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DISPERSIVE EFFECT OF CROSS-AGING WITH ARCHIMEDEAN COPULAS

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Abstract

In this paper, we compare conditional distributions derived from bivariate archimedean copulas in terms of their respective variability using the dispersive stochastic order. Specifically, we consider the effect of increasing the second component on the variability of the conditional distribution of the first component. Characterizations are provided in terms of the generator and of the marginal distributions. Several examples involving standard parametric copulas such as Clayton and Frank are discussed.

Key words and phrases: copula, dependence, dispersive order.

Classification: 60E15 (Inequalities; stochastic orderings)

1 Introduction and motivation

In this paper, we consider random couples (X_1, X_2) with joint distribution function $F_{\boldsymbol{X}}$ of the form

$$F_{\boldsymbol{X}}(x_1, x_2) = C_\phi\big(F_1(x_1), F_2(x_2)\big) \tag{1.1}$$

where for $t \in \mathbb{R}$, $F_i(t) = \Pr[X_i \leq t]$, $i = 1, 2$, and C_ϕ is the archimedean copula with generator ϕ defined as

$$C_\phi(u_1, u_2) = \begin{cases} \phi^{[-1]}\big(\phi(u_1) + \phi(u_2)\big) \text{ if } \phi(u_1) + \phi(u_2) \leq \phi(0), \\ 0 \text{ otherwise,} \end{cases} \tag{1.2}$$

for $0 \leq u_1, u_2 \leq 1$. The generator $\phi : [0, 1] \to \mathbb{R}^+$ entering (1.2) is a continuous, possibly infinite, strictly decreasing convex function such that $\phi(1) = 0$. The pseudo-inverse of ϕ is the function $\phi^{[-1]}$ given by

$$\phi^{[-1]}(t) = \begin{cases} \phi^{-1}(t) \text{ for } 0 \leq t \leq \phi(0), \\ 0 \text{ for } \phi(0) \leq t \leq +\infty. \end{cases} \tag{1.3}$$

Clearly, $\phi^{[-1]}$ is continuous and non-increasing on $\mathbb{R}^+$, and strictly decreasing on $[0, \phi(0)]$. For a strict generator (i.e. a generator ϕ such that $\lim_{t \to 0} \phi(t) = +\infty$), $\phi^{[-1]}$ is just the inverse ϕ^{-1} of ϕ. For a non-strict generator (i.e. a generator ϕ such that $\phi(0) < +\infty$), $\phi^{[-1]}$ coincides with the inverse ϕ^{-1} of ϕ on $[0, \phi(0)]$ and is set equal to 0 after $\phi(0)$. Throughout the paper, we assume that the marginal distribution functions F_1 and F_2 are continuous and strictly increasing on their support. If needed, we also assume that F_1 and F_2 possess probability density functions, denoted as f_1 and f_2, respectively.

Archimedean copulas (1.2) enjoy numerous convenient mathematical properties and are therefore appreciated for modelling or simulating bivariate data. See, e.g., Nelsen (2006, Chapter 4) for a review. In particular, archimedean copulas naturally appear in relation with frailty models for the joint distribution of two survival times depending on the same latent factor (the generator being then the inverse of the Laplace transform of this latent factor).

Here, we examine the behavior of one component of a random vector when the other component gets larger, or ages, that is, we compare the distribution of X_1 given $X_2 = x_2$ to the distribution of X_1 given $X_2 = x_2'$ with $x_2 < x_2'$. We refer to this phenomenon as cross-aging. Random couples with joint distribution function (1.1) often possess positive dependence properties, ensuring that X_1 "increases" in X_2, and vice versa. Formally, recall the definition of stochastic dominance, which translates into mathematical terms the intuitive idea of "being larger than" for random variables: given the random variables X and Y, X is said to be smaller than Y in the stochastic dominance (denoted as $X \preceq_{\text{st}} Y$) if $\Pr[X \leq t] \geq \Pr[Y \geq t]$ for all t. Whatever the threshold t, it is thus more likely that X falls below t compared to Y. Given a random variable X and an event A, let us denote as $[X|A]$ a random variable with distribution function $x \mapsto \Pr[X \leq x|A]$. Under appropriate conditions on the generator ϕ, we have that $[X_1|X_2 = x_2] \preceq_{\text{st}} [X_1|X_2 = x_2']$ for $x_2 \leq x_2'$ when (X_1, X_2) has distribution function (1.1). This property is referred to as conditional increasingness in the literature. If C_ϕ is obtained from the frailty construction (i.e. the generator ϕ is the inverse of a Laplace transform) then the above stochastic inequality holds with the stronger likelihood ratio order replacing $\preceq_{\rm st}$. This property is known in the literature as total positivity of degree 2 (TP2) and is fulfilled by most parametric families of archimedean copulas. For more results in that direction, we refer the interested readers, e.g., to Denuit et al. (2005, Chapter 5).

Whereas X_1 generally "increases" in X_2, a natural question that has to the best of our knowledge not yet been addressed in the literature concerns the variability of X_1 given X_2. When X_2 is known to increase, does X_1 become more or less variable? In this paper, we answer this question using the dispersive order which turns out to be the appropriate tool to study the variability of conditionals derived from archimedean copulas provided some conditions are met.

The paper proceeds as follows. Section 2 recalls basic facts about dispersive order. In Section 3, we examine the case of unit uniform marginals. The strictness of the generators turns out to play an important role in the analysis conducted there. Then, in Section 4, we allow for arbitrary marginals. It turns out that this general case is not a direct consequence of the preceding section as the marginal behavior does matter. In Section 5, we establish comparative results where cross-aging (in the dispersive sense) provides the appropriate theoretical argument. The final Section 6 concludes.

2 Dispersive order

The dispersive order can be used for comparing spread among probability distributions. Considering two random variables X and Y, X is smaller than Y in the dispersive order when the difference between any two quantiles of X is smaller than the difference between the corresponding quantiles of Y. The dispersive order has a long history in statistics. We refer the reader e.g. to the review paper by Jeon et al. (2006) as well as to the reference book by Shaked and Shanthikumar (2007) for a detailed presentation of the stochastic order relation $\preceq_{\rm disp}$. In the context of lifetime distributions, it has been used by Belzunce et al. (1996) and Pellerey and Shaked (1997) to characterize IFR and DFR distributions.

Define the generalized inverse (or quantile function) of the distribution function F for $\alpha \in (0,1)$ by

$$F^{-1}(\alpha) = \inf\left\{x \in \mathbb{R} \middle| F(x) \geq \alpha\right\}.$$

Recall that given the random variables X and Y with distribution functions F_X and F_Y and inverses F_X^{-1} and F_Y^{-1}, respectively, X is said to be smaller than Y in the dispersive order (denoted as $X \preceq_{\rm disp} Y$) if

$$F_X^{-1}(\beta) - F_X^{-1}(\alpha) \leq F_Y^{-1}(\beta) - F_Y^{-1}(\alpha) \quad \text{whenever } 0 < \alpha \leq \beta < 1 \tag{2.1}$$
$$\Leftrightarrow \alpha \mapsto F_Y^{-1}(\alpha) - F_X^{-1}(\alpha) \text{ non-decreasing on } [0,1].$$

It is clear that the order $\preceq_{\rm disp}$ indeed corresponds to a comparison of X and Y by variability because it requires the difference between any two quantiles of X to be smaller than the corresponding difference in quantiles of Y. It is easy to prove that $X \preceq_{\rm disp} Y$ implies $\mathbb{V}\text{ar}[X] \leq \mathbb{V}\text{ar}[Y]$. In addition to the definition (2.1), the following characterization is useful:

$$X \preceq_{\rm disp} Y \Leftrightarrow x \mapsto F_Y^{-1}(F_X(x)) - x \text{ non-decreasing.} \tag{2.2}$$

See formula (3.B.10) in Shaked and Shanthikumar (2007).

3 Unit uniform marginals

If the support of the conditionals is finite with common endpoints then it is easy to see that the dispersive order cannot hold. This can be deduced from (2.1): $F_Y^{-1} - F_X^{-1}$ cannot be monotone on $[0,1]$ if $F_X^{-1}(0) = F_Y^{-1}(0) > -\infty$ and $F_X^{-1}(1) = F_Y^{-1}(1) < +\infty$. Therefore, in case the support is bounded, we need different endpoints for a possible comparison in terms of the dispersive order. This is only possible if the generator is non-strict, as shown next.

Let (U_1, U_2) be a couple of random variables with joint distribution function C_ϕ given in (1.2). The distribution function of $[U_1|U_2 = u_2]$ is given by

$$\Pr[U_1 \leq u_1|U_2 = u_2] = \begin{cases} \frac{\phi'(u_2)}{\phi' \circ \phi^{-1}(\phi(u_1)+\phi(u_2))} \text{ if } u_1 \geq \phi^{-1}\left(\phi(0) - \phi(u_2)\right), \\ 0 \text{ if } u_1 < \phi^{-1}\left(\phi(0) - \phi(u_2)\right). \end{cases}$$

The corresponding quantile function is given by

$$\psi_{u_2}(\alpha) = \begin{cases} \phi^{-1}\left(\phi\left((\phi')^{-1}\left(\frac{\phi'(u_2)}{\alpha}\right)\right) - \phi(u_2)\right) \text{ if } \alpha \geq \frac{\phi'(u_2)}{\phi'(0^+)}, \\ \phi^{-1}\left(\phi(0) - \phi(u_2)\right) \text{ if } \alpha < \frac{\phi'(u_2)}{\phi'(0^+)}. \end{cases}$$

Note that for strict generator the quantile function simplifies to

$$\psi_{u_2}(\alpha) = \phi^{-1}\left(\phi\left((\phi')^{-1}\left(\frac{\phi'(u_2)}{\alpha}\right)\right) - \phi(u_2)\right).$$

Now, for $\alpha = 1$ we get $\phi^{-1}(0) = 1$ whatever ϕ (be it strict or not) and the conditioning value u_2. On the contrary, letting α tend to 0 gives 0 if ϕ is strict, whatever u_2, but the limit may depend on ϕ and on u_2 if ϕ is non-strict. More precisely, when ϕ is non strict the support is $\left[\phi^{-1}\left(\phi(0) - \phi(u_2)\right), 1\right]$. Therefore, for a strict generator ϕ, the support for both $[U_1|U_2 = u_2]$ and $[U_1|U_2 = u_2']$ is the interval $[0,1]$ and no dispersive order relation can hold whereas if ϕ is non strict then a dispersive comparison may be possible.

We also consider conditionals of the form $[U_1|U_2 \leq u_2]$ in this paper. The distribution function of $[U_1|U_2 \leq u_2]$ is

$$\Pr[U_1 \leq u_1|U_2 \leq u_2] = \begin{cases} \frac{\phi^{-1}(\phi(u_1)+\phi(u_2))}{u_2} \text{ if } u_1 \geq \phi^{-1}\left(\phi(0) - \phi(u_2)\right), \\ 0 \text{ if } u_1 \leq \phi^{-1}\left(\phi(0) - \phi(u_2)\right). \end{cases}$$

The corresponding quantile function is given by

$$\psi^*_{u_2}(\alpha) = \phi^{-1}\big(\phi(u_2\alpha) - \phi(u_2)\big).$$

Also here, we see that for $\alpha = 1$ we get $\phi^{-1}(0) = 1$ whereas the limit for α tending to 0 is 0 for a strict generator but may depend on ϕ and on u_2 if the generator is non-strict. Hence, no dispersive order relation is possible between $[U_1|U_2 \leq u_2]$ and $[U_1|U_2 \leq u_2']$ if ϕ is strict.

The next result investigates the effect of increasing one component of the archimedean vector.

Proposition 3.1. *Let (U_1, U_2) be a couple of unit uniform random variables with archimedean copula with non-strict generator ϕ. Then,* *(i) The stochastic inequality* $[U_1|U_2 = u_2] \preceq_{disp} [U_1|U_2 = u_2']$ *holds for* $u_2 < u_2' \in [0,1]$ *if, and only if,* $\alpha \mapsto \psi_{u_2'} \circ \psi_{u_2}^{-1}(\alpha) - \alpha$ *is non-decreasing or, equivalently, if and only if* $\frac{\partial}{\partial\alpha}\psi_{u_2}(\alpha) \le \frac{\partial}{\partial\alpha}\psi_{u_2'}(\alpha)$ *for all* α.

(ii) The stochastic inequality $[U_1|U_2 \le u_2] \preceq_{disp} [U_1|U_2 \le u_2']$ *holds for* $u_2 < u_2' \in [0,1]$ *if, and only if,* $\alpha \mapsto \psi^*_{u_2'} \circ \psi_{u_2}^{*-1}(\alpha) - \alpha$ *is non-decreasing or, equivalently, if and only if* $\frac{\partial}{\partial\alpha}\psi^*_{u_2}(\alpha) \le \frac{\partial}{\partial\alpha}\psi^*_{u_2'}(\alpha)$ *for all* α.

Proof. The first part of the statements in (i)-(ii) is a direct application of (2.2). To prove the second part of the statement in (i), note that

$$\begin{aligned} & \psi_{u_2'} \circ \psi_{u_2}^{-1}(\alpha) - \alpha \text{ is non-decreasing in } \alpha \\ \Leftrightarrow\ & \frac{\frac{\partial}{\partial\alpha}\psi_{u_2'}(\psi_{u_2}^{-1}(\alpha)) - \frac{\partial}{\partial\alpha}\psi_{u_2}(\psi_{u_2}^{-1}(\alpha))}{\frac{\partial}{\partial\alpha}\psi_{u_2}(\psi_{u_2}^{-1}(\alpha))} \ge 0 \text{ for all } \alpha \\ \Leftrightarrow\ & \frac{\partial}{\partial\alpha}\psi_{u_2'}(\psi_{u_2}^{-1}(\alpha)) - \frac{\partial}{\partial\alpha}\psi_{u_2}(\psi_{u_2}^{-1}(\alpha)) \ge 0 \text{ for all } \alpha \\ \Leftrightarrow\ & \frac{\partial}{\partial\alpha}\psi_{u_2}(\alpha) \le \frac{\partial}{\partial\alpha}\psi_{u_2'}(\alpha) \text{ for all } \alpha \end{aligned}$$

which ends the proof of (i). The reasoning leading to (ii) is similar. □

Let us now examine an example.

Example 3.2 (Family 7 in Table 4.1 of Nelsen (2006))**.** Consider the generator $\phi_\theta(t) = -\ln\big(\theta t + (1-\theta)\big)$ indexed by $\theta \in (0,1]$. The corresponding copula function is

$$C_{\phi_\theta}(u_1,u_2) = \max\Big\{\theta u_1 u_2 + (1-\theta)(u_1+u_2-1), 0\Big\}.$$

Consider (U_1,U_2) with joint distribution function C_{ϕ_θ}. For $\theta = 1$, we get the lower bound copula $C(u_1,u_2) = \max\{u_1+u_2-1, 0\}$ which gives the minimum in the dispersive sense since $[U_1|U_2 = u_2]$ is constantly equal to $1-u_2$ in that case. For $\theta = 0$, we get the independent case.

For fixed $\theta \in (0,1]$, we have

$$\psi_{u_2}(\alpha) = \begin{cases} \frac{\alpha-1+\theta}{\theta} \text{ if } \alpha \ge \frac{1-\theta}{\theta u_2+1-\theta}, \\ \frac{(1-u_2)(1-\theta)}{\theta u_2+1-\theta} \text{ if } \alpha < \frac{1-\theta}{\theta u_2+1-\theta}, \end{cases}$$

so that for $t' < t$,

$$\psi_{t'}(\alpha) - \psi_t(\alpha) = \begin{cases} 0 \text{ if } \alpha \ge \frac{1-\theta}{\theta t'+1-\theta}, \\ \frac{(1-\theta)(1-t')}{\theta t'+1-\theta} - \frac{\alpha-1+\theta}{\theta} \text{ if } \frac{1-\theta}{\theta t+1-\theta} \le \alpha \le \frac{1-\theta}{\theta t'+1-\theta}, \\ \frac{(1-\theta)(1-t')}{\theta t'+1-\theta} - \frac{(1-\theta)(1-t)}{\theta t+1-\theta} \text{ if } \alpha \le \frac{1-\theta}{\theta t+1-\theta}. \end{cases}$$

Since $\psi_{t'} - \psi_t$ is non-increasing in α, we finally get by (2.1)

$$u_2 < u_2' \quad \Rightarrow \quad [U_1|U_2 = u_2] \preceq_{\text{disp}} [U_1|U_2 = u_2'].$$

Now, as

$$\frac{\partial}{\partial\alpha}\psi_t^*(\alpha) = \frac{t}{\theta t + 1 - \theta}$$

increases in t, we also have

$$u_2 < u_2' \quad \Rightarrow \quad [U_1|U_2 \le u_2] \preceq_{\text{disp}} [U_1|U_2 \le u_2'].$$

For this copula, we thus see that increasing the second component increases the conditional distribution in the dispersive order.

4 Arbitrary marginals

Now that we have an effective condition for the conditionals to be ordered in the dispersive order for the unit uniform case, it is natural to wonder whether this condition also applies to random couples with arbitrary marginals connected through an archimedean copula. However, the results obtained in Section 3 do not allow to treat this more general situation. The reason is that the implication $X \preceq_{\text{disp}} Y \Rightarrow g(X) \preceq_{\text{disp}} g(Y)$ is not necessarily true for increasing transformations g unless additional assumptions about the shape of the function g and the respective distributions of X and Y are fulfilled.

There is nevertheless one particular case where the results derived in Section 3 extend to other marginals than unit uniform ones, as discussed next.

Proposition 4.1. *Let (X_1, X_2) be a random vector with distribution function* (1.1). *If F_1 is concave then $[X_1|X_2 = x_2]$ increases in x_2 in the $\preceq_{disp}$-sense if the non-strict generator ϕ fulfills the condition of Proposition 3.1(i). Similarly, if F_1 is concave then $[X_1|X_2 \le x_2]$ increases in x_2 in the $\preceq_{disp}$-sense if the non-strict generator ϕ fulfills the condition of Proposition 3.1(ii).*

Proof. Assume that the condition in Proposition 3.1(i) is met by ϕ. If we define $U_i = F_i(X_i)$, $i = 1, 2$, then (U_1, U_2) fulfills the conditions of Proposition 3.1(i). Since the common right endpoint of the supports of $[U_1|U_2 = u_2]$ and of $[U_1|U_2 = u_2']$ is 1, we have $[U_1|U_2 = u_2'] \preceq_{\text{st}} [U_1|U_2 = u_2]$. Also, we have from Proposition 3.1(i) that $[U_1|U_2 = u_2] \preceq_{\text{disp}} [U_1|U_2 = u_2']$. From Theorem 3.B.10 in Shaked and Shanthikumar (2007), we see that provided and F_1^{-1} is convex (or, equivalently, F_1 is concave, that is, the corresponding probability density function is decreasing), we have

$$\begin{aligned}[U_1|U_2 = u_2] \preceq_{\text{disp}} [U_1|U_2 = u_2'] \;&\Rightarrow\; [F_1^{-1}(U_1)|U_2 = u_2] \preceq_{\text{disp}} [F_1^{-1}(U_1)|U_2 = u_2'] \\ &\Leftrightarrow\; [X_1|X_2 = x_2] \preceq_{\text{disp}} [X_1|X_2 = x_2'].\end{aligned}$$

The same type of result holds for the other conditioning. □

Concave distribution functions are unimodal about 0 (i.e. they possess decreasing densities). For such distributions, the assumptions of Proposition 3.1 are thus enough to ensure that the conditionals are ordered in the $\preceq_{\text{disp}}$-sense. Concave distribution functions arise in a number of ways in applied probability. In particular, all the DFR (for decreasing failure rate) distributions have concave distribution functions. Moreover, this class is closed under change of scale, power transformation, left truncation, limits, mixtures and the formation of arbitrary series systems.

Let us now consider arbitrary marginals. Note that switching from unit uniform to arbitrary marginals allows us to consider a strict generator ϕ as long as the supports of the conditional distributions do not coincide with some bounded interval. For instance, considering a strict generator ϕ with marginals F_1 and F_2 with common support $(0, +\infty)$ makes a $\preceq_{\text{disp}}$ comparison possible. For these reasons, we do not repeat the conditions on the generator, keeping in mind that we exclude the case with identical bounded supports in the next result.

Proposition 4.2. *Let $\boldsymbol{X} = (X_1, X_2)$ be a random vector with distribution function* (1.1). *Then,*

(i) the stochastic inequality $[X_1|X_2 = x_2] \preceq_{disp} [X_1|X_2 = x_2']$ holds for $x_2 < x_2'$ if, and only if,

$$\alpha \mapsto F_1^{-1}\big(\psi_{F_2(x_2')}(\alpha)\big) - F_1^{-1}\big(\psi_{F_2(x_2)}(\alpha)\big) \text{ non-decreasing on } [0,1]. \tag{4.1}$$

(ii) the stochastic inequality $[X_1|X_2 \leq x_2] \preceq_{disp} [X_1|X_2 \leq x_2']$ holds for $x_2 < x_2'$ if, and only if,

$$\alpha \mapsto F_1^{-1}\big(\psi^*_{F_2(x_2')}(\alpha)\big) - F_1^{-1}\big(\psi^*_{F_2(x_2)}(\alpha)\big) \text{ non-decreasing on } [0,1]. \tag{4.2}$$

Proof. The result is a consequence of (2.1)-(2.2) together with Proposition 3.1. Define $U_i = F_i(X_i)$, $i = 1, 2$. Considering (i), the conditional distribution of X_1 is given by

$$\begin{aligned}\Pr[X_1 \leq x_1 | X_2 = x_2] &= \Pr[F_1^{-1}(U_1) \leq x_1 | F_2^{-1}(U_2) = x_2] \\ &= \Pr[U_1 \leq F_1(x_1) | U_2 = F_2(x_2)] \\ &= \psi^{-1}_{F_2(x_2)}(F_1(x_1))\end{aligned}$$

so that the corresponding quantile function is $F_1^{-1} \circ \psi_{F_2(x_2)}$. The proof for (ii) is similar. □

Note that only F_1 matters in Proposition 4.2, not F_2. This comes from the fact that the condition $X_2 = x_2$ or $X_2 \leq x_2$ can equivalently be expressed in terms of $U_2 = F_2(X_2)$, coming back to the unit uniform distribution whatever F_2.

Another way to state the results in Proposition 4.2 consists in imposing that the first derivative of (4.1)-(4.2) is non-negative. For instance, this gives for (4.1)

$$x_2 \mapsto \frac{\frac{\partial}{\partial \alpha}\psi_{F_2(x_2)}}{f_1\Big(F_1^{-1}\big(\psi_{F_2(x_2)}(\alpha)\big)\Big)} \text{ non-decreasing,}$$

where f_1 denotes the probability density function corresponding to F_1.

Let us now consider a couple of examples involving standard families of parametric archimedean copulas.

Example 4.3. Consider Frank's copula given by

$$C_{\phi_\theta}(u_1, u_2) = -\frac{1}{\theta} \ln \left(1 + \frac{(\exp(-\theta u_1) - 1)(\exp(-\theta u_2) - 1)}{\exp(-\theta) - 1}\right), \quad \theta \neq 0.$$

This is an archimedean copula with generator $\phi_\theta(t) = \ln(e^{-\theta} - 1) - \ln(e^{-t\theta} - 1)$. Then, we obtain

$$\psi_{F_2(x_2)}(\alpha) = \frac{1}{\theta} \ln\left(\frac{e^{-\theta F_2(x_2)} + \alpha(1 - e^{-\theta F_2(x_2)})}{e^{-\theta F_2(x_2)} + \alpha(e^{-\theta} - e^{-\theta F_2(x_2)})}\right) \tag{4.3}$$

and

$$\psi^*_{F_2(x_2)}(\alpha) \quad = \quad \frac{1}{\theta} \ln\left(\frac{e^{-\theta F_2(x_2)} - 1}{e^{-\theta F_2(x_2)} - 1 + (e^{-\theta F_2(x_2)\alpha} - 1)(e^{-\theta} - 1)}\right). \tag{4.4}$$

For instance, with unit Exponential marginal F_1, that is, $F_1(x) = 1 - \exp(-x)$, we get

$$F_1^{-1}\big(\psi_{F_2(x_2')}(\alpha)\big) - F_1^{-1}\big(\psi_{F_2(x_2)}(\alpha)\big) = -\ln\left(\frac{1 - \psi_{F_2(x_2')}(\alpha)}{1 - \psi_{F_2(x_2)}(\alpha)}\right) \tag{4.5}$$

and

$$F_1^{-1}\big(\psi^*_{F_2(x_2')}(\alpha)\big) - F_1^{-1}\big(\psi^*_{F_2(x_2)}(\alpha)\big) = -\ln\left(\frac{1 - \psi^*_{F_2(x_2')}(\alpha)}{1 - \psi^*_{F_2(x_2)}(\alpha)}\right) \tag{4.6}$$

For $\theta \geq 0$, Frank's copulas express positive dependence, i.e. large values of one component tend to be associated with large values of the other one. Considering $F_2(x_2) = 0.25$ and $F_2(x_2') = 0.75$, Figure 4.1 (top left panel) shows that the difference (4.5) is increasing for values of θ corresponding to Kendall's τ equal to 0.1, 0.4, 0.7, and 0.9, respectively. This means that increasing the value of X_2 makes X_1 more variable in the $\preceq_{\text{disp}}$-sense. On the contrary, for $\theta \leq 0$, the dependence is negative, that is, large values of one component tend to be associated with small values of the other one. For such θs, we see from Figure 4.1 (top right panel) that the difference (4.5) is now decreasing. Increasing X_2 now makes X_1 less variable in the $\preceq_{\text{disp}}$-sense. Moving from the center of the distribution to the tails does not modify the conclusion, as it can be seen from Figure 4.2 (top panels) where we consider x_2 and x_2' such that $F_2(x_2) = 0.99$ and $F_2(x_2') = 0.995$. Figure 4.3 shows the difference (4.5) as a function of α and θ. The different behavior according to the sign of θ is clearly visible there.

In addition to the unit Exponential case, we also consider in Figures 4.1-4.2 the case of Pareto marginal F_1, that is, $F_1(x) = 1 - x^{-a}$ for $x > 1$ and some $a > 0$, standard Normal marginal F_1, and Gamma marginal F_1. We can see there that the results obtain in the unit Exponential case are also valid in the Pareto case. However, no dispersive order relation holds in the Normal case whereas in the Gamma case, the dispersive order relation is valid only for sufficiently high correlation. This illustrates the effect of marginal distributions on cross-aging.

Example 4.4. Consider Clayton's copula defined by

$$C_{\phi_\theta}(u_1, u_2) = \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-1/\theta}, \ \ \theta > 0. \tag{4.7}$$

This copula belongs to the archimedean class, generated by $\phi_\theta(t) = \frac{t^{-\theta}-1}{\theta}$, $\theta > 0$, which is strict. In this case, we find $\phi_\theta^{-1}(t) = (\theta t + 1)^{-1/\theta}$ and

$$\psi_{F_2(x_2)}(\alpha) = \Big(1 + (F_2(x_2))^{-\theta}\left(\alpha^{-\theta/(\theta+1)} - 1\right)\Big)^{-1/\theta} \tag{4.8}$$

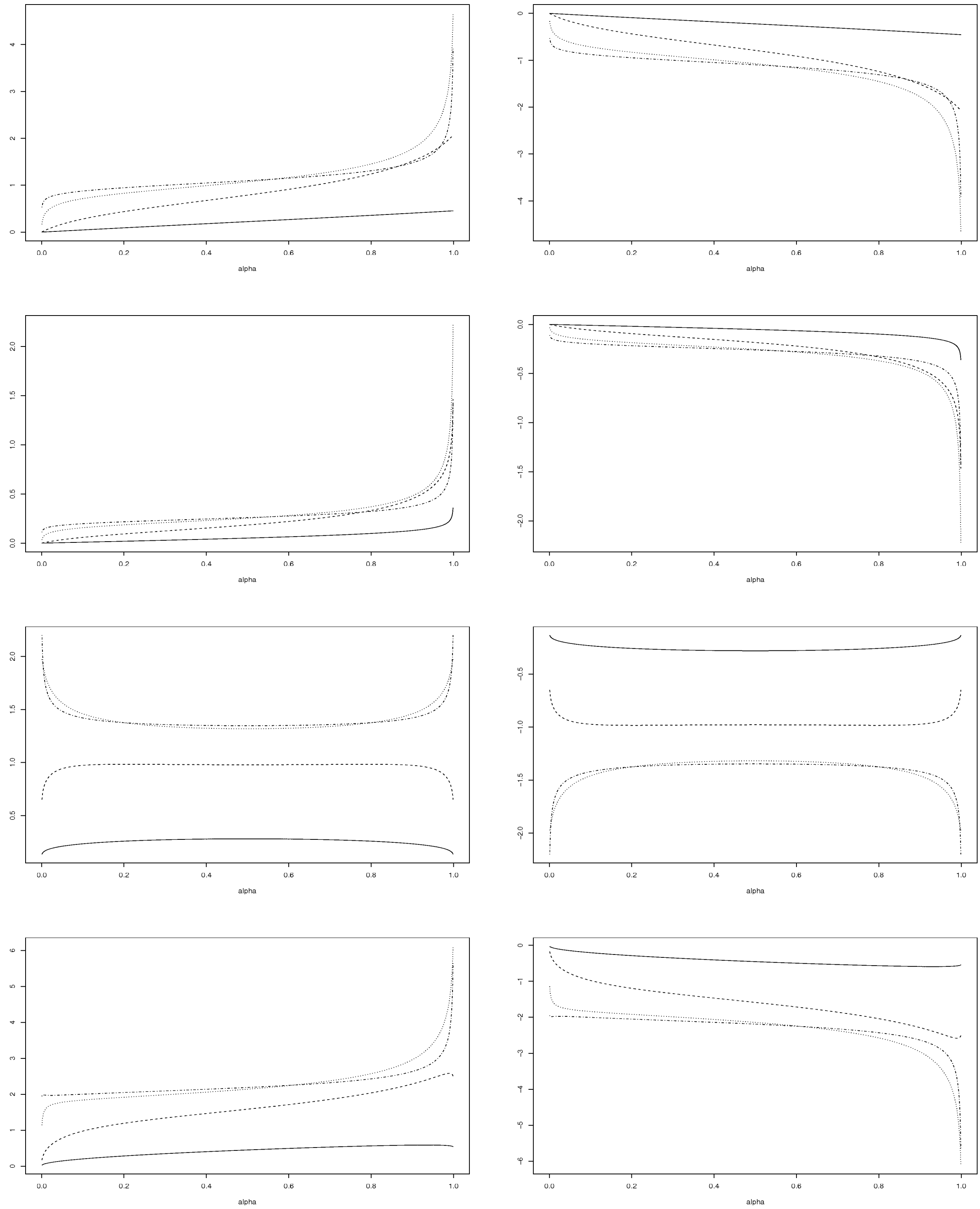

Figure 4.1: Graph of (4.1) for Frank copula with x_2 and x'_2 such that $F_2(x_2) = 0.25$ and $F_2(x'_2) = 0.75$ and values of θ corresponding to Kendall's τ equal to 0.1 (solid), 0.4 (dashed), 0.7 (dotted), and 0.9 (dotdash) in the left panels and to -0.1 (solid), -0.4 (dashed), -0.7 (dotted), and -0.9 (dotdash) in the right panels. From top to bottom: unit Exponential marginal F_1, Pareto marginal F_1 (with $a = 5$), standard Normal marginal F_1, and Gamma marginal F_1 (with shape parameter 3 and scale parameter 1, that is, with mean and variance equal to 3).

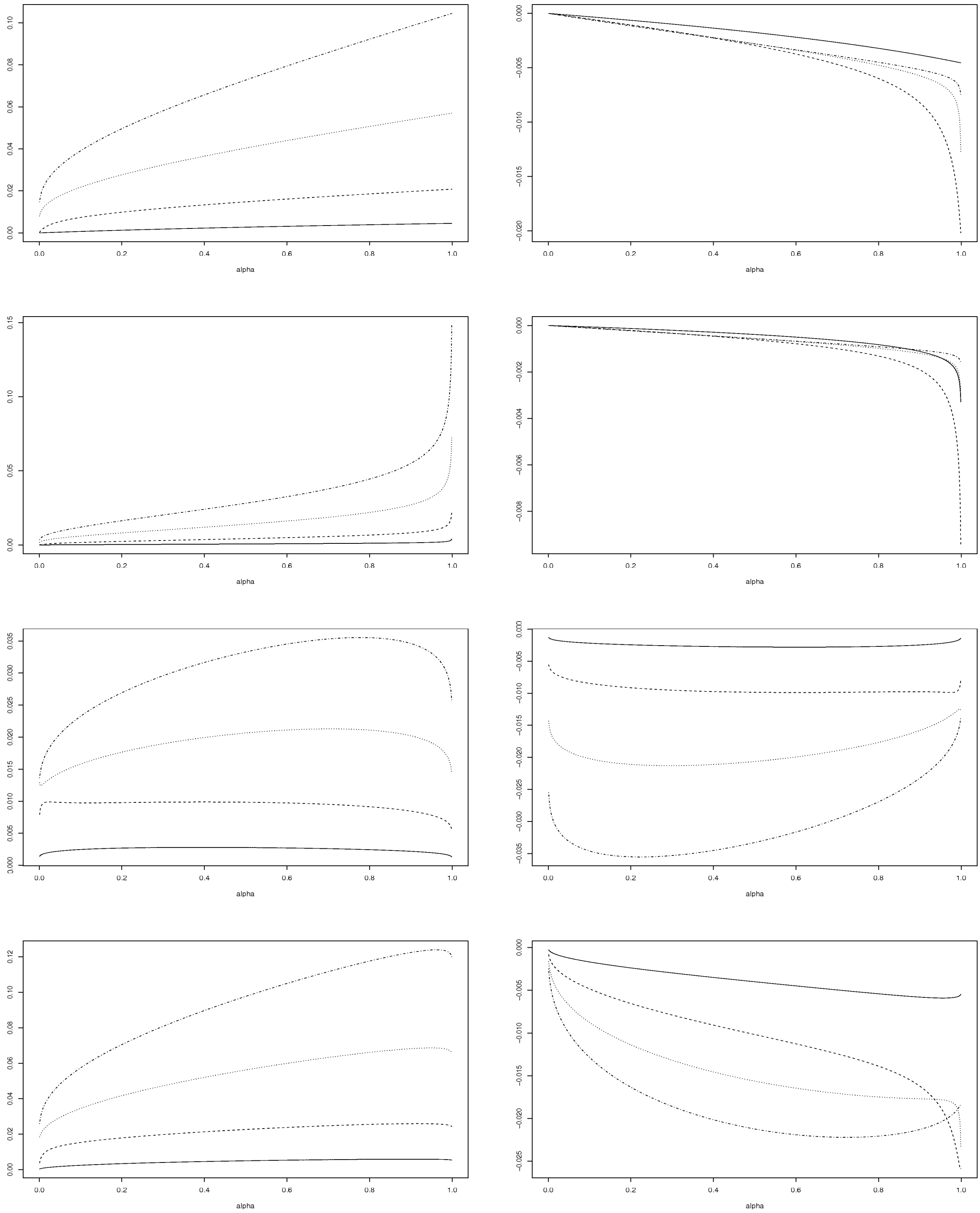

Figure 4.2: Graph of (4.1) for Frank copula with x_2 and x_2' such that $F_2(x_2) = 0.99$ and $F_2(x_2') = 0.995$ and values of θ corresponding to Kendall's τ equal to 0.1 (solid), 0.4 (dashed), 0.7 (dotted), and 0.9 (dotdash) in the left panels and to -0.1 (solid), -0.4 (dashed), -0.7 (dotted), and -0.9 (dotdash) in the right panels. From top to bottom: unit Exponential marginal F_1, Pareto marginal F_1 (with $a = 5$), standard Normal marginal F_1, and Gamma marginal F_1 (with shape parameter 3 and scale parameter 1, that is, with mean and variance equal to 3).

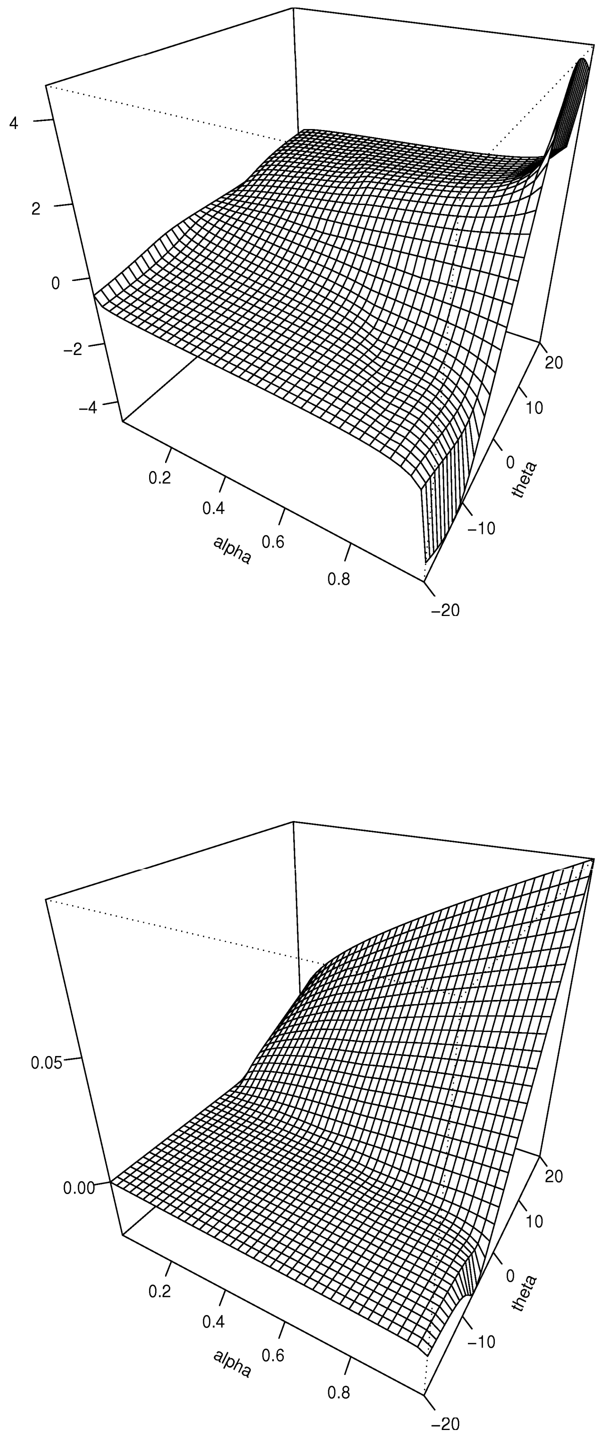

Figure 4.3: Graph of (4.5) as a function of α and θ for Frank copula with x_2 and x_2' such that $F_2(x_2) = 0.25$ and $F_2(x_2') = 0.25$ (top panel) and with x_2 and x_2' such that $F_2(x_2) = 0.99$ and $F_2(x_2') = 0.995$ (bottom panel) with unit Exponential marginal F_1.

and

$$\psi^*_{F_2(x_2)}(\alpha) = \Big((F_2(x_2)\alpha)^{-\theta} - F_2(x_2)^{-\theta} + 1\Big)^{-1/\theta}. \tag{4.9}$$

For instance, with Pareto marginal F_1, we get

$$F_1^{-1}\big(\psi_{F_2(x_2')}(\alpha)\big) - F_1^{-1}\big(\psi_{F_2(x_2)}(\alpha)\big) = \Big(1 - \psi_{F_2(x_2')}(\alpha)\Big)^{-1/a} - \Big(1 - \psi_{F_2(x_2)}(\alpha)\Big)^{-1/a} \tag{4.10}$$

and

$$F_1^{-1}\big(\psi^*_{F_2(x_2')}(\alpha)\big) - F_1^{-1}\big(\psi^*_{F_2(x_2)}(\alpha)\big) = \Big(1 - \psi^*_{F_2(x_2')}(\alpha)\Big)^{-1/a} - \Big(1 - \psi^*_{F_2(x_2)}(\alpha)\Big)^{-1/a}. \tag{4.11}$$

Note that the dependence expressed by Clayton copula (4.7) is always positive (an extension of (4.7) to negative θs is possible but is not considered here). The limiting case $\theta = 0$ corresponds to independence and increasing θ strengthens the positive relationship between the two components of the random couple. Figure 4.4 is the counterpart of Figure 4.1 and Figure 4.5 is the counterpart of Figure 4.3 for Clayton copula. The conclusions drawn for Frank copulas in the case $\theta > 0$ still apply to Clayton copulas.

5 Conditional comparison of random vectors with identical copulas

Consider two random couples, (X_1, X_2) and (Y_1, Y_2), say, sharing the same archimedean copula C_ϕ. We assume that (X_1, X_2) possesses the dispersive cross-aging property and we would like to compare conditional distributions $[X_1|X_2 = x_2]$ and $[Y_1|Y_2 = x_2]$ when the marginals are ordered. The next result provides an answer to this problem.

Proposition 5.1. *Let (X_1, X_2) and (Y_1, Y_2) be two random couples with the same archimedean copula C_ϕ. Assume that $[X_1|X_2 = x_2] \preceq_{disp} [X_1|X_2 = x_2']$ holds for all $x_2 \leq x_2'$. Then,*

$$X_1 \preceq_{disp} Y_1 \text{ and } Y_2 \preceq_{st} X_2 \Rightarrow [X_1|X_2 = x_2] \preceq_{disp} [Y_1|Y_2 = x_2] \text{ for all } x_2.$$

Proof. Denote as F_i the distribution function of X_i, $i = 1, 2$, and as G_i the distribution function of Y_i, $i = 1, 2$. Clearly,

$$\begin{aligned} G_1^{-1}\left(\psi_{G_2(x_2)}(\alpha)\right) - F_1^{-1}\left(\psi_{F_2(x_2)}(\alpha)\right) &= \Big(G_1^{-1}\left(\psi_{G_2(x_2)}(\alpha)\right) - F_1^{-1}\left(\psi_{G_2(x_2)}(\alpha)\right)\Big) \\ &\quad + \Big(F_1^{-1}\left(\psi_{G_2(x_2)}(\alpha)\right) - F_1^{-1}\left(\psi_{F_2(x_2)}(\alpha)\right)\Big). \end{aligned} \tag{5.1}$$

Since $X_1 \preceq_{\text{disp}} Y_1$ we know that $\alpha \mapsto G_1^{-1}(\alpha) - F_1^{-1}(\alpha)$ is non-decreasing. This, in turn, implies that $\alpha \mapsto G_1^{-1}(\psi_{G_2(x_2)}(\alpha)) - F_1^{-1}(\psi_{G_2(x_2)}(\alpha))$ is non-decreasing, since $\alpha \mapsto \psi_{G_2(x_2)}(\alpha)$ is non-decreasing. The function inside the first bracket of (5.1) is thus non-decreasing. Let us now consider the function inside the second bracket of (5.1). Putting $x_2' = F_2^{-1}(G_2(x_2)$, we have $x_2 \leq x_2'$ since $Y_2 \preceq_{\text{st}} X_2$. Now using the fact that $[X_1|X_2 = x_2] \preceq_{\text{disp}} [X_1|X_2 = x_2']$, we see that $F_1^{-1}(\psi_{F_2(x_2')}(\alpha)) - F_1^{-1}(\psi_{F_2(x_2)}(\alpha))$ is non-decreasing and coincides with $F_1^{-1}(\psi_{G_2(x_2)}(\alpha)) - F_1^{-1}(\psi_{F_2(x_2)}(\alpha)$. Hence, $G_1^{-1}(\psi_{G_2(x_2)}(\alpha)) - F_1^{-1}(\psi_{F_2(x_2)}(\alpha))$ appears as the sum of two non-decreasing functions and is therefore also non-decreasing, which ends the proof. □

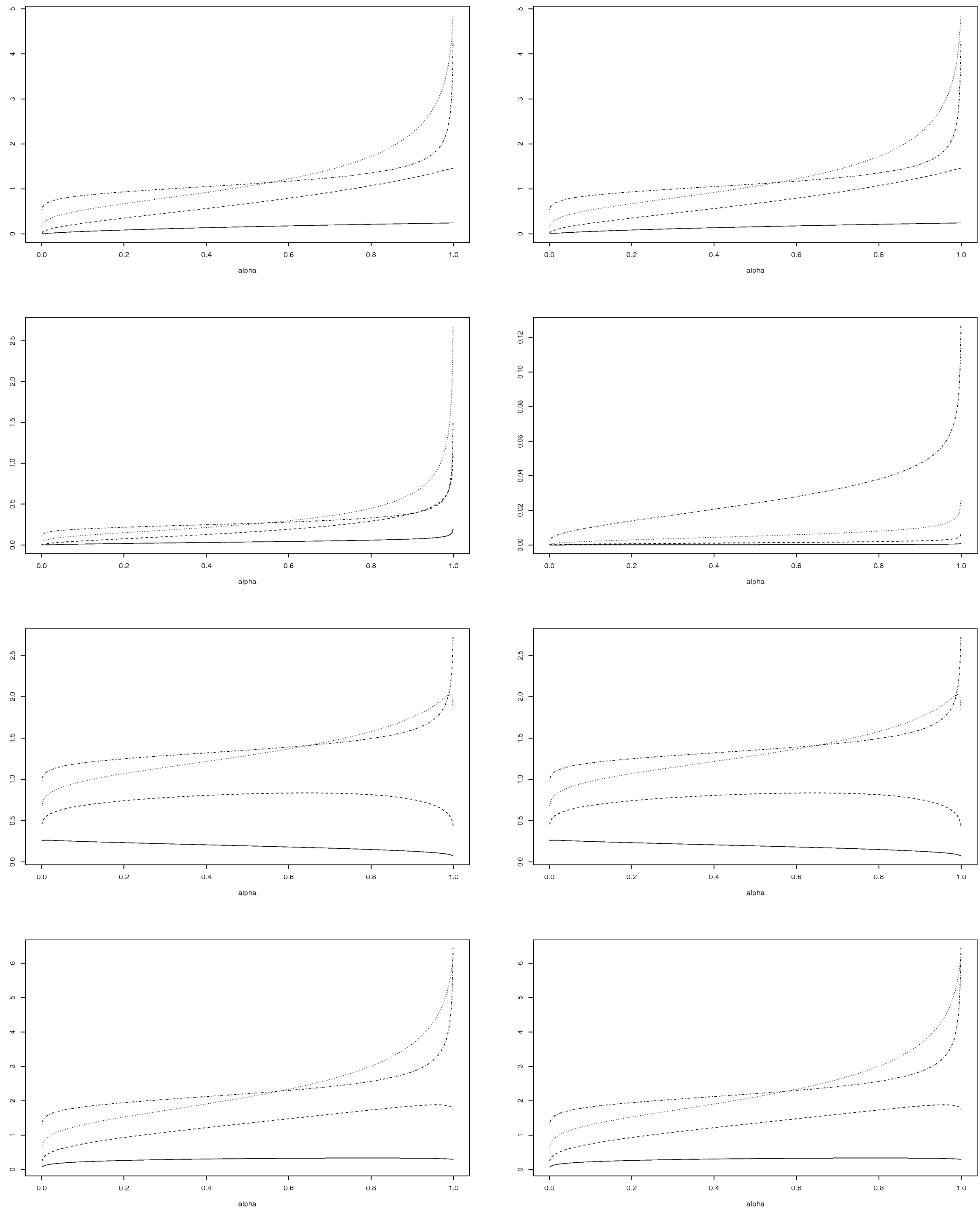

Figure 4.4: Graph of (4.1) for $F_2(x_2) = 0.25$ and $F_2(x_2') = 0.75$ (left panel) and $F_2(x_2) = 0.99$ and $F_2(x_2') = 0.995$ (right panel) for Clayton copula and values of θ corresponding to Kendall's τ equal to (solid), 0.4 (dashed), 0.7 (dotted), and 0.9 (dotdash). From top to bottom: unit Exponential marginal F_1, Pareto marginal F_1 (with $a = 5$), standard Normal marginal F_1, and Gamma marginal F_1 (with shape parameter 3 and scale parameter 1, that is, with mean and variance equal to 3).

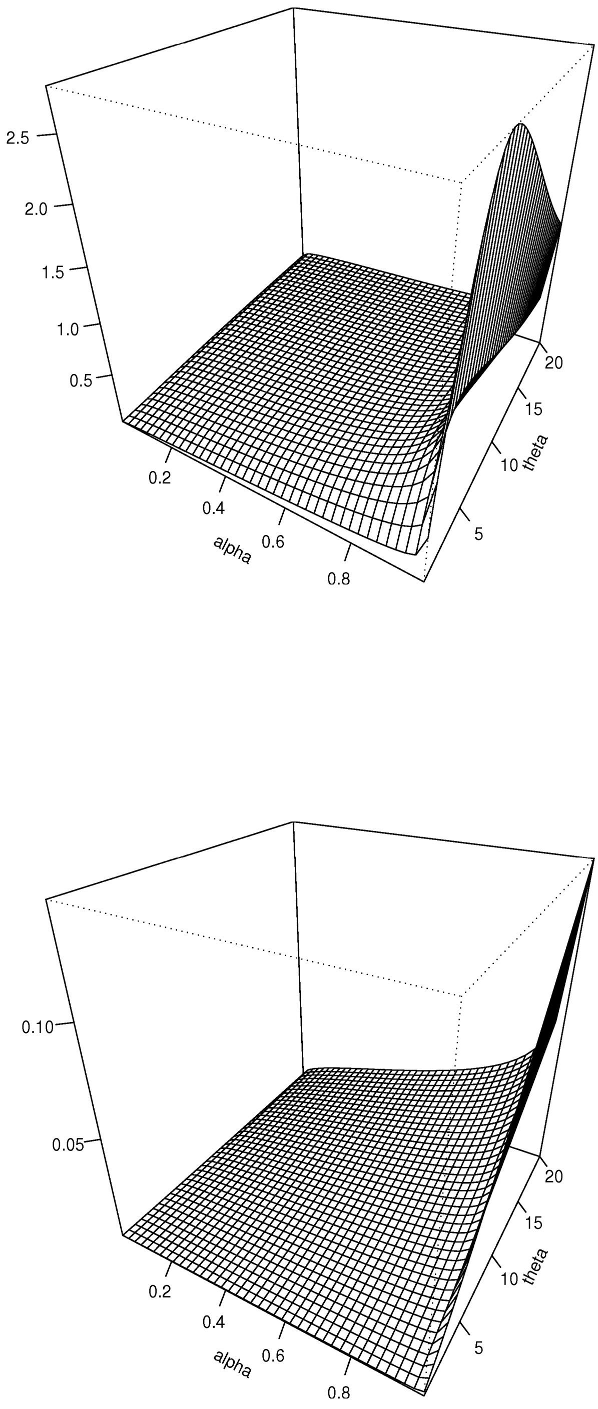

Figure 4.5: Graph of (4.10) as a function of α and θ for Clayton copula with x_2 and x_2' such that $F_2(x_2) = 0.25$ and $F_2(x_2') = 0.75$ (top panel) and with x_2 and x_2' such that $F_2(x_2) = 0.99$ and $F_2(x_2') = 0.995$ (bottom panel) with Pareto marginal F_1 ($a = 5$).

A similar result holds for conditional distributions $[X_1|X_2 \leq x_2]$ and $[Y_1|Y_2 \leq x_2]$. Taking $F_2 = G_2$, we see that provided (X_1, X_2) possesses the dispersive cross-aging property, increasing the first marginal distribution in the $\preceq_{\text{disp}}$-sense also increases the conditional distributions in the $\preceq_{\text{disp}}$-sense.

6 Conclusion

In this paper, we have established necessary and sufficient conditions for dispersive inequalities between conditionals of bivariate distribution functions built from archimedean copulas, a phenomenon called dispersive cross-aging. Given the importance of the dispersive stochastic order relation in many applications, the results derived in this paper allow for a deeper understanding of the dependence structure induced by archimedean copulas. The conditions derived in this paper are easy to verify (at least numerically) and are satisfied by standard copulas including Clayton and Frank families.

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