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MINIMAL ENTROPY MARTINGALE MEASURE IN A SEMI-MARKOV REGIME SWITCHING COX-ROSS-RUBINSTEIN MODEL

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Minimal entropy martingale measure in a semi-Markov regime switching Cox-Ross-Rubinstein model

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Abstract

In this paper, we present a discrete time regime switching Cox-Ross-Rubinstein model where the regime switches are governed by a discrete time semi-Markov process. We model the evolution of the price of a stock and look at the pricing and hedging of options on that stock. We discuss and derive conditions for the model to be arbitrage free and relate this to the notion of martingale measure. We show that the market is incomplete i.e. that there exists an infinite number of martingale measures. This leads us to select a martingale measure. We choose to study and characterize the notion of minimal entropy martingale measure in this context. Finally, we conclude.

1 Introduction

It is quite clear that the search for realistic and tractable models to be used for the pricing and hedging of financial derivatives has become one of the most active topics of research in modern quantitative finance. Historically, we can cite the Cox-Ross-Rubinstein model in discrete time (see [3]) and the Black-Scholes model in continuous time (see [1]). Since then, many alternative approaches have been put forward: stochastic volatility models, local volatility models, jump-diffusion, Lévy processes, regime switching, etc.

The idea behind regime switching models is to allow the parameters that govern the evolution of the assets to switch between different states according to some underlying (hidden or not) stochastic process. The rationale behind this is that the economic environment is not stable but changes at some unpredictable times and this should be taken into account when pricing financial instruments.

The paper that triggered the current interest in regime switching models is the paper by Hamilton ([13]). This paper focuses on Markov switching time series. Since then, regime switching models have gained in popularity and have been applied in many other fields but most of the litterature focuses on Markov switching models. There is now a long list of papers that deal with applications of regime switching models in modern quantitative finance. Amongst others let us cite [5], [9], [10] and

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[16]. There is also one paper that briefly presents a discrete-time Markov switching model for option pricing ([15]).

As was previsously mentionned, most of the litterature focuses on Markov switching models. However, the Markov nature of financial phenomena can be criticized (see [14], [7], [8], [4] and [6]). An extension that has been suggested to tackle these problems is the class of semi-Markov regime switching models (see [2]). There are many advantages in considering semi-Markov switching as opposed to homogeneous Markov switching. First, they allow for more flexibility in the distribution of the duration times in each state. This is coherent with the ideas in [4] and [6]. Second, semi-Markov switching models don't necessarily possess the Markov property. This is in line with the ideas and results presented in [14]. In this paper, the authors argue and present results that reject the Markov property. For more discussion of the Markov property (or the lack of it) in financial data, see [7] and [8]. And third, because homogeneous Markov processes are a subset of semi-Markov processes, semi-Markov regime switching models should always be at least as good as homogeneous Markov switching models.

Of course, some authors have now looked at semi-Markov regime switching models and their applications in finance. A nice feature of semi-Markov models is that they are intuitively coherent and easy to understand whilst remaining parcimonious. Some of the papers that deal with this are given by [12], [17], [18]. These papers are set in a continuous time framework.

Very few papers deal with discrete time semi-Markov regime switching models. Our paper studies this subject in the context of a generalization of the well-known Cox-Ross-Rubinstein model. The idea is to let the parameters be governed by an underlying homogeneous semi-Markov process. First, we provide a sound mathematical background for this model. We then discuss the notion of absence of arbitrage and show the existence of equivalent martingale measures. We then discuss the notion of market incompleteness in our framework. This leads us to introduce the notion of minimal entropy martingale measure. We characterize this measure in the context of our model. Our result shows that, in our setting, the minimal entropy martingale measure is a product of two terms. One term representing the unhedgeable risk associated to the semi-Markov switching process and the second term representing the hedgeable risk associated to the binomial type market movements (a term very similar to what exists in the simple binomial model). Finally, the paper concludes with a summary and some comments on our results.

2 Mathematical framework

Let us consider a discrete time financial market built on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We suppose that financial transactions can only take place at fixed times 0,1,2,...,T^*. We note $\mathbb{T} = \{0, 1, ..., T^*\}$.

We define the set $E = \{1, ..., m\}$ for m finite and $\mathcal{E}$ as the sigma-algebra on E. We suppose our probability space carries a pair of processes (X_n, T_n) taking values in $E \times \mathbb{N}$. We suppose (X_n, T_n) is a homogeneous Markov renewal process of semi-Markov kernel Q i.e. that for all n, X_n, j and t

$$\mathbb{P}(X_{n+1} = j; T_{n+1} - T_n \leq t | X_0, T_0, ..., X_n, T_n) = \mathbb{P}(X_{n+1} = j; T_{n+1} - T_n \leq t | X_n) := Q_{X_n j}(t)$$

We define ν_t as $\nu_t = \max_n\{T_n \leq t\}$. Then, the semi-Markov process Y of kernel Q is defined by

$$Y_t = X_{\nu_t}$$

where we suppose that Y_0 is known and non-random. The process Y will control the "regime" or "state" of the economy.

Similarly, the process K_t is defined as

$$K_t = t - T_{\nu_t}$$

and represents the time elapsed since the last jump. We suppose K_0 is known and non-random.

To every state $i \in E$ we associate two positive constants u_i and d_i such that $u_i > d_i$ and $u_i \neq u_j$ and $d_i \neq d_j$ for $i \neq j$.

We define $\mathcal{F}_t = \sigma(\zeta_s, Y_s,\ s \leq t)$. This choice of filtration is very important as it implies that Y is observable. This is in line with the ideas in [17].

Our probability space is also assumed to carry an adapted process ζ_t. Conditional on the past, this process behaves like a "binomial" process. We impose that

$$\mathbb{P}(\zeta_{t+1} = u_{Y_t}|\mathcal{F}_t) = 1 - \mathbb{P}(\zeta_{t+1} = d_{Y_t}|\mathcal{F}_t)$$

This process describes the evolution of the market from time t to $t+1$. We will say that the market goes up (resp. down) if $\zeta_{t+1} = u_{Y_t}$ (resp. $\zeta_{t+1} = d_{Y_t}$).

We suppose that ζ_{t+1} and Y_{t+1} are conditionally independent given $\mathcal{F}_t$; that is $\forall t \geq 1$:

$$\mathbb{P}[\zeta_{t+1} = u_{Y_t}, Y_{t+1} = j|\mathcal{F}_t] = \mathbb{P}[\zeta_{t+1} = u_{Y_t}|\mathcal{F}_t]\mathbb{P}[Y_{t+1} = j|\mathcal{F}_t]$$

We note $P[Y_{t+1} = j|\mathcal{F}_t] = \upsilon_t^j$. Because of the structure of process Y, it follows that $\upsilon_t^j = P[Y_{t+1} = j|Y_t, K_t]$. Furthermore, using the properties of semi-Markov processes, we can write

$$\upsilon_t^j = \frac{Q_{Y_t j}(K_t + 1) - Q_{Y_t j}(K_t)}{1 - \sum_{j=1}^m Q_{Y_t j}(K_t)}$$

Let us introduce some notation. We define $\mathbb{P}[\zeta_{t+1} = u_{Y_t}|\mathcal{F}_t] := z_t$. It is clear that

$$\pi_t^j := \mathbb{P}[\zeta_{t+1} = u_{Y_t}, Y_{t+1} = j|\mathcal{F}_t] = z_t \upsilon_t^j$$

Furthermore, we define

$$\kappa_t^j := (1 - z_t)\upsilon_t^j = \mathbb{P}[\zeta_{t+1} = d_{Y_t}, Y_{t+1} = j|\mathcal{F}_t]$$

It is then quite clear that whatever value t:

$$\sum_{j=1}^m [\pi_t^j + \kappa_t^j] = 1$$

When given that $Y_t = i$ and $K_t = k$, the pair of processes Y and K can take m different values at time $t+1$. Indeed, either $(Y_{t+1}, K_{t+1}) = (i, k+1)$, either $(Y_{t+1}, K_{t+1}) = (j, 0)$ for every $j \neq i$. However, the system composed of process

ζ, Y and K can take $2m$ different values. These values are all determined by the following sets of events (where j can be equal to i):

$$\begin{aligned} A_{t+1}^{j,u} &= \{\omega \in \Omega : \ Y_{t+1} = j, \zeta_{t+1} = u_{Y_t}\} \\ A_{t+1}^{j,d} &= \{\omega \in \Omega : \ Y_{t+1} = j, \zeta_{t+1} = d_{Y_t}\} \end{aligned}$$

2.1 The market model

The market is supposed to carry two assets, a riskless asset S_t^0 and a risky asset S_t^1 (we will denote by S_t the vector stochastic process $S_t = (S_t^0, S_t^1)$). The riskless asset is supposed to have a constant rate of return r and we suppose $S_0^0 = 1$ and r as known. We have

$$S_t^0 = (1+r)^t$$

The price S_t^1 of the risky asset is supposed to be known at time zero (i.e. S_0^1 is known) and has the following dynamics

$$S_{t+1}^1 = \zeta_{t+1} S_t^1$$

So, given the state of the system at time t and specifically given that $Y_t = i$, the risky asset can only take two possible values at time $t+1$ namely $u_i S_t^1$ or $d_i S_t^1$.

Remark 2.1. This model is a regime switching Cox-Ross-Rubinstein model. However, in our model, because of the possibility of regime switches, the binomial lattice is not path independent in the sense that if the market goes up and then down, the risky asset might not have the same value as if the market had gone down and then up.

Remark 2.2. Let us note that the price of the risky asset is the same when events $A^{i,u}$ and $A^{j,u}$ (or $A^{i,d}$ and $A^{j,d}$) happen i.e., when the market goes up (resp. down), the price of the risky asset is $S_{t+1}^1 = u_{Y_t} S_t^1$ (resp. $S_{t+1}^1 = d_{Y_t} S_t^1$) in the next period whether there is a regime change or not (see figure 1). The effect of the regime switch will only be felt in the next time period.

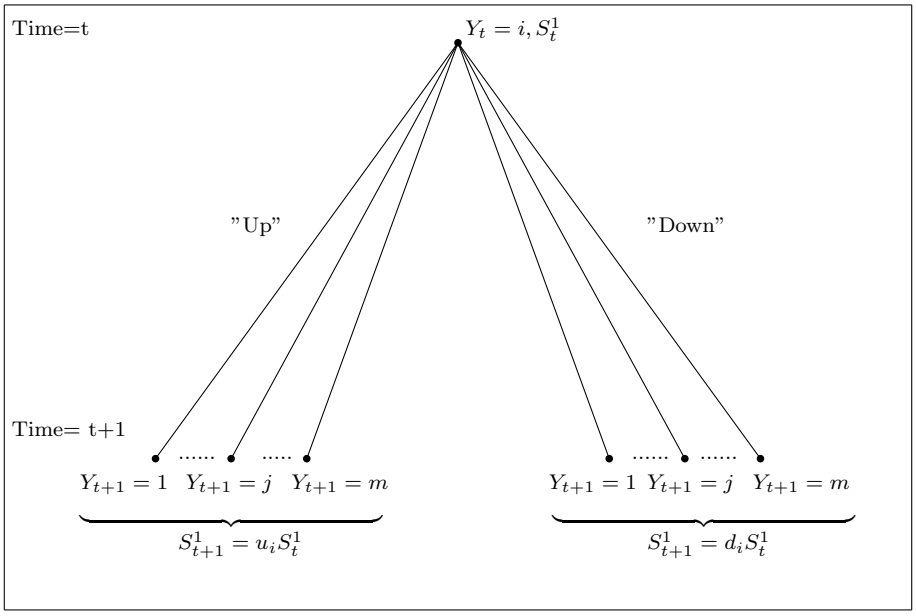

Figure 1: Regime switching semi-Markov binomial tree

3 Absence of arbitrage and martingale measures

In our market, a trading strategy will be a vector of adapted stochastic processes $\phi_t = (\phi_t^0, \phi_t^1)$ where ϕ_t^i represents the number of units of asset S_t^i held at time t.

To each trading strategy, we can associate a portfolio whose wealth $V_t(\phi)$ at time t is given by

$$V_t(\phi) = \phi_t^0 S_t^0 + \phi_t^1 S_t^1 := \phi_t \cdot S_t$$

Definition 3.1. *A trading strategy is called self-financing if*

$$\phi_{t-1} \cdot S_t = \phi_t \cdot S_t$$

Definition 3.2. *A self-financing trading strategy is is an arbitrage strategy if* $\mathbb{P}(V_0(\phi) = 0) = 1$ *and*

$$\mathbb{P}(V_T(\phi) \geq 0) = 1 \quad \textit{and} \quad \mathbb{P}(V_T(\phi) > 0) > 0$$

A standard approach in the litterature is to discuss under which conditions is a market arbitrage free. In our case, a necessary condition is given by the following proposition.

Proposition 3.3. *In order to avoid arbitrage opportunities, we need to impose that* $u_i > 1 + r > d_i$ *for every* $i \in E$.

Proof. We will show that if $u_i > d_i > 1+r$ then we can build an arbitrage portfolio. Indeed, at time $t = 0$, borrow S_0^1. Buy one unit of risky asset. At time t=1, sell the asset for S_1^1. Reimburse $(1 + r)S_0^1$. This yields a sure profit of $S_1^1 - S_0^1(1 + r) > 0$. The same type of reasoning applies in the case $1 + r > u_i > d_i$. □

It is well known that absence of arbitrage is linked to the existence of equivalent martingale measures. We will develop these ideas in what follows.

For each $j \in E$, let us define stochastic processes p_t^j and q_t^j such that for every $t \in \mathbb{T}$, we have $\sum_{j=1}^m (p_t^j + q_t^j) = 1$. Let D_t be defined as

$$D_t = \prod_{s=0}^{t-1} \Big(\sum_{j=1}^m \Big[\frac{p_s^j}{\pi_s^j} \mathbb{1}_{A_{s+1}^{j,u}} + \frac{q_s^j}{\kappa_s^j} \mathbb{1}_{A_{s+1}^{j,d}} \Big] \Big) \tag{1}$$

Lemma 3.4. *Let D_t be defined by (1), then $D_t > 0$ for all t, $\mathbb{E}[D_t] = 1$ and*

$$\mathbb{E}[D_{t+1} | \mathcal{F}_t] = D_t$$

Proof. The first statement is obvious from the definition of D_t and of the quantities involved in this definition. The third statement is immediate from the definition of the $A_{s+1}^{j,u}$, $A_{s+1}^{j,d}$, p_t^j and q_t^j and the fact that

$$D_t = D_{t-1} \Big(\sum_{j=1}^m \Big[\frac{p_{t-1}^j}{\pi_{t-1}^j} \mathbb{1}_{A_t^{j,u}} + \frac{q_{t-1}^j}{\kappa_{t-1}^j} \mathbb{1}_{A_t^{j,d}} \Big] \Big)$$

The second statement comes from calculating $\mathbb{E}(D_0)$ and using statement three. □

This result allows us to think of D_t as a density process. This process will be used to introduce equivalent measures.

Definition 3.5. *Define $\mathbb{P}^*$ as the equivalent measure with density D_{T^*} with respect to $\mathbb{P}$.*

We will show that under one condition, the measure $\mathbb{P}^*$ is an equivalent martingale measure. This means that under this measure, every asset (properly discounted) should behave as a martingale.

Theorem 3.6. *Let p_t^* be the stochastic process defined by $p_t^* = \frac{1+r-d_{Y_t}}{u_{Y_t}-d_{Y_t}}$. Then, if for all t we have $\sum_{j=1}^m p_t^j = p_t^*$, the measure $\mathbb{P}^*$ is an equivalent $\mathcal{F}_t$-martingale measure.*

Proof. By definition, $\mathbb{P}^*$ is an equivalent measure. It remains to be shown that under this measure, every discounted asset behaves as a martingale. The case of the riskless asset is trivial. For the risky asset, we have

$$\begin{aligned}
\mathbb{E}^*[\frac{1}{1+r} S_{t+1}^1 | \mathcal{F}_t] &= \frac{1}{1+r} S_t^1 \frac{1}{D_t} \mathbb{E}[\zeta_{t+1} D_{t+1} | \mathcal{F}_t] \\
&= \frac{S_t^1}{1+r} \mathbb{E}\Big[u_{Y_t} \sum_{j=1}^m \frac{p_t^j}{\pi_t^j} \mathbb{1}_{A_{t+1}^{j,u}} + d_{Y_t} \sum_{j=1}^m \frac{q_t^j}{\kappa_t^j} \mathbb{1}_{A_{t+1}^{j,d}} | \mathcal{F}_t \Big] \\
&= \frac{S_t^1}{1+r} (u_{Y_t} \sum_{j=1}^m p_t^j + d_{Y_t} \sum_{j=1}^m q_t^j) \\
&= \frac{S_t^1}{1+r} (u_{Y_t} p_t^* + d_{Y_t}(1 - p_t^*)) \\
&= S_t^1
\end{aligned}$$

The last equality follows by definition of p_t^*. □

Because we have shown that there exist martingale measures and thanks to the equivalence between existence of martingale measures and absence of arbitrage, we have the following result:

Corollary 3.7. *The market is arbitrage free.*

Remark 3.8. Although we have existence of martingale measures, we do not have uniqueness. Indeed, the only condition we have is $\sum_{j=1}^{m} p_t^j = p_t^*$ for all t. This doesn't specify the values of the p_t^j's and as soon as $m > 1$, this means we have **an infinite number** of martingale measures.

The next result will give more insight on the meaning of the process p_t^*.

Proposition 3.9. *The value p_t^* is the probability that the market "goes up" at time $t+1$ under the equivalent martingale measure $\mathbb{P}^*$ i.e. $p_t^* = \mathbb{P}^*[\zeta_{t+1} = u_{Y_t}|\mathcal{F}_t]$*

Proof. Let us calculate $\mathbb{P}^*[A_{t+1}^{j,u}|\mathcal{F}_t]$.

$$\begin{aligned}
\mathbb{P}^*[A_{t+1}^{j,u}|\mathcal{F}_t] = \mathbb{P}^*[Y_{t+1} = j, \zeta_{t+1} = u_{Y_t}|\mathcal{F}_t] &= \mathbb{E}^*\left[\mathbb{1}_{A_{t+1}^{j,u}}|\mathcal{F}_t\right] \\
&= \frac{1}{D_t}\mathbb{E}\left[\mathbb{1}_{A_{t+1}^{j,u}} D_{t+1}|\mathcal{F}_t\right] \\
&= \mathbb{E}\left[\frac{p_t^j}{\pi_t^j}\mathbb{1}_{A_{t+1}^{j,u}}|\mathcal{F}_t\right] \\
&= p_t^j
\end{aligned}$$

Remember that $\sum_{j=1}^{m} p_t^j = p_t^*$. This means that $p_t^* = \mathbb{P}^*[\zeta_{t+1} = u_{Y_t}|\mathcal{F}_t]$. □

So, p_t^* gives a description of the conditional distribution of ζ_{t+1} (and so of S_{t+1}^1) under the equivalent martingale measure.

4 Market incompleteness

This section will discuss the notion of market (in-)completeness in our framework. For this, we need to introduce some further definitions.

Definition 4.1. *A path-independent contingent claim of maturity T is a square integrable $\mathcal{F}_T$-measurable random variable. Such contingent claims will be denoted by H and its value at maturity is given by H_T.*

The aim is to see if and when such a contingent claim be hedged in our market model. This is the notion of market completeness.

Definition 4.2. *A market is complete if for every contingent claim H_T, there exists a self-financing strategy such that $V_T(\phi) = H_T$. Such a strategy is called a replicating strategy.*

On an intuitive level, a market will be complete if there is an "equilibrium" between the number of sources of risk and the number of assets available for hedging in the market. In our case, because we have an extra source of uncertainty due to the semi-Markov process, we expect the market to be incomplete.

On a more formal level, in discrete time, a market is complete if and only if there exist a unique equivalent martingale measure. By remark 3.8, we know that the number of equivalent martingale measures is infinite. This yields

Theorem 4.3. *Let H_T be a contingent claim. Then, in general, there exist no self-financing strategy such that $V_T(\phi) = H_T$ and so the market is incomplete.*

Proof. Direct using remark 3.8 and the second fundamental theorem of finance. □

Although straightforward, the last theorem is a very strong and general result. Nevertheless, we will study an interesting example in a one period framework.

4.1 An interesting example

The next proposition shows that, in a one period model, when the contingent claim only depends on S_1^1 (for example the standard european call and put options), the claim is perfectly replicable with a self-financing strategy.

Proposition 4.4. *Let $H_1 := H(S_1^1)$ be a contingent claim. To be precise we suppose that $H_1 = H(S_1^1)$ is a $\sigma(S_1^1)$-measurable square integrable random variable. Then, there exist a self-financing replicating strategy. Furthermore, the contingent claim has a unique arbitrage-free price.*

Proof. We will establish the existence of the self-financing replicating strategy. Let $S_0^0 = 1$, S_0^1 and $Y_0 = i$ be known and let $H(S_1^1)$ be a contingent claim. Because S_1^1 is independent of Y_1, for the market model to be complete, we would need to find a pair (ϕ_0^0, ϕ_0^1) such that

$$\begin{aligned} \phi_0^0(1+r) + \phi_0^1 S_0^1 u_i &= H(S_0^1 u_i) \\ \phi_0^0(1+r) + \phi_0^1 S_0^1 d_i &= H(S_0^1 d_i) \end{aligned}$$

The solution of this system is given by

$$\begin{aligned} \phi_0^0 &= \frac{u_i H(S_0^1 d_i) - d_i H(S_0^1 u_i)}{(1+r)(u_i - d_i)} \\ \phi_0^1 &= \frac{H(S_0^1 u_i) - H(S_0^1 d_i)}{S_0^1 u_i - S_0^1 d_i} \end{aligned}$$

By the argument of absence of arbitrage, the price at time 0 of the contingent claim has to be the same as that of the replicating portfolio i.e. the price is given by $V_0(\phi) = \phi_0^0 + \phi_0^1 S_0^1$. □

Remark 4.5. Let us notice that $V_0(\phi)$ can be rewritten as

$$\begin{aligned} V_0(\phi) &= \frac{1}{1+r}(p_0^* H(S_0^1 u_i) + (1-p_0^*)H(S_0^1 d_i)) \\ &= \frac{1}{1+r}\mathbb{E}^*[H(S_1^1)|\mathcal{F}_0] \end{aligned}$$

Remark 4.6. We have shown that there exists a large class of contingent claims (all those of the form $H = H(S_1^1)$) that can be perfectly hedged and priced in a one period model. However, this is not in contradiction with the second fundemental theorem. Indeed, this theorem states that every contingent claim should be replicable by a self-financing strategy. In our model, given our filtration, this means that we should also be able to perfectly hedge claims of the form $H(S_1^1, Y_1)$ and this is not the case.

5 Minimal entropy martingale measures

As shown in theorem 4.3, general claims lead to an incomplete market so there is no way to perfectly hedge these claims with a self-financing strategy. This is linked to the non-uniqueness of equivalent martingale measures. We want to choose a specific martingale measure according to a criterion. Our choice is the martingale measure that minimizes relative entropy, the so-called minimal entropy martingale measure. The aim of this section is to provide a characterization of the minimal entropy martingale measure in our context. For more about these measures consult [11].

Definition 5.1. *Let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures. The relative entropy $I(\mathbb{P}, \mathbb{Q})$ is defined as:*

$$I(\mathbb{P}, \mathbb{Q}) = \begin{cases} \mathbb{E}^{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}} \ln(\frac{d\mathbb{Q}}{d\mathbb{P}})], & \text{if } \mathbb{Q} << \mathbb{P} \\ +\infty & \text{otherwise} \end{cases}$$

Definition 5.2. *A measure $\mathbb{Q}$ is called an minimal entropy martingale measure if it minimizes the relative entropy over the set of all equivalent martingale measures.*

Our aim is to characterize this measure in our context. We will first study this problem in a one-period framework.

5.1 One period model

From the definitions, the aim is to find the p_0^j and q_0^j (for all j) that minimize

$$\sum_{j=1}^{m}(p_0^j \ln\left(\frac{p_0^j}{\pi_0^j}\right) + q_0^j \ln\left(\frac{q_0^j}{\kappa_0^j}\right))$$

but subject to the constraints (that ensure that the measure will be an equivalent martingale measure):

$$\sum_{j=1}^{m}(p_0^j + q_0^j) = 1 \tag{2}$$

$$u_{Y_0}\sum_{j=1}^{m} p_0^j + d_{Y_0}\sum_{j=1}^{m} q_0^j = 1 + r \tag{3}$$

This is a problem of optimization under constraints. We will solve this by the method of lagrange multipliers. An explicit solution can be obtained as is shown in the next theorem.

Theorem 5.3. *In the one-period model, the minimal entropy martingale measure is given by*

$$p_0^j = \frac{\pi_0^j}{\sum_{j=1}^m \pi_0^j} \left(\frac{(1+r)-d_{Y_0}}{u_{Y_0}-d_{Y_0}} \right)$$

$$q_0^j = \frac{\kappa_0^j}{\sum_{j=1}^m \kappa_0^j} \left(\frac{u_{Y_0}-(1+r)}{u_{Y_0}-d_{Y_0}} \right)$$

Proof. The lagrangian L for the problem is (using equation (1)):

$$\begin{aligned} L &= E(D_1 \ln(D_1)) + \lambda(\sum_{j=1}^m (p_0^j + q_0^j) - 1) + \gamma(u_{Y_0} \sum_{j=1}^m p_0^j + d_{Y_0} \sum_{j=1}^m q_0^j - (1+r)) \\ &= \sum_{j=1}^m (p_0^j \ln\left(\frac{p_0^j}{\pi_0^j}\right) + q_0^j \ln\left(\frac{q_0^j}{\kappa_0^j}\right)) + \lambda(\sum_{j=1}^m (p_0^j + q_0^j) - 1) + \gamma(u_{Y_0} \sum_{j=1}^m p_0^j + d_{Y_0} \sum_{j=1}^m q_0^j - (1+r)) \end{aligned}$$

The partial differentials give us (for every j):

$$p_0^j = \pi_0^j \exp(-(1+\lambda+\gamma u_{Y_0})) \tag{4}$$

$$q_0^j = \kappa_0^j \exp(-(1+\lambda+\gamma d_{Y_0})) \tag{5}$$

Using these relations and the partial differentials with respect to λ and γ yields

$$\exp(-\gamma u_{Y_0}) \sum_{j=1}^m \pi_0^j + \exp(-\gamma d_{Y_0}) \sum_{j=1}^m \kappa_0^j = \exp(1+\lambda) \tag{6}$$

$$u_{Y_0} \exp(-\gamma u_{Y_0}) \sum_{j=1}^m \pi_0^j + d_{Y_0} \exp(-\gamma d_{Y_0}) \sum_{j=1}^m \kappa_0^j = (1+r)\exp(1+\lambda)$$

Mixing the last two equations yields

$$\exp(-\gamma d_{Y_0}) = \exp(-\gamma u_{Y_0}) \frac{(u_{Y_0} - (1+r)) \sum_{j=1}^m \pi_0^j}{(1+r-d_{Y_0}) \sum_{j=1}^m \kappa_0^j} \tag{7}$$

From equations 4, 5, 6 and 7, we get the desired result. □

Corollary 5.4. *The minimal entropy martingale measure satisfies*

$$p_0^j = \upsilon_0^j \left(\frac{(1+r)-d_{Y_0}}{u_{Y_0}-d_{Y_0}} \right)$$

$$q_0^j = \upsilon_0^j \left(\frac{u_{Y_0}-(1+r)}{u_{Y_0}-d_{Y_0}} \right)$$

Proof. This follows by definition of π_0^j, κ_0^j and υ_0^j. □

5.2 Multiperiod model

The aim is to specify the minimal entropy martingale measure in a multiperiod model. This boils down to finding the p_t^j and q_t^j in equation 1 such that we minimize the relative entropy and under the constraint that these parameters define an equivalent martingale measure. Let us formalize this in the following problem.

Problem 5.5. Let

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \prod_{s=0}^{T^*-1} \Big(\sum_{j=1}^{m} \Big[\frac{p_s^j}{\pi_s^j} \mathbb{1}_{A_{s+1}^{j,u}} + \frac{q_s^j}{\kappa_s^j} \mathbb{1}_{A_{s+1}^{j,d}} \Big] \Big)$$

The aim is to minimize

$$E^{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}} \ln(\frac{d\mathbb{Q}}{d\mathbb{P}})]$$

over p_s^j and q_s^j under the constraints that for every t

$$\begin{aligned} \sum_{j=1}^{m} (p_t^j + q_t^j) &= 1 \\ u_{Y_t} \sum_{j=1}^{m} p_t^j + d_{Y_t} \sum_{j=1}^{m} q_t^j &= 1 + r \\ q_t^j; p_t^j &> 0 \end{aligned}$$

The results found are very similar to those in the one period model.

Theorem 5.6. *The solution to problem 5.5 is given by (for every t):*

$$p_t^j = \frac{\pi_t^j}{\sum_{j=1}^{m} \pi_t^j} \left(\frac{(1+r) - d_{Y_t}}{u_{Y_t} - d_{Y_t}} \right)$$

$$q_t^j = \frac{\kappa_t^j}{\sum_{j=1}^{m} \kappa_t^j} \left(\frac{u_{Y_t} - (1+r)}{u_{Y_t} - d_{Y_t}} \right)$$

Proof. We will give the proof in the two period case. The proof extends easily to n periods.

Let us rewrite the entropy in the two period setting. Using equation (1), this boils down to

$$I(\mathbb{P}, \mathbb{Q}) = E^{\mathbb{P}}[D_1 \ln(D_1) \Big(\sum_{j=1}^{m} \Big[\frac{p_1^j}{\pi_1^j} \mathbb{1}_{A_2^{j,u}} + \frac{q_1^j}{\kappa_1^j} \mathbb{1}_{A_2^{j,d}} \Big] \Big)$$

$$+ D_1 \Big(\sum_{j=1}^{m} \Big[\frac{p_1^j}{\pi_1^j} \mathbb{1}_{A_2^{j,u}} + \frac{q_1^j}{\kappa_1^j} \mathbb{1}_{A_2^{j,d}} \Big] \Big) \ln \Big(\sum_{j=1}^{m} \Big[\frac{p_1^j}{\pi_1^j} \mathbb{1}_{A_2^{j,u}} + \frac{q_1^j}{\kappa_1^j} \mathbb{1}_{A_2^{j,d}} \Big] \Big)]$$

We now condition on $\mathcal{F}_1$. This yields

$$I(\mathbb{P}, \mathbb{Q}) = E^{\mathbb{P}}[D_1 \ln(D_1) E[\Big(\sum_{j=1}^{m} \Big[\frac{p_1^j}{\pi_1^j} \mathbb{1}_{A_2^{j,u}} + \frac{q_1^j}{\kappa_1^j} \mathbb{1}_{A_2^{j,d}} \Big] \Big) | \mathcal{F}_1]$$

$$+ D_1 E[\Big(\sum_{j=1}^{m} \Big[\frac{p_1^j}{\pi_1^j} \mathbb{1}_{A_2^{j,u}} + \frac{q_1^j}{\kappa_1^j} \mathbb{1}_{A_2^{j,d}} \Big] \Big) \ln \Big(\sum_{j=1}^{m} \Big[\frac{p_1^j}{\pi_1^j} \mathbb{1}_{A_2^{j,u}} + \frac{q_1^j}{\kappa_1^j} \mathbb{1}_{A_2^{j,d}} \Big] \Big) | \mathcal{F}_1]]$$

Using the properties of the D_i's and theorem 5.3 we get

$$\begin{aligned} I(\mathbb{P},\mathbb{Q}) &= E^{\mathbb{P}}[D_1\ln(D_1)+D_1\sum_{j=1}^{m}(p_1^j\ln(\frac{p_1^j}{\pi_1^j})+q_1^j\ln(\frac{q_1^j}{\kappa_1^j}))] \\ &= \sum_{j=1}^{m}(p_0^j\ln(\frac{p_0^j}{\pi_0^j})+q_0^j\ln(\frac{q_0^j}{\kappa_0^j}))+\sum_{j=1}^{m}(p_1^j\ln(\frac{p_1^j}{\pi_1^j})+q_1^j\ln(\frac{q_1^j}{\kappa_1^j})) \\ &= \sum_{i=0}^{1}\sum_{j=1}^{m}(p_i^j\ln(\frac{p_i^j}{\pi_i^j})+q_i^j\ln(\frac{q_i^j}{\kappa_i^j})) \end{aligned}$$

The lagrangian L of our problem can then be written

$$L=\sum_{i=0}^{1}(\sum_{j=1}^{m}(p_i^j\ln\left(\frac{p_i^j}{\pi_i^j}\right)+q_i^j\ln\left(\frac{q_i^j}{\kappa_i^j}\right))+\lambda_i(\sum_{j=1}^{m}(p_i^j+q_i^j)-1)+\gamma_i(u_{Y_i}\sum_{j=1}^{m}p_i^j+d_{Y_i}\sum_{j=1}^{m}q_i^j-(1+r))))$$

From here onwards, the procedure is exactly similar to that in theorem 5.3. Following this yields the desired result. □

This theorem gives us the minimal entropy martingale measure in our context. This measure can be obtained by "putting together" the minimal entropy martingale measures for every period and by keeping track of the evolution of the processes and of the information structure.

Corollary 5.7. *The minimal entropy martingale measure is given by*

$$p_t^j = \upsilon_t^j\left(\frac{(1+r)-d_{Y_t}}{u_{Y_t}-d_{Y_t}}\right)$$

$$q_t^j = \upsilon_t^j\left(\frac{u_{Y_t}-(1+r)}{u_{Y_t}-d_{Y_t}}\right)$$

Proof. Trivial by the last theorem and the definitions of π_t^j and κ_t^j. □

Corollary 5.4 and 5.7 tell us that the minimal martingale measure which is the equivalent martingale measure "closest" to $\mathbb{P}$ is the product of two terms. The first term is related to the state risk or unhedgeable risk. Because of the minimal entropy property, in order to remain closest to the real world measure, this first term remains exactly the same as the real-world measure. However, the measure not only needs to be closest to $\mathbb{P}$ but also an equivalent martingale measure. The second term ensures this. Indeed, it is associated to the market risk or the hedgeable risk, and is similar to the risk neutral measure in the classical binomial model except it may depend on the state. This guarantees that the measure is an equivalent martingale measure but closest to the real world probabilities.

Remark 5.8. Because of the form of the minimal entropy martingale measure, we see that our "beliefs" about the probability of the market going up or down (i.e. the probability of these events under the real-world measure) will not influence the arbitrage price of the derivative (if priced under the minimal entropy measure). However, because of the first term, this is not true for our beliefs about the evolution of the semi-Markov process (i.e. the probabilities related to Y under $\mathbb{P}$). Indeed, under the minimal entropy martingale measure, the semi-Markov process behaves in

exactly the same way as under the physical measure. There is absolutely no change in the probabilities related to process Y. This means that our beliefs about the behavior of Y under P will play a capital role in pricing under the minimal entropy martingale measure.

Remark 5.9. The results of corollary 5.4 and 5.7 are only possible because we assumed the conditional independence between Y and ζ. If we do not assume this, the best we can hope for are the results in theorem 5.3 and 5.6.

6 Conclusions

We have presented and discussed a semi-Markov regime switching Cox-Ross-Rubinstein model for the evolution of the stock and its link to derivative pricing. We have seen that equivalent martingale measures exist in this framework and this implies that the market is arbitrage free. However, the number of equivalent martingale measures is infinite.

As was expected due to the extra source of uncertainty (the semi-Markov process) and the infinite number of martingale measures, the market is incomplete in the general case. However, in the one-period framework when the contingent claim is independent of the semi-Markov process, we can show that although the number of martingale measures is infinite, there exist a self-financing replicating strategy. This shows that there is a class of contingent claims that can be perfectly hedged but not all since general claims of the form $H = H(S_1^1, Y_1)$ can't be perfectly hedged in our model.

Because of this market incompleteness and its link to non-uniqueness of the martingale measure, we need to select a martingale measure in order to be able to price contingent claims. This selection has to be made according to some criteria. We decided to choose the minimal entropy martingale measure i.e. the martingale measure "closest" to the original real-world measure. We managed to give a complete characterization of this measure that seems well suited for potential applications. The results show that this measure is the product of two terms. The first term is linked to the unhedgeable risk associated with the semi-Markov process Y. Actually, we show that under the minimal entropy martingale measure, the evolution of Y remains the same as under measure P. The second term is linked with the hedgeable risk, the market risk, and is very similar to the martingale measure of classical binomial models.

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