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PRUDENCE, TEMPERANCE, EDGINESS, AND HIGHER DEGREE RISK APPORTIONMENT AS DECREASING CORRELATION AVERSION

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PRUDENCE, TEMPERANCE, EDGINESS, AND RISK APPORTIONMENT AS DECREASING SENSITIVITY TO DETRIMENTAL CHANGES

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Abstract

This paper shows that the notions of prudence, temperance, edginess, and, more generally, risk apportionment of any degree are the consequences of the natural idea that the sensitivity to detrimental changes should decrease with initial wealth. In the setting of Epstein & Tanny (1980), this turns out to be equivalent to the supermodularity of the expected utility for some specific 4-state lotteries.

JEL classification: D81

Key words: expected utility, wealth effect, supermodularity, stochastic dominance.

1 Introduction and motivation

In this paper we show that the natural feeling of a sensitivity to detrimental changes decreasing with initial wealth can be used to explain the notions of prudence, temperance, and edginess which are now often used in the analysis of risky choices besides that of risk aversion. Formally, in the expected utility model prudence, temperance, and edginess are defined respectively by a positive third derivative, by a negative fourth derivative, and by a positive fifth derivative of the utility function. Note that these concepts appear at least indirectly in non-expected utility models (Bleichrodt & Eeckhoudt (2005)). These assumptions are traditionally justified by reference to a specific decision problem: the analysis of precautionary savings for prudence[1] (Kimball (1990)), the demand for risky assets in the presence of background risks for temperance (Kimball (1992), Gollier & Pratt (1996)), and the reactivity to multiple risks on precautionary motives for edginess (Lajeri-Chaherli (2004)). This explanation of the sign of the third, the fourth, and the fifth derivatives of the utility function based upon specific decision models is in sharp contrast with the usual interpretation of the negative sign of the second derivative which relies on a very broad type of preference unrelated to a specific choice problem. In this paper, we show that risk apportionment of any degree can be interpreted as a lower sensitivity to detrimental changes when the decision-maker gets richer. This sensitivity is measured by an expected utility premium, that is, by means of the difference between the expected utilities after and before the detrimental change. In other words, the expected utility premium measures the loss in expected utility induced by the detrimental change. Then, using the elementary correlation increasing transformation defined by Epstein & Tanny (1980) we show that prudence, temperance, and edginess are based on the natural idea that aversion to probability spreads in specific 4-state lotteries should decrease as wealth increases. From a mathematical point of view, this amounts to require that the expected utility is supermodular in the initial wealth level and Epstein-Tanny correlation parameter when the decision-maker is faced with these specific lotteries.

[1] The role of prudence has also been recently illustrated in other contexts: self-protection activities (Chiu (2005)), optimal audits (Fagart & Sinclair-Desgagné (2007)).

Starting from a different premise, Eeckhoudt & Schlesinger (2006) and Eeckhoudt et al. (2009) also justify prudence and temperance, as well as general risk apportionments, on the basis of another general preference. In the first paper they state it as a preference for "pain disaggregation" while in the second one they rely upon the tendancy to "combine good with bad". Notice that these two papers include references to previous papers that had partially used similar ideas. In the present paper, we show that prudence and temperance, as well as general risk apportionments, all result from the natural tendency of getting less sensitive to detrimental changes as wealth increases. The same idea is then used with specific 4-state lotteries where these notions follow from the aversion to probability spreads decreasing with initial wealth.

The present work is organized as follows. In Section 2, we first introduce some concepts needed in the paper. Then, we prove that the signs of the successive derivatives of the utility function control the monotonicity of the aversion to detrimental changes. In Section 3, we present the concept of an "elementary correlation increasing transformation" and we recall the seminal result by Epstein & Tanny (1980) relating risk aversion to "(positive) correlation aversion". Measuring the dislike for correlation by the approach based on utility premium developed after Friedman & Savage (1948) (see also Eeckhoudt & Schlesinger (2006)), we show that prudence, like risk aversion, is a consequence of the intuitive idea that a decision-maker should be less sensitive to an increase in correlation when he gets richer. This result is then extended to general risk apportionments in Section 4 by means of specific 4-state lotteries. In that context, we show that decision-makers dislike probability spreads, i.e. transfers of probability mass from the inner cases to the outer cases. Given the importance of the concept of temperance and edginess, their equivalence to probability spread aversion is discussed in details in Section 4. The closing Section 5 briefly concludes the paper.

2 Decreasing sensitivity to detrimental changes

2.1 Notation

Henceforth, we denote as u', u'', and u''' the first derivative, the second derivative, and the third derivative of the utility function u. More generally, we write $u^{(n)}$ for the nth derivative of u, $n = 1, 2, 3, 4, \ldots$; the notations u', u'', and u''' and $u^{(1)}$, $u^{(2)}$, and $u^{(3)}$, respectively, will be used interchangeably. As decision-makers are usually assumed to be non-satiated and risk-averse, u is non-decreasing and concave. If u is differentiable, this means that $u' \geq 0$ and $u'' \leq 0$.

More recently, it has been shown that higher derivatives of u also matter. Therefore, let us consider the non-decreasing utility functions with derivatives of degrees 1 to s of alternating signs. This property is satisfied by the utility functions most commonly used in mathematical economics including all the completely monotone utility functions such as the logarithmic, exponential and power utility functions. Formally, let us define the class $\mathcal{U}_{s-\text{icv}}$, $s = 1, 2, \ldots$, of the regular s-increasing concave functions as the class containing all the utility functions u such that $(-1)^{k+1}u^{(k)} \geq 0$ for $k = 1, \ldots, s$. To get all the s-increasing concave utilities, we need to supplement $\mathcal{U}_{s-\text{icv}}$ with all the pointwise limits of elements in $\mathcal{U}_{s-\text{icv}}$. This gives the class $\overline{\mathcal{U}}_{s-\text{icv}}$ of all the utilities such that $(-1)^{k+1}u^{(k)} \geq 0$ for $k = 1, \ldots, s-2$ and $(-1)^{s-2}u^{(s-2)}$ is non-decreasing and concave.

The class $\overline{\mathcal{U}}_{s-\text{icv}}$ can be characterized by sign properties of divided differences. Recall that the kth divided difference, $k = 1, 2, \ldots$, of the function u at distinct points x_0, x_1, $\ldots$, x_k, denoted by $[x_0, x_1, \ldots, x_k]u$, is defined recursively by

$$[x_0, x_1, \ldots, x_k]u = \frac{[x_1, x_2, \ldots, x_k]u - [x_0, x_1, \ldots, x_{k-1}]u}{x_k - x_0}, \tag{2.1}$$

starting from $[x_i]u = u(x_i)$, $i = 0, 1 \ldots, k$. These divided differences extend derivatives to less regular functions. Then, $u \in \overline{\mathcal{U}}_{s-\text{icv}}$ if, and only if, $(-1)^{k+1}[x_0, x_1, \ldots, x_k]u \geq 0$ for any distinct $x_0, x_1, \ldots, x_k$, $k = 1, 2, \ldots, s$.

The class $\overline{\mathcal{U}}_{s-\text{icv}}$ of the s-increasing concave functions is the largest class of functions u for which the implication $X \preceq_{s-\text{icv}} Y \Rightarrow \mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)]$ holds true for every pair (X, Y)

of ordered random variables. For this reason, $\overline{\mathcal{U}}_{s-\text{icv}}$ is often called the maximal generator of the order $\preceq_{s-\text{icv}}$. This means that $\overline{\mathcal{U}}_{s-\text{icv}}$ corresponds to the largest class of decision-makers whose preferences are in accordance with $\preceq_{s-\text{icv}}$. We refer the reader, e.g., to Denuit, De Vijlder & Lefèvre (1999) for more details about the maximal generator of $\preceq_{s-\text{icv}}$.

Letting s tend to $+\infty$ gives utilities with all odd derivatives positive and all even derivatives negative. In this case, utility functions are completely monotone and express mixed risk aversion, as studied in Caballé & Pomansky (1996).

2.2 Higher degree stochastic dominance relations

The common preferences of all the decision-makers with s-increasing concave utility functions generate the s-increasing concave dominance rule, called the s-increasing concave order. More precisely, given two random variables X and Y, X is said to be smaller than Y in the s-increasing concave order, denoted by $X \preceq_{s-\text{icv}} Y$ when

$$\begin{aligned} & \mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)] \text{ for all } u \text{ in } \overline{\mathcal{U}}_{s-\text{icv}} \\ \Leftrightarrow \quad & \mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)] \text{ for all } u \text{ in } \mathcal{U}_{s-\text{icv}}, \end{aligned}$$

provided the expectations exist. For more details about these orders, we refer the interested readers to Denuit, Lefèvre & Shaked (1998) and Denuit, De Vijlder & Lefèvre (1999).

These orders are closely related to the sth degree increase in risk of Ekern (1980), denoted here as $\preceq_{s-\text{cv}}$. Specifically,

$$\left.\begin{array}{r} X \preceq_{s-\text{icv}} Y \\ \mathbb{E}[X^k] = \mathbb{E}[Y^k] \\ \text{for } k = 1, 2, \ldots, s-1 \end{array}\right\} \Leftrightarrow \mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)] \text{ for all } u \text{ such that } (-1)^{s+1} u^{(s)} \geq 0.$$

If we define as $\mathcal{U}_{s-\text{cv}}$ the class of the regular s-concave utilities, i.e. those with $(-1)^{s+1}u^{(s)} \geq 0$, and as $\overline{\mathcal{U}}_{s-\text{cv}}$ the class of all the s-concave utilities, i.e. those such that $(-1)^{s-2}u^{(s-2)}$ is concave we can then define the s-concave orders $\preceq_{s-\text{cv}}$ as

$$\begin{aligned} X \preceq_{s-\text{cv}} Y \quad \Leftrightarrow \quad & \mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)] \text{ for all } u \text{ in } \overline{\mathcal{U}}_{s-\text{cv}} \\ \Leftrightarrow \quad & \mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)] \text{ for all } u \text{ in } \mathcal{U}_{s-\text{cv}} \\ \Leftrightarrow \quad & X \preceq_{s-\text{icv}} Y \text{ and } \mathbb{E}[X^k] = \mathbb{E}[Y^k] \text{ for } k = 1, 2, \ldots, s-1. \end{aligned}$$

Utility functions in $\mathcal{U}_{s-\text{cv}}$ are those which satisfy risk apportionment of degree s in the terminology of Eeckhoudt & Schlesinger (2006).

2.3 Aversion to detrimental changes and s-increasing utility functions

The following result is at the core of our analysis. It states that a decision-maker with a s-increasing concave utility function becomes less sensitive to detrimental changes as wealth increases. Using this result, we will be able to examine how the preference for risk apportionment is changing with wealth.

We know from the proof of Theorem 3 in Eeckhoudt, Schlesinger & Tsetlin (2009) that given $u \in \mathcal{U}_{(s+t)-\text{icv}}$ and $X \preceq_{s-\text{icv}} Y$ the function g defined by

$$g(w) = \mathbb{E}[u(w+X)] - \mathbb{E}[u(w+Y)] \tag{2.2}$$

belongs to $\mathcal{U}_{t-\text{icv}}$. In the next result, we study the equivalence between the non-decreasingness of g and $u \in \overline{\mathcal{U}}_{(s+1)-\text{icv}}$, thus allowing for utilities with non-differentiable $u^{(s-2)}$.

Proposition 2.1. *Consider $X \preceq_{s-icv} Y$, $u \in \overline{\mathcal{U}}_{s-icv}$, and g defined in* (2.2). *Then, $u \in \overline{\mathcal{U}}_{(s+1)-icv} \Rightarrow g$ non-decreasing. Conversely, if whatever X and Y such that $X \preceq_{s-icv} Y$, g defined in* (2.2) *is non-decreasing then $u \in \overline{\mathcal{U}}_{(s+1)-icv}$.*

Proof. Note that $u \in \overline{\mathcal{U}}_{(s+1)-\text{icv}} \subset \overline{\mathcal{U}}_{s-\text{icv}} \Rightarrow g \leq 0$. Define for $h \geq 0$ the function $\Delta_h u$ as $\Delta_h u(w) = u(w+h) - u(w)$. Then,

$$[x_0, \ldots, x_k](-\Delta_h u) = [x_0, \ldots, x_k]u - [x_0+h, \ldots, x_k+h]u.$$

Recall that if $u \in \overline{\mathcal{U}}_{(2j+1)-\text{cv}}$ then $[x_0, \ldots, x_{2j}]u$ is non-decreasing in $x_0, \ldots, x_{2j}$ whereas if $u \in \overline{\mathcal{U}}_{(2j)-\text{cv}}$ then $[x_0, \ldots, x_{2j-1}]u$ is non-increasing in $x_0, \ldots, x_{2j-1}$. Hence, $[x_0, \ldots, x_k](-\Delta_h u)$ is non-negative if $k = 2j-1$ and non-positive if $k = 2j$ for $k = 1, \ldots, s$, that is, $-\Delta_h u \in \overline{\mathcal{U}}_{s-\text{icv}}$. Hence,

$$\begin{aligned} X \preceq_{s-\text{icv}} Y \quad &\Rightarrow \quad \mathbb{E}[-\Delta_h u(w+X)] \leq \mathbb{E}[-\Delta_h u(w+Y)] \\ &\Leftrightarrow \quad \mathbb{E}[u(w+X)] - \mathbb{E}[u(w+h+X)] \leq \mathbb{E}[u(w+Y)] - \mathbb{E}[u(w+h+Y)] \\ &\Leftrightarrow \quad g(w) \leq g(w+h) \end{aligned}$$

so that g is non-decreasing.

Let us now establish the converse. The non-decreasingness of g ensures that $g(w+h) \geq g(w)$ for any $h \geq 0$ which in turn means that

$$\mathbb{E}[-\Delta_h u(w+X)] \leq \mathbb{E}[-\Delta_h u(w+Y)]$$

holds for any $h \geq 0$ and for any ordered pair (X, Y). This implies $-\Delta_h u \in \overline{\mathcal{U}}_{s-\text{icv}}$ since $\overline{\mathcal{U}}_{s-\text{icv}}$ is the maximal generator of $\preceq_{s-\text{icv}}$ (else, proceeding as in Denuit, De Vijlder & Lefèvre (1999) it would be possible to construct two random variables X and Y such that $X \preceq_{s-\text{icv}} Y$ but $\mathbb{E}[-\Delta_h u(w+X)] > \mathbb{E}[-\Delta_h u(w+Y)]$, contradicting our assumption). Thus, $-\Delta_h u \in \overline{\mathcal{U}}_{s-\text{icv}}$ for any $h \geq 0$. To prove that then $u \in \overline{\mathcal{U}}_{(s+1)-\text{icv}}$, we need to establish that $(-1)^{s-1} u^{(s-1)}$ is non-decreasing and concave, or equivalently that the increments

$$(-1)^{s-1}\Big(u^{(s-1)}(w+h) - u^{(s-1)}(w)\Big)$$

of $(-1)^{s-1} u^{(s-1)}$ are non-negative and non-increasing. This is indeed the case since $-\Delta_h u \in \overline{\mathcal{U}}_{s-\text{icv}}$ implies that

$$w \mapsto (-1)^{s-2}(-\Delta_h u)^{(s-2)}(w) = (-1)^{s-1}\Big(u^{(s-2)}(w+h) - u^{(s-2)}(w)\Big)$$

is non-decreasing and concave. □

We see that the pain $\mathbb{E}[u(w+X)] - \mathbb{E}[u(w+Y)]$ caused by the deterioration of Y into X decreases as the initial wealth w increases. The decision-maker thus becomes less sensitive to detrimental changes of Y into X as he gets richer.

3 Correlation aversion, risk aversion and prudence

3.1 Elementary correlation increasing transformation

Let us recall the concept of an "elementary correlation increasing transformation". This concept links correlation aversion to risk aversion, as was already shown in Epstein & Tanny (1980, Theorem 4). Let I_1 and I_2 be a couple of binary random variables such that

$$\Pr[I_i = 0] = 1 - \Pr[I_i = 1] = p_i, \;\; i = 1, 2.$$

Without loss of generality, we assume that $p_1 \leq p_2$. Let us now consider ρ such that $-p_1p_2 \leq \rho \leq p_1(1-p_2)$ and define the joint distribution of (I_1, I_2) as

$$\begin{aligned}
\Pr[I_1 = 0, I_2 = 0] &= p_1p_2 + \rho \\
\Pr[I_1 = 1, I_2 = 0] &= (1-p_1)p_2 - \rho \\
\Pr[I_1 = 0, I_2 = 1] &= p_1(1-p_2) - \rho \\
\Pr[I_1 = 1, I_2 = 1] &= (1-p_1)(1-p_2) + \rho.
\end{aligned}$$

Compared to the case when I_1 and I_2 are mutually independent, we see that ρ is added to the probability mass at (0,0) and (1,1), whereas the same quantity is subtracted from the probability mass at (0,1) and (1,0). Clearly,

$$\mathbb{C}\text{ov}[I_1, I_2] = \Pr[I_1 = 1, I_2 = 1] - \Pr[I_1 = 1]\Pr[I_2 = 1] = \rho$$

so that ρ can be considered as a correlation parameter.

When ρ increases we face a correlation increasing transformation as defined by EPSTEIN & TANNY (1980)[2] and a correlation averse decision-maker should then dislike an increase in ρ. Let us now prove that correlation aversion implies risk aversion in the expected utility model. Consider a decision-maker with utility function u and initial wealth w facing the risky outcome $a_1I_1 + a_2I_2$ for some non-negative constants a_1 and a_2. Clearly, $a_1I_1 + a_2I_2$ corresponds to the 4-state lottery

$$a_1I_1 + a_2I_2 = \begin{cases} 0 \text{ with probability } p_1p_2 + \rho \\ a_1 \text{ with probability } (1-p_1)p_2 - \rho \\ a_2 \text{ with probability } p_1(1-p_2) - \rho \\ a_1 + a_2 \text{ with probability } (1-p_1)(1-p_2) + \rho. \end{cases}$$

The corresponding expected utility is

$$\begin{aligned}
\mathbb{U}(w, \rho) &= \mathbb{E}[u(w + a_1I_1 + a_2I_2)] \\
&= (p_1p_2 + \rho)u(w) + ((1-p_1)p_2 - \rho)u(w + a_1) \qquad (3.1) \\
&\quad +(p_1(1-p_2) - \rho)u(w + a_2) + ((1-p_1)(1-p_2) + \rho)u(w + a_1 + a_2).
\end{aligned}$$

[2]A similar set-up is used by DOHERTY & SCHLESINGER (1983) but their objective was quite different from ours.

It is easily seen that $\mathbb{U}(w,\rho)$ non-increasing in $\rho \Leftrightarrow u$ is concave. Indeed the partial derivative of $\mathbb{U}(w,\rho)$ with respect to ρ equals

$$\frac{\partial}{\partial \rho}\mathbb{U}(w,\rho) = u(w+a_1+a_2) - u(w+a_1) - (u(w+a_2) - u(w)) \tag{3.2}$$

which is non-positive when u is concave (so that marginal utility is non-increasing). This shows that an increase in the correlation parameter ρ is welfare deteriorating for a risk-averse decision-maker, as pointed out by Epstein & Tanny (1980).

Note that increasing ρ increases the correlation between the random variables a_1I_1 and a_2I_2 faced by the decision-maker. Considering the 4-state lottery $a_1I_1 + a_2I_2$, we also see that increasing ρ transfers some probability mass from the inner outcomes a_1 and a_2 to the outer outcomes 0 and $a_1 + a_2$. Any risk-averse decision-maker dislikes such a probability spread, i.e. an increase in the probability of getting the outer outcomes and a corresponding decrease in the probability of getting the inner ones.

As explained in the introduction, we measure here the strength of dislike for correlation by means of a correlation utility premium defined as

$$CUP(w,\rho) = \mathbb{U}(w,\rho) - \mathbb{U}(w,0).$$

In words, $CUP(w,\rho)$ measures the degree of "pain" associated with facing the correlation ρ, where pain is measured by the loss in expected utility resulting from the correlation ρ between the random variables a_1I_1 and a_2I_2 compared to independence. Considering (3.1), we see that

$$CUP(w,\rho) = \rho\Big(u(w+a_1+a_2) - u(w+a_1) - u(w+a_2) + u(w)\Big)$$

so that $\frac{\partial CUP(w,\rho)}{\partial \rho} \leq 0$ for all w, a_1, $a_2 \Leftrightarrow u$ is concave.

Remark 3.1. As pointed out by Friedman & Savage (1948) for the cost of risk, there are also to ways for measuring the impact of the correlation. The first way refers to a monetary measure, the *correlation premium* $\pi(w,\rho)$ such that $\mathbb{U}(w,\rho) = \mathbb{U}(w - \pi(w,\rho), 0)$. Here, $\pi(w,\rho)$ is the amount of money that the agent is ready to pay to eliminate the correlation level between risks. The second way refers to a non monetary measure, the "correlation utility premium" $CUP(w,\rho)$ defined above. It measures the degree of "pain" e.g. the disutility associated with facing the correlation ρ. Note that

$$\text{sign}\big(-CUP(w,\rho)\big) = \text{sign}\big(\pi(w,\rho)\big).$$

We refer the reader, e.g., to Jindapon & Neilson (2007) for an extensive discussion about these two ways of measuring the cost of a deterioration in the decision-maker's wealth. In this paper, we only consider Friedman-Savage utility premiums.

To propose an interpretation more grounded on observable data for the sensitivity to an increase in correlation, let us consider the willingness to pay to decrease the correlation. Should ρ be transformed into ρ_0 with $\rho_0 < \rho$, expected utility would remain constant if the wealth level w were changed by a compensating variation v such that:

$$\mathbb{U}(w,\rho) = \mathbb{U}(w - v, \rho_0).$$

When ρ_0 is marginally changed around ρ, the willingness to pay is given by a total differentiation of equation (3.1), that is,

$$WTP_\rho = \frac{dw}{d\rho} = -\frac{\frac{\partial \mathbb{U}(w,\rho)}{\partial \rho}}{\frac{\partial \mathbb{U}(w,\rho)}{\partial w}}.$$

Thus, WTP_ρ is defined by the marginal rate of substitution between wealth w and the correlation level ρ. It captures the tradeoff between a change in wealth and a change in correlation level. Notice that the sign of WTP_ρ is the sign of $-\frac{\partial \mathbb{U}(w,\rho)}{\partial \rho}$ that coincides with the sign of $-\frac{\partial CUP(w,\rho)}{\partial \rho}$.

3.2 Prudence

As indicated in the introduction prudence is defined by the non-negativity of the third derivative of the utility function. It is usually justified by reference to the decision of building up precautionary savings in order to better face future income risk. We now show that in fact prudence, like risk aversion, can be justified by the decision-maker's attitude to an increase in the correlation parameter ρ.

In order to stress the intuitive nature of the concept of prudence, let us notice that it is pretty reasonable to assume that a decision-maker becomes less sensitive to an increase in the correlation parameter when he is richer, i.e. an increase in the initial wealth w should moderate the negative impact of an higher value of ρ on welfare.

To analyze the implications of this assumption, let us consider the random variable $I_1 a_1 + I_2 a_2$ where I_1 and I_2 are as described in Section 3.1. For given positive values of a_1 and a_2, an increase in ρ should reduce welfare less when w is large since then it affects a smaller share of the initial wealth. Considering that under correlation aversion the derivative of the expected utility $\mathbb{U}(w, \rho)$ with respect to ρ is negative and that this derivative should approach 0 as w increases, this means that we expect

$$\frac{\partial^2}{\partial w \partial \rho} \mathbb{U}(w, \rho) = \frac{\partial}{\partial w} \left(\frac{\partial}{\partial \rho} \mathbb{E}[u(w + a_1 I_1 + a_2 I_2)] \right) \geq 0, \tag{3.3}$$

that is, the function $(w, \rho) \mapsto \mathbb{U}(w, \rho)$ is supermodular. This derivative equals

$$\frac{\partial^2}{\partial w \partial \rho} \mathbb{U}(w, \rho) = u'(w) + u'(w + a_1 + a_2) - u'(w + a_1) - u'(w + a_2), \tag{3.4}$$

which is non-negative when u' is convex $\Leftrightarrow u \in \overline{\mathcal{U}}_{3-\mathrm{cv}}$. If u is thrice differentiable, u' is convex $\Leftrightarrow u''' \geq 0 \Leftrightarrow u \in \mathcal{U}_{3-\mathrm{cv}}$. Consequently, prudence can also be interpreted as an implication of the lower sensitivity to an increase in ρ due to increased initial wealth w.

4 Risk apportionment of higher degrees

4.1 Decreasing aversion to probability spreads in 4-state lotteries

Considering Section 3, we know that risk aversion means that the decision-maker dislikes an increase in the correlation parameter ρ when final wealth is given by $w + a_1 I_1 + a_2 I_2$ and that prudence means that the decision-maker is less sensitive to an increase in ρ when he gets richer. This section shows that the same idea can be used to characterize temperance, edginess, and higher degree risk apportionment, substituting more general lotteries for $w + a_1 I_1 + a_2 I_2$. Specifically, we show that any risk apportionment can be defined as a lower sensitivity to an increase in the correlation parameter ρ as wealth increases.

Recall that according to Eeckhoudt & Schlesinger (2006), preferences are said to satisfy risk apportionment of degree s if $(-1)^{s+1}u^{(s)} \geq 0 \Leftrightarrow u \in \mathcal{U}_{s-\text{cv}}$. This notion extends prudence, temperance, and edginess to any degree s and can be defined by means of comparison of specific lotteries. Here, we show that risk apportionment can be alternatively characterized by supermodularity of the expected utility viewed as a function of initial wealth w and correlation parameter ρ.

We are now ready to state our main result.

Proposition 4.1. *Assume that the decision-maker is faced with the final wealth*

$$w + (1 - I_1)X_1 + I_1 Y_1 + (1 - I_2)X_2 + I_2 Y_2$$

where (I_1, I_2) is as described in Section 3.1. The non-negative random variables X_1, X_2, Y_1, and Y_2 are assumed to be mutually independent, independent from (I_1, I_2), and such that $X_1 \preceq_{s_1-icv} Y_1$ and $X_2 \preceq_{s_2-icv} Y_2$. Then,

$$u \in \overline{\mathcal{U}}_{(s_1+s_2+1)-icv} \Rightarrow \mathbb{U}(w, \rho) \text{ is supermodular.}$$

Conversely, if $\mathbb{U}(w, \rho)$ is supermodular whatever (I_1, I_2), X_1, X_2, Y_1, and Y_2 fulfilling the requirements listed above then $u \in \overline{\mathcal{U}}_{(s_1+s_2+1)-icv}$.

Proof. The final wealth $w + (1 - I_1)X_1 + I_1 Y_1 + (1 - I_2)X_2 + I_2 Y_2$ can be seen as a lottery with the following four outcomes:

$$w+(1-I_1)X_1+I_1Y_1+(1-I_2)X_2+I_2Y_2 = \begin{cases} w + X_1 + X_2 \text{ with probability } p_1p_2 + \rho, \\ w + X_1 + Y_2 \text{ with probability } p_1(1 - p_2) - \rho, \\ w + Y_1 + X_2 \text{ with probability } (1 - p_1)p_2 - \rho, \\ w + Y_1 + Y_2 \text{ with probability } (1 - p_1)(1 - p_2) + \rho. \end{cases}$$

Let us now consider another random vector $((1-I_1')X_1+I_1'Y_1, (1-I_2')X_2+I_2'Y_2)$ where (I_1', I_2') has the same distribution as (I_1, I_2), except that the correlation parameter ρ is replaced with $\rho' > \rho$, that is,

$$\begin{aligned} \Pr[I_1' = 0, I_2' = 0] &= p_1p_2 + \rho' \\ \Pr[I_1' = 1, I_2' = 0] &= (1 - p_1)p_2 - \rho' \\ \Pr[I_1' = 0, I_2' = 1] &= p_1(1 - p_2) - \rho' \\ \Pr[I_1' = 1, I_2' = 1] &= (1 - p_1)(1 - p_2) + \rho'. \end{aligned}$$

Taking

$$X = w + (1 - I_1')X_1 + I_1'Y_1 + (1 - I_2')X_2 + I_2'Y_2$$

and

$$Y = w + (1 - I_1)X_1 + I_1Y_1 + (1 - I_2)X_2 + I_2Y_2$$

we know from Proposition 2.1 in DENUIT, EECKHOUDT & REY (2009) that $X \preceq_{(s_1+s_2)-\text{icv}} Y$. Invoking our Proposition 2.1 then ends the proof. □

Taking $s_1 = s_2 = 1$, and noting that $X_i = 0 \preceq_{1-\text{icv}} a_i = Y_i$ holds for $i = 1, 2$ (since the a_i's are non-negative), we get the result established in Section 3.2 for prudence.

Let us discuss the meaning of Proposition 4.1. The stochastic order relation $X \preceq_{(s_1+s_2)-\text{icv}} Y$ expresses the preference between a pair of 4-state lotteries offering either $w + X_1 + X_2$, $w + X_1 + Y_2$, $w + Y_1 + X_2$, or $w + Y_1 + Y_2$. Because $X \preceq_{(s_1+s_2)-\text{icv}} Y$, any decision-maker with a utility function $u \in \overline{\mathcal{U}}_{(s_1+s_2)-\text{icv}}$ dislikes a simultaneous increase in the probability of getting the extreme outcomes $w + X_1 + X_2$ (the worst one) and $w + Y_1 + Y_2$ (the best one) and a corresponding decrease in the probability of getting the intermediate outcomes $w+X_1+Y_2$ and $w+Y_1+X_2$. Proposition 4.1 states that the pain caused by such a probability mass shift is decreasing in the initial wealth level w provided $u \in \overline{\mathcal{U}}_{(s_1+s_2+1)-\text{icv}}$. Hence, the extent to which the decision-maker dislikes a spread in the probabilities from the inner cases $w + X_1 + Y_2$ and $w + Y_1 + X_2$ to the outer cases $w + X_1 + X_2$ and $w + Y_1 + Y_2$ is decreasing with wealth w.

Note that even if the correlation parameter ρ controls the amount of dependence between I_1 and I_2, Proposition 4.1 does not really deal with correlation aversion. Among the different terms in the final wealth, some are positively related, such as I_1Y_1 and I_2Y_2 or $(1 - I_1)X_1$ and $(1 - I_2)X_2$, but others are negatively related, like $(1 - I_1)X_1$ and and I_2Y_2, for instance. Of course, $(1 - I_1)X_1$ and I_1Y_1 are mutually exclusive (that is, only one of them can be nonzero), an extreme form of negative dependence studied in DHAENE & DENUIT (1999). We will come back to this issue in the next sections where the special cases of temperance and edginess are discussed in details.

4.2 Temperance

Temperance, defined by $u^{(4)} \leq 0 \Leftrightarrow u \in \mathcal{U}_{4-\mathrm{cv}}$, was introduced by Kimball (1992) in a context of risk management in the presence of background risk. A decision maker is temperant when "an unavoidable (background) risk leads him to reduce exposure to another risk even if the two risks are statistically independent". Note that again the definition is given in the context of a specific decision problem, and not as the expression of a preference.

As it was the case for prudence, temperance also can be interpreted as an implication of the lower sensitivity to an increase in the correlation parameter ρ due to an increase in initial wealth. This is a consequence of Proposition 4.1 taking $Y_i = a_i \geq 0$ for $i = 1, 2$, $X_2 = 0$ and $X_1 \geq 0$ independent of (I_1, I_2) and such that $\mathbb{E}[X_1] \leq a_1$. The final wealth faced by the decision-maker is $w + (1 - I_1)X_1 + I_1a_1 + I_2a_2$. Note that $(1 - I_1)X_1 + I_1a_1$ can be interpreted as a lottery giving X_1 with probability p_1 and a_1 with probability $1 - p_1$. Since $X_1 \preceq_{2-\mathrm{icv}} a_1$ and $0 \preceq_{1-\mathrm{icv}} a_2$ we are in a position to apply Proposition 4.1 with $s_1 = 2$ and $s_2 = 1$. For such X_1, a_1 and a_2, an increase in ρ reduces welfare less for a larger value of w, that is, the second mixed derivative of the expected utility

$$\mathbb{U}(w, \rho) = \mathbb{E}[u(w + (1 - I_1)X_1 + a_1I_1 + a_2I_2)] \tag{4.1}$$

with respect to w and ρ is non-negative if, and only if, the decision-maker is temperant. Like prudence, we see that temperance is the consequence of a lower sensitivity to a change in the correlation parameter ρ when wealth increases.

In Epstein & Tanny (1980) as well as in our Section 3, a correlation increasing transformation is applied to the pair (I_1, I_2). It indeed increases the correlation between the variables I_1 and I_2 of interest. In (4.1), increasing ρ increases correlation between a_1I_1 and a_2I_2. However, increasing ρ decreases the correlation between $(1 - I_1)X_1$ and a_2I_2 since

$$\mathbb{C}\mathrm{ov}[(1 - I_1)X_1, a_2I_2] \;=\; -a_2\mathbb{E}[X_1]\mathbb{C}\mathrm{ov}[I_1, I_2] = -\rho a_2\mathbb{E}[X_1].$$

Therefore, the interpretation given here to temperance does not really refer to correlation aversion, but to a more subtle relationship between the underlying random variables as explained in Section 4.1.

4.3 Edginess

Edginess, defined by $u^{(5)} \geq 0 \Leftrightarrow u \in \mathcal{U}_{5-\text{cv}}$, was introduced by LAJERI-CHAHERLI (2004) in a context of multiple risks in a two-period model. Specifically, edginess captures the reactivity to multiple risks on precautionary motives. It is a necessary condition to have preferences exhibiting *standard prudence* or *precautionary vulnerability* (we refer to LAJERI-CHAHERLI (2004) for more details). Like prudence and temperance, edginess can be interpreted as the consequence of a lower sensitivity to a change in the correlation parameter ρ when wealth increases.

To illustrate this, let us consider I_1 and I_2 as defined in Section 3.1. Let us apply Proposition 4.1 with $Y_i = a_i \geq 0$ for $i = 1, 2$ and two independent non-negative random variables X_1 and X_2 such that $\mathbb{E}[X_i] \leq a_i$ holds for $i = 1, 2$. Then, as $X_i \preceq_{2-\text{icv}} a_i$ is valid for $i = 1, 2$, we are in a position to apply Proposition 4.1 with $s_1 = s_2 = 2$.

For such X_1, X_2, a_1 and a_2, an increase in the correlation parameter ρ reduces welfare less for a larger value of w, that is, the second mixed derivative of the expected utility

$$\mathbb{U}(w, \rho) = \mathbb{E}[u(w + (1 - I_1)X_1 + a_1 I_1 + (1 - I_2)X_2 + a_2 I_2)] \tag{4.2}$$

with respect to w and ρ is non-negative if, and only if, the decision-maker exhibits edginess. Like prudence and temperance, edginess can thus be defined as a lower sensitivity to a change in the correlation parameter as wealth increases.

5 Conclusion

Very often in decision problems, many results depend upon the signs of successive derivatives of the utility function. The present paper has provided new and unified interpretations of these signs. It is first shown that a decision-maker whose non-decreasing utility function has derivatives alternating in signs becomes less sensitive to detrimental changes as he gets richer. This underlies many aspects of a decision-maker's behavior under risk, including risk aversion, prudence, temperance, and edginess. Exactly as risk aversion that has been presented from the very beginning as a form of preference independently of the context in which risk arises, the more recent notions of prudence, temperance, and edginess (and more generally the notion of risk apportionment of any degree) are defined here using the idea of aversion to detrimental changes decreasing in wealth. Thus, these notions appear as natural as that of risk aversion.

This paper then considers a class of 4-state lotteries with a simple dependence structure indexed by a single correlation parameter ρ. Risk apportionment turns out to be equivalent to decreasing aversion to probability spreads, that is, to shifts of the probability mass from the inner to the outer lottery outcomes. This allows us to provide a better understanding of the meaning of the sign of the successive derivatives of a utility function, complementing previous studies. Our contribution may also be adapted to experimental testing.

In order to deal with general correlation increasing transformation in the sense of Epstein & Tanny (1980), we need bivariate stochastic dominance relations, as explained next. Consider a utility function u defined on the real plane and denote as $u^{(i,j)}$ the (i,j)th mixed partial derivative of u with respect to x_1 and x_2, that is, $u^{(i,j)} = \frac{\partial^{i+j}}{\partial x_1^i \partial x_2^j} u$. Then, (X_1, X_2) is said to be smaller than (Y_1, Y_2) in the bivariate (s_1, s_2)-increasing concave order, denoted by $(X_1, X_2) \preceq_{(s_1,s_2)-\text{icv}} (Y_1, Y_2)$, when $\mathbb{E}[u(X_1, X_2)] \leq \mathbb{E}[u(Y_1, Y_2)]$ for all the utility functions u such that $(-1)^{k_1+k_2+1} u^{(k_1,k_2)} \geq 0$ for all $k_1 = 0, \ldots, s_1$, $k_2 = 0, \ldots, s_2$, with $k_1 + k_2 \geq 1$. See Denuit, Eeckhoudt & Rey (2009) and the references therein for more details. For $s_1 = s_2 = 1$ we get the general increasing transformation of Epstein & Tanny (1980). Since the bivariate function $(x_1, x_2) \mapsto u(w + \alpha_1 x_1 + \alpha_2 x_2)$ has derivatives exhibiting the required signs whatever w, α_1 and $\alpha_2 \geq 0$ when $u \in \mathcal{U}_{(s_1+s_2)-\text{icv}}$, we have that

$$(X_1, X_2) \preceq_{(s_1,s_2)-\text{icv}} (Y_1, Y_2) \Rightarrow w + \alpha_1 X_1 + \alpha_2 X_2 \preceq_{(s_1+s_2)-\text{icv}} w + \alpha_1 Y_1 + \alpha_2 Y_2,$$

$$\text{for all } w, \alpha_1 \text{ and } \alpha_2 \geq 0.$$

For $s_1 = s_2 = 1$ we get that $w + \alpha_1 X_1 + \alpha_2 X_2$ precedes $w + \alpha_1 Y_1 + \alpha_2 Y_2$ in second degree stochastic dominance.

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