*Proof.* Let us fix $\delta_i = \delta_j$, then whatever value we give to $p_i$ (with $p_i \in ]0;1[$), there exist at least one solution $p_j$ with $p_j = p_i$ for every $\tau$ (this is obvious from equation (12)). □

*Remark* 5.7. The existence of solutions to this system of second order equations remains unclear in the case where $\delta_i \neq \delta_j$.

Despite theorem 5.6, the situation remains unsatisfactory. Indeed, the only viable model we can guarantee is a model where $p_i = p_j$ and $\delta_i = \delta_j$. This means that $u_i = u_j$ and $d_i = d_j$ for all pair $(i, j)$ and for all $\tau$. So, it is as if there is only one state since in all states the evolution of the term structure is governed by the same values. The only thing that changes from state to state in this situation is the probability of changing states but this has little influence since all states yield the same perturbation.

In conclusion, if we apply our intuitive condition in the presence of regime changes the only "viable" model we can guarantee is a model where every regime is "the same" and so the whole regime switching structure become useless. So, in order for regime changes to have a real meaning we can't apply the path independence condition in the presence of regime switches.

# 6 Measure selection

As pointed out in section 4, due to the extra source of uncertainty, we have an infinite number of martingale measures and so an incomplete market. Although many ways of dealing with market incompleteness have been put forward, few papers discuss measure selection in interest rate models. Two noticeable exceptions are given by [1] and [2].

To make our model useful, we also need to discuss a method for selecting a martingale measure according to some criterion. We choose to focus on the minimal entropy martingale measure. The idea is to select among all equivalent martingale measures the one "closest" (in terms of entropy) to the real-world measure.

More formally, let us define the relative entropy.

**Definition 6.1.** *Let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures. The relative entropy $I(\mathbb{P},\mathbb{Q})$ is defined as:*

$$I(\mathbb{P},\mathbb{Q}) = \begin{cases} \mathbb{E}^{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}}\ln(\frac{d\mathbb{Q}}{d\mathbb{P}})], & \textit{if } \mathbb{Q} << \mathbb{P} \\ +\infty & \textit{otherwise} \end{cases}$$

**Definition 6.2.** *A measure $\mathbb{Q}$ is called an minimal entropy martingale measure if it minimizes the relative entropy over the set of all equivalent martingale measures.*

Because of our characterization of equivalent martingale measures by the density process $D_t$, characterizing the minimal entropy martingale measure is equivalent to giving the parameters $p_t^j$ and $q_t^j$ in the density process associated to the minimal entropy martingale measure. Before proving the general case, we demonstrate the one-period case..

**Proposition 6.3.** *In the one-period model, the minimal entropy martingale measure is given by*

$$p_0^j = \upsilon_0^j \left(\frac{1-d_{Y_0}}{u_{Y_0}-d_{Y_0}}\right)$$

$$q_0^j = \upsilon_0^j \left(\frac{u_{Y_0}-1}{u_{Y_0}-d_{Y_0}}\right)$$

*Proof.* From the definitions, the aim is to find the $p_0^j$ and $q_0^j$ (for all $j$) that minimize

$$\sum_{j=1}^{m}(p_0^j \ln\left(\frac{p_0^j}{\pi_0^j}\right) + q_0^j \ln\left(\frac{q_0^j}{\kappa_0^j}\right))$$

but subject to the constraints (that ensure that the measure will be an equivalent martingale measure):

$$\sum_{j=1}^{m}(p_0^j + q_0^j) = 1 \tag{13}$$

$$u_{Y_0}\sum_{j=1}^{m} p_0^j + d_{Y_0}\sum_{j=1}^{m} q_0^j = 1 \tag{14}$$

This is a problem of optimization under constraints. We will solve this by the method of Lagrange multipliers. An explicit solution can be obtained by using Lagrange multipliers.

The lagrangian $L$ for the problem is (using equation (5)):

$$\begin{aligned} L &= E(D_1 \ln(D_1)) + \lambda(\sum_{j=1}^{m}(p_0^j + q_0^j) - 1) + \gamma(u_{Y_0}\sum_{j=1}^{m} p_0^j + d_{Y_0}\sum_{j=1}^{m} q_0^j - 1) \\ &= \sum_{j=1}^{m}(p_0^j \ln\left(\frac{p_0^j}{\pi_0^j}\right) + q_0^j \ln\left(\frac{q_0^j}{\kappa_0^j}\right)) + \lambda(\sum_{j=1}^{m}(p_0^j + q_0^j) - 1) + \gamma(u_{Y_0}\sum_{j=1}^{m} p_0^j + d_{Y_0}\sum_{j=1}^{m} q_0^j - 1) \end{aligned}$$

The partial differentials give us (for every $j$):

$$p_0^j = \pi_0^j \exp(-(1 + \lambda + \gamma u_{Y_0})) \tag{15}$$

$$q_0^j = \kappa_0^j \exp(-(1 + \lambda + \gamma d_{Y_0})) \tag{16}$$

Using these relations and the partial differentials with respect to $\lambda$ and $\gamma$ yields

$$\begin{aligned} \exp(-\gamma u_{Y_0})\sum_{j=1}^{m}\pi_0^j + \exp(-\gamma d_{Y_0})\sum_{j=1}^{m}\kappa_0^j &= \exp(1+\lambda) \quad (17) \\ u_{Y_0}\exp(-\gamma u_{Y_0})\sum_{j=1}^{m}\pi_0^j + d_{Y_0}\exp(-\gamma d_{Y_0})\sum_{j=1}^{m}\kappa_0^j &= \exp(1+\lambda) \end{aligned}$$

Mixing the last two equations yields

$$\exp(-\gamma d_{Y_0}) = \exp(-\gamma u_{Y_0})\frac{(u_{Y_0} - 1)\sum_{j=1}^{m}\pi_0^j}{(1 - d_{Y_0})\sum_{j=1}^{m}\kappa_0^j} \tag{18}$$

From equations 15, 16, 17 and 18, and the definitions of $\pi_0^j$, $\kappa_0^j$ and $\upsilon_0^j$ we get the desired result. $\square$

In the general case, the aim is to minimize

$$E^{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}}\ln(\frac{d\mathbb{Q}}{d\mathbb{P}})]$$

over $p_s^j$ and $q_s^j$ under the constraints that for every $t$

$$\begin{aligned} \sum_{j=1}^{m}(p_t^j + q_t^j) &= 1 \\ u_{Y_t}\sum_{j=1}^{m} p_t^j + d_{Y_t}\sum_{j=1}^{m} q_t^j &= 1 \\ q_t^j ; p_t^j &> 0 \end{aligned}$$

We will first give an expression for the n-period Lagrangian and then give the general solution to our problem.

**Lemma 6.4.** *The n-period Lagrangian is given by*

$$L=\sum_{i=0}^{n-1}(\sum_{j=1}^{m}(p_i^j\ln\left(\frac{p_i^j}{\pi_i^j}\right)+q_i^j\ln\left(\frac{q_i^j}{\kappa_i^j}\right))+\lambda_i(\sum_{j=1}^{m}(p_i^j+q_i^j)-1)+\gamma_i(u_{Y_i}\sum_{j=1}^{m}p_i^j+d_{Y_i}\sum_{j=1}^{m}q_i^j-1))$$

*Proof.* We will give a proof by induction. This result is clearly verified for $n=1$ as can be seen in proposition 6.3. Let us then suppose this result holds for $n-1$, we will show this implies the result for $n$.

By definition of $D_n$, it follows that

$$I(\mathbb{P},\mathbb{Q})=E^{\mathbb{P}}[D_{n-1}\ln(D_{n-1})\Big(\sum_{j=1}^{m}\Big[\frac{p_{n-1}^j}{\pi_{n-1}^j}\mathbb{1}_{A_n^{j,u}}+\frac{q_{n-1}^j}{\kappa_{n-1}^j}\mathbb{1}_{A_n^{j,d}}\Big]\Big)$$
$$+D_{n-1}\Big(\sum_{j=1}^{m}\Big[\frac{p_{n-1}^j}{\pi_{n-1}^j}\mathbb{1}_{A_n^{j,u}}+\frac{q_{n-1}^j}{\kappa_{n-1}^j}\mathbb{1}_{A_n^{j,d}}\Big]\Big)\ln\Big(\sum_{j=1}^{m}\Big[\frac{p_{n-1}^j}{\pi_{n-1}^j}\mathbb{1}_{A_n^{j,u}}+\frac{q_{n-1}^j}{\kappa_{n-1}^j}\mathbb{1}_{A_n^{j,d}}\Big]\Big)]$$

We now condition on $\mathcal{F}_{n-1}$. This yields

$$I(\mathbb{P},\mathbb{Q})=E^{\mathbb{P}}[D_{n-1}\ln(D_{n-1})E[\sum_{j=1}^{m}\Big[\frac{p_{n-1}^j}{\pi_{n-1}^j}\mathbb{1}_{A_n^{j,u}}+\frac{q_{n-1}^j}{\kappa_{n-1}^j}\mathbb{1}_{A_n^{j,d}}\Big]|\mathcal{F}_{n-1}]$$
$$+D_{n-1}E[\Big(\sum_{j=1}^{m}\Big[\frac{p_{n-1}^j}{\pi_{n-1}^j}\mathbb{1}_{A_n^{j,u}}+\frac{q_{n-1}^j}{\kappa_{n-1}^j}\mathbb{1}_{A_n^{j,d}}\Big]\Big)\ln\Big(\sum_{j=1}^{m}\Big[\frac{p_{n-1}^j}{\pi_{n-1}^j}\mathbb{1}_{A_n^{j,u}}+\frac{q_{n-1}^j}{\kappa_{n-1}^j}\mathbb{1}_{A_n^{j,d}}\Big]\Big)|\mathcal{F}_{n-1}]]$$

This yields

$$I(\mathbb{P},\mathbb{Q})=E^{\mathbb{P}}[D_{n-1}\ln(D_{n-1})]+\sum_{j=1}^{m}(p_{n-1}^j\ln(\frac{p_{n-1}^j}{\pi_{n-1}^j})+q_{n-1}^j\ln(\frac{q_{n-1}^j}{\kappa_{n-1}^j}))$$

Using our hypothesis about the Lagrangian for $n-1$ periods and the multiple constraints for every period, we get the desired result. □

**Theorem 6.5.** *The minimal entropy martingale measure is characterized by*

$$p_t^j=\upsilon_t^j\left(\frac{1-d_{Y_t}}{u_{Y_t}-d_{Y_t}}\right)$$
$$q_t^j=\upsilon_t^j\left(\frac{u_{Y_t}-1}{u_{Y_t}-d_{Y_t}}\right)$$

*Proof.* We start from the last lemma and the expression for the Lagrangian. The next step is to take the partial differentials of the lagrangian $L$ with respect to the $p_i$'s, $q_i$'s, $\gamma_i$'s and $\lambda_i$'s for every $i$ and to set the equations equal to zero. Mixing these equations in the same way as in proposition 6.3 yields the desired result. Given our conditions on $u_i$ and $d_i$, the result also clearly satisfies the condition of positivity. □

*Remark* 6.6. The last result implies that the minimal entropy martingale measure is composed of two terms. The first term is the same as under the physical measure (this is due to the fact this is the minimal entropy measure). This term is linked to the evolution of the underlying process $Y_t$ and represents the unhedgeable risk associated to the regime switches in the market. The second term is linked to the market risk and ensures the measure is an equivalent martingale measure (this term is very similar to the equivalent measure in the classical Ho and Lee model).

# 7 Some numerical examples

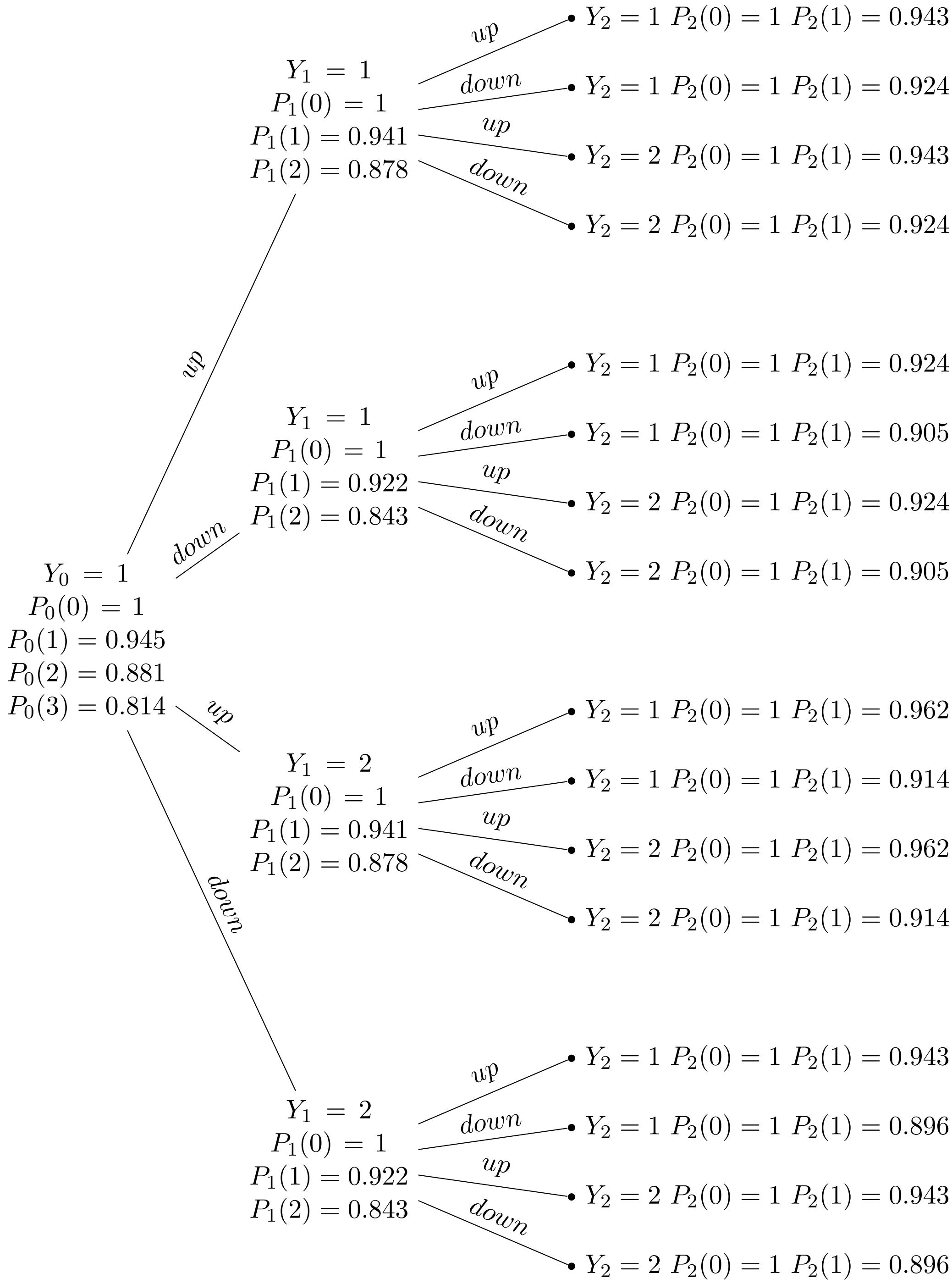


Figure 2: Evolution of bond prices in our example

This section will provide some simple examples of application of our model in the Markov and semi-Markov switching case. Our aim is to use our model to price some assets in these frameworks.

We place ourselves in a four period (including time zero) model with two states. We suppose that $Y_0 = 1$ and $K_0 = 4$. As in the classical Ho and Lee model, we have to specify some parameters (in our case a value for each state). We give the following values to the parameters of the switching Ho and Lee model: $p_1 = 0.5$, $p_2 = 0.4$, $\delta_1 = 0.98$ and $\delta_2 = 0.95$.

Again, as in the Ho and Lee model, we take as given an initial set of prices for the zero-coupon bonds. The values are: $P_0(0) = 1$, $P_0(1) = 0,945$, $P_0(2) = 0.881$ and $P_0(3) = 0,814$.

Our aim is to price a European option of maturity $t = 2$ and strike price $S = 0.93$ on a zero-coupon bond of maturity $t = 3$ (for more on bond option pricing consult [22] and [24]). We will do this in the case of Markov and semi-Markov switching and we choose to use the minimal entropy martingale measure as a pricing measure.

Given our model for the evolution of the term-structure and our chosen parameter values, the prices of the zero-coupon bonds evolve as in figure 2 (the figure only presents the first three time periods (including time zero) since in the last period the price of the remaining bond is trivially equal to one).

## 7.1 Markov switching framework

We consider a two-state Markov chain with transition matrix given by

$$P = \begin{pmatrix} 0.9 & 0.1 \\ 0.05 & 0.95 \end{pmatrix}$$

As was previously mentionned, a Markov chain is a special case of a semi-Markov chain. Let us define $q_{ij}(t) = \mathbb{P}(X_{n+1} = j; T_{n+1} - T_n = t | X_n)$. This quantity completely defines a semi-Markov kernel since $Q_{ij}(t) = \sum_{k=0}^{t} q_{ij}(k)$. It can be shown (see [28]) that a Markov chain is a semi-Markov chain with $q_{ij}(t)$ satisfying the following equalities

$$q_{ij}(t) = \begin{cases} p_{ij}(p_{ii}^{t-1}), \text{if } j \neq i \text{ and t} \geq 1 \\ 0, \text{elsewhere} \end{cases}$$

where $p_{ij}$ are the entries of the transition matrix $P$.

Using this, the fact we work with the minimal entropy martingale measure and the definition of $v_t^j$, we can obtain the price of the option at different points in time. This is done by using the standard martingale pricing theory i.e. by using backward induction (see [22] for more details). This is illustrated in figure 3. Indeed, the last "column" simply presents the different potential payoffs of the option at maturity (the maximum between zero and the value of the bond minus the strike price). Then, the time $t = 1$ prices are simply obtained as a discounted average of the payoffs. We can repeat this argument to obtain the initial price of the option.

The values noted above certain branches of the tree are simply the probabilities of transition from one node to the following. We have only noted those values that are useful in our example.

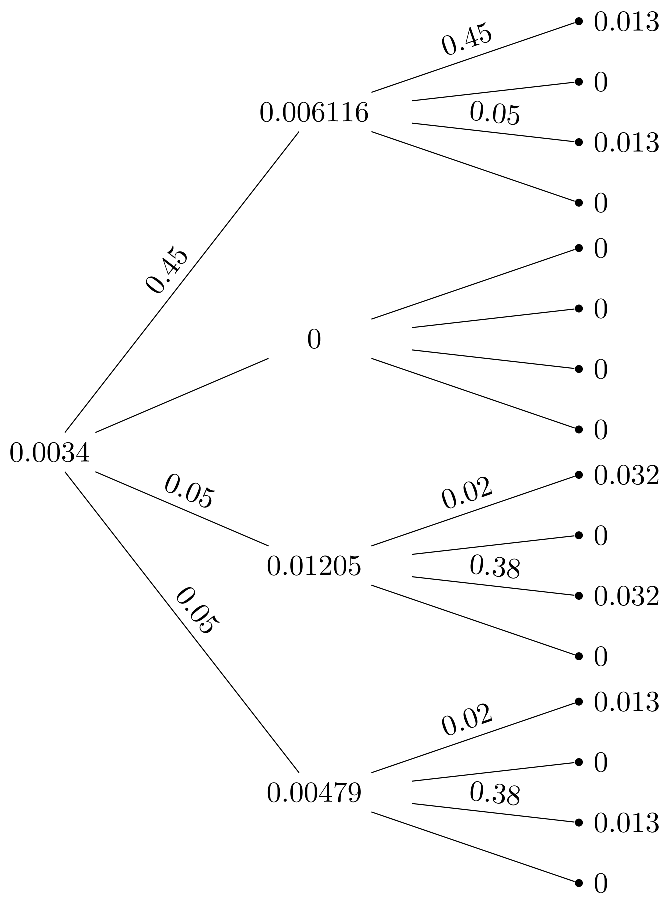


Figure 3: Evolution of the price of the bond option of maturity $t = 2$, strike price $S = 0.93$ on a zero-coupon bond of maturity $t = 3$ in the Markov switching case.

### 7.2 Semi-Markov switching framework

As in the Markov switching case, we use $q_{ij}(t) = \mathbb{P}(X_{n+1} = j; T_{n+1} - T_n = t | X_n)$. In this example, we use a discrete-time Weibull distribution for the duration distribution (for more details about this distribution consult [28] and the references cited therein). To this end, let us define some constants $\alpha_{12} = 0.3$, $\beta_{12} = 0.5$, $\alpha_{21} = 0.5$ and $\beta_{21} = 0.7$. Then, following [28], we impose $q_{ij}(0) = 0$ and

$$q_{ij}(t) = \begin{cases} \alpha_{ij}^{(t-1)^{\beta_{ij}}} - \alpha_{ij}^{t^{\beta_{ij}}}, \text{ if j} \neq i \text{ and } t \geq 1 \\ 0, \text{ elsewhere} \end{cases}$$

Using this, the definition of our minimal entropy martingale measure and the same backward induction argument as in the Markov case, we are able to obtain the tree of prices for our option (see figure 4).

*Remark* 7.1. From figures 3 and 4, we can observe that the prices obtained in the Markov switching and semi-Markov switching cases are very different due to the differences in the transition probabilities. What is also noticeable is that, as pointed in the introduction, the Markov case is simply a particular example of semi-Markov switching and our methodology applies directly to this case.

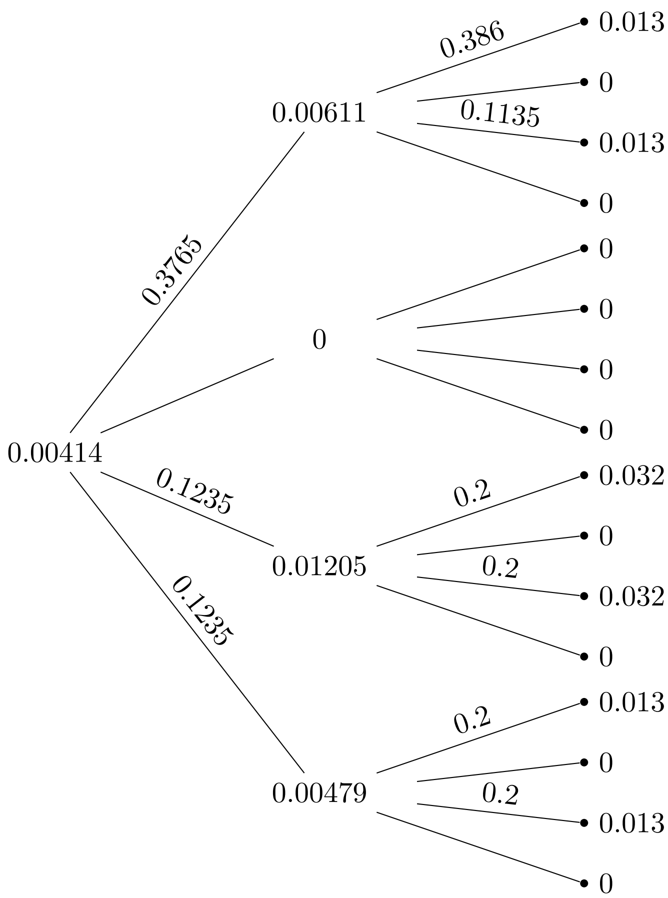


Figure 4: Evolution of the price of the bond option of maturity $t = 2$, strike price $S = 0.93$ on a zero-coupon bond of maturity $t = 3$ in the semi-Markov switching case.

# 8 Conclusions

The model presented here allows for a more realistic approach of discrete time models of the term structure. Indeed, it seems reasonable to assume that the parameters governing a model should not remain constant through time. This feature was not present in the Ho and Lee model.

Our model can also be made to be consistant with the notion of absence of arbitrage. These ideas have been explored and the link with the notion of martingale measure clarified.

The classical model by Ho and Lee was built so as to have recombining paths. This is a desirable property from a computational point of view. Our model keeps this property when there are no regime switches but this property is of no interest in the presence of regime switching.

We treat the market incompleteness by using the minimal entropy martingale measure approach. We give an explicit characterization of this measure in our context. This should prove very useful in practical implementations of our model.

Our last section, shows an example of a potential application of our model in the case of a European bond option. We use our model to price this asset. We choose the minimal entropy martingale measure as pricing measure. We study the pricing of this option in both the Markov and semi-Markov framework. What is noticeable is that the prices obtained are different due to the difference in transition probabilities. Another feature of interest is that the Markov switching is a particular case of our model and as such our model should always perform at least as well as the Markov switching model.

In practice, the model of Ho and Lee requires three basic inputs: an initial term structure $P_0(\tau)$ (this is exogeneously given) and the knowledge of two constants $p$ and $\delta$. Choosing the constant $p$ is in fact equivalent to choosing the risk-neutral measure. In our model, we still need the initial term structure $P_0(\tau)$ (again this is exogeneously given), but we also need the specification of the semi-Markov process $Y$ (this should be estimated/calibrated from market data but should be viewed as an exogeneous input). As far as model parameters are concerned, if we work with the minimal entropy martingale measure, we need to introduce a parameter $\delta_i$ and a parameter $p_i$ for each state $i \in E$.

It is worth noticing that the model of Ho and Lee is also just a special case of our model (just choose all the $\delta_i$'s and $p_i$'s to be equal) and as such, our model should always perform at least as well as the model by Ho and Lee.

## References


[1] Biagini F.; "A quadratic approach for interest rate models in incomplete markets", Proceedings of the workshop in mathematical finance, Konstanz, (2001)

[2] Biagini F.; "Mean-Variance hedging for interest rate models with stochastic volatility", Decisions in Economics and Finance, **25**, (2002): 1-17

[3] Bulla J., Bulla I.; "Stylized facts of financial time series and hidden semi-Markov models", Computational Statistics & Data Analysis, **51**(4), (2006): 2192-2209

[4] Chryssaphinou O., Karaliopoulou M., Limnios N.; "On discrete-time semi-Markov chains and applications in word occurences", Communications in Statistics: Theory and Methods, **37**, (2008): 1306-1322

[5] Çinlar E. ; *Introduction to stochastic processes*, Prentice-Hall, Englewood Cliffs, New-Jersey, 1975

[6] Cox J.C., Ingersoll Jr.J.E., Ross S.A.; "A theory of the term structure of interest rates", Econometrica, **53**, (1985): 385-408

[7] Dahlquist M., Gray S.F.; "Regime switching and interest rates in the European monetary system", Journal of International Economics, **50**, (2000): 399-419

[8] D'Amico G., Janssen J., Manca R.; "European and american options: the semi-Markov case", Physica A, Statistical Mechanics and its Applications, **388**, (2009): 3181-3194

[9] D'Amico G., Janssen J., Manca R.; "Initial and final backward and forward discrete time non-homogeneous semi-Markov credit risk models", Methodology and Computing in Applied Probability, **12**(2), (2010): 215-225

[10] Diebold F., Rudebusch G.; "A nonparametric investigation of duration dependence in the American business cycle", Journal of Political Economy, **98**, (1990): 596- 616

[11] Di Masi G.B., Kabanov Y.M., Runggaldier W.J.; "Mean-variance hedging of options on stocks with Markov volatilities", Theory of Probability and Its Applications, **39**, (1994): 211-222

[12] Durland J., McCurdy T.; "Duration dependent transitions in a Markov model of U.S. GNP growth", Journal of Business and Economics Statistics, **12**, (1994): 279-288

[13] Easly D., O'Hara M.; "Price, trade size and information in securities markets", Journal of Financial Economics, **19**, (1987): 69-90

[14] Easly D., O'Hara M.; "Time and the process of security price adjustment", Journal of Finance, **47**, (1992): 577-605

[15] Elliott R.J., van der Hoek J.; "An application of hidden Markov models to asset allocation problems", Finance and Stochastics, **3**, (1997): 229-238

[16] Elliott R.J., Chan L., Siu T.K.; "Option pricing and Esscher transform under regime switching", Annals of Finance, **1**(4), (2005): 423-432

[17] Ghosh M.K., Goswami A.; "Option pricing in a semi-Markov modulated market" ; SIAM Journal on control and Optimization, **48**, (2009): 1519-1541

[18] Goldfeld S.M., Quandt R.E.; "A Markov model for switching regressions", Journal of Econometrics, **1**, (1973): 3-16

[19] Hamilton J.D.; "A new approach to economic analysis of non-stationnary time series and the business cycle", Econometrica, **57**, (1989): 357-384

[20] Hunt J.S., Devolder P.; "Minimal entropy martingale measure in a semi-Markov regime switching Cox-Ross-Rubinstein model", preprint, ISBA, UCL, (2010)

[21] Hunt J.S.; "A short note on continuous-time Markov and semi-Markov processes", discussion paper (DP-1003), ISBA, UCL, (2010)

[22] Ho T.S.Y., Lee S.B.; "Term structure movements and pricing interest rate contingent claims", Journal of Finance, **41**(5), (1986): 1011-1029

[23] Hong Y., Li H.; "Nonparametric specification testing for continuous-time models with applications to term structure of interest rates", Review of Financial Studies, **18**, (2005): 37-84

[24] Hull J., White A.; "Pricing interest-rate derivative securities", Review of Financial Studies, **3**, (1990): 573-592

[25] Kijima M., Yoshida T.; "A simple option pricing model with Markovian volatilities", Journal of the Operations Research Society of Japan , **36**(3), (1993): 149-166

[26] Landén C. ; "Bond pricing in a hidden Markov model of the short rate"; Finance and Stochastics, **4**, (2000): 371-389

[27] Limnios N., Oprisan G.; *Semi-Markov processes and reliability*, Birkhauser, Boston, 2001

[28] Limnios N., Barbu V.; *Semi-Markov chains and hidden semi-Markov models towards applications*, Springer, New-York, 2008

[29] Rogers L.C.G, Jobert A.; "Option pricing with Markov-modulated dynamics"; SIAM Journal on Control and Optimization, **44**, (2006): 2063-2078

[30] Silvestrov D., Stenberg F. ; "A pricing process with stochastic volatility controlled by a semi-Markov process"; Communications in Statistics: theory and methods, **33** (3), (2004): 591-608

[31] Svishchuk A.V. ; "Hedging of options under mean-square criterion and semi-Markov volatility"; Ukrainian Journal of Mathematics, **47** (7), (1995): 1119-1127

[32] Vasicek O.; "An equilibrium characterisation of the term structure"; Journal of Financial Economics, **5**, (1977): 177-188

[33] Wu S., Zeng Y.; "The term structure of interest rates under regime shifts and jumps"; Economics Letters, **93**, (2006): 215-221