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Adding independent risks in an insurance portfolio: which shape for the insurer's preferences?

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ADDING INDEPENDENT RISKS IN AN INSURANCE PORTFOLIO: WHICH SHAPE FOR THE INSURERS' PREFERENCES?

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Abstract

Many papers adopted the expected utility paradigm to analyze insurance decisions. Insurance companies manage policies by adding independent risks. however the impact of this on the insurer's expected utility is not completely clear. Indeed, it is not true that risk aversion toward the additional loss generated by a new policy included in an insurance portfolio is a decreasing function of the number of contracts already underwritten (i.e. the "fallacy of large numbers'). This paper shows that most commonly used utility functions do not necessarily positively value the aggregation of independent risks so that they are not eligible for insurers. This casts some doubt about conclusions drawn in the papers postulating such completely monotonic utilities for guiding insurers' choices. Finally, we show that the sufficient conditions for adding risks that can be found in the literature need to be refined by restricting the domain of definition of the insurer's utility function.

Key words and phrases: Expected utility, compensating premium, prudence, temperance.

1 Introduction and motivation

Many explanations for the creation of insurance portfolios are based on a loose application of the law of large numbers and of the central-limit theorem. Most authors use these fundamental results of probability theory to show that the average loss per policyholder becomes more concentrated around the mean as the size of the portfolio increases. However, the insurer is not so much interested in the average loss per policy, but rather in the total payout. As pointed out by Brockett (1983), large deviations theorems are the appropriate tools to study exceedance probabilities for the insurer's total payout as the portfolio size increases.

Smith & Kane (1994) explained that insurance is made possible by the inclusion of safety loadings, i.e. excesses of the premiums paid over the corresponding expected losses. In the case of independent risks, such loadings ensure that the insolvency probability becomes negligible when the size of the portfolio is large enough. Contributions in excess of the insured's expected loss create capacity to absorb deviations from the expected outcomes. This explains why insurance is beneficial.

As insurance policies are purchased to protect policyholders against adverse financial contingencies, insolvency risk plays a special role in the insurance industry. Risk capital is held to assure policyholders that claims can be paid even if larger than expected. Increasing the number n of independent losses $L_1, L_2, \ldots, L_n$ is expected to decrease the probability of loosing money so that the collective of n policies may be found acceptable. This, however, confuses risk with insolvency. Increasing the size n of the portfolio may increase the risk even though it lessens the probability of insolvency and lowers expected loss. Even if insolvency is avoided with a probability approaching 1, a loss is still possible and its disutility may be considerable. An expected utility maximizer must take this disutility into account.

The present paper explores the impact of adding risks in the classical expected utility paradigm. This allows us to examine the shape of an insurer's preferences. Let us assume that the insurer possesses some utility function $u(\cdot)$ and that he acts in order to maximize the expected utility. The prospect X with distribution function F_X is then evaluated by

$$\mathbb{U}[X] = \int_{-\infty}^{+\infty} u(x) dF_X(x) = \mathbb{E}[u(X)]$$

and X is preferred over Y if $\mathbb{U}[X] \geq \mathbb{U}[Y]$. Typically, X represents the net loss, i.e. the pure premium minus the total claim amount in this paper (so that X has zero mean). Formally, $X = \mathbb{E}[L] - L$ where L denotes the loss related to some insurance contract and $\mathbb{E}[L]$ is the corresponding expected loss (often called pure premium in the actuarial literature).

Insurers are often considered to be risk neutral, that is, their utility function is assumed to be linear. This is reasonable as long as all the assumptions underlying the law of large numbers apply. In such a case, the insurer may indeed be seen as a kind of intermediary who collects and disperses funds amongst the policyholders, and who diversifies the insurance risk among a large set of shareholders. See, e.g., Eeckhoudt et al. (2005, Chapter 3) for a discussion.

However, the assumptions underlying the law of large numbers are quite restrictive. Even if the individual insurance risks remain independent (an assumption that will be kept throughout this paper), the number of policies is limited in practice and insurance portfolios

can be quite heterogeneous. Considerable safety loadings have then to be included in the premium, making it unfair and questioning the insurer's risk neutrality.

It appears extremely difficult to elicit an analytical expression for the utility function of a particular insurer. Nevertheless, it is possible to infer which general properties have to hold. Every utility function has to be non-decreasing, indicating that more money is always better. Furthermore, insurers have to exhibit some sort of risk averse behavior. Henceforth, every utility function will be assumed to be non-decreasing (non-satiation) and concave (risk aversion). Denoting as $u^{(k)}$ the kth derivative of u, assumed to exist, this means that we restrict ourselves to utilities such that $u^{(1)} \geq 0$ and $u^{(2)} \leq 0$. This paper studies additional features of the utility function u making insurance possible.

The present paper concentrates on the situation of an insurance company aggregating risks in a portfolio. Section 2 discusses the acceptance property. Insurers exhibiting this kind of behavior positively value the aggregation of independent risks. Diamond (1984) indicates sufficient conditions for the acceptance property to hold. Section 3 shows that these sufficient conditions put some restrictions on the domain of definition of the insurer's utility function. We provide some examples of utility functions satisfying these conditions. The final Section 4 discusses the implications of these results for insurance economics.

2 Adding risks and acceptance property

2.1 Compensating premium

Denote as κ the initial capital owned by the insurance company. Whatever the independent net losses $X_i = \mathbb{E}[L_i] - L_i$, $i = 1, 2, \ldots$, with zero mean,

$$\mathbb{U}\left[\kappa + \sum_{i=1}^{n+1} X_i\right] \leq \mathbb{U}\left[\kappa + \sum_{i=1}^{n} X_i\right]$$

so that without safety loading, no rational decision-maker would agree to increase the number of policies. Moreover, adding a policy which does not contribute any loading deteriorates the situation of the insurance company, even if all the other policies pay a premium exceeding their expected loss. Hence, we see that no insurance company is willing to cover risk in exchange of only the pure premium but requires some extra money.

The insurer is ready to cover a net loss $X = \mathbb{E}[L] - L$ for a price at least equal to the solution $\pi[X]$ of the equation

$$\mathbb{U}\big[\kappa + \pi[X] + X\big] = \mathbb{U}[\kappa] = u(\kappa). \tag{2.1}$$

The premium charged to the policyholder for covering X is then $\mathbb{E}[L]+\pi[X]$. In Eeckhoudt & Gollier (2001), $\pi[X]$ is called the compensating premium. It is the price to be paid to compensate for the bearing of X, when initially the insurer holds the deterministic capital κ. Condition (2.1) expresses that $\pi[X]$ is fair in terms of utility: the right-hand side of (2.1) represents the utility without X; the left-hand side of (2.1) represents the expected utility of the insurer covering the net loss X. Therefore (2.1) means that, provided an amount of $\pi[X]$ is obtained, the expected utility of wealth with X is equal to the utility without X:

(2.1) can be interpreted as an equality between the expected utility of the income $\pi[X]+X$ and the utility of not accepting X. If u is concave then $\pi[X]$ in (2.1) is non-negative.

2.2 Utility functions with the acceptance property

Samuelson (1963) termed adding risks a "fallacy of large numbers" because it is not true for all risk averse utility functions that the risk aversion toward the nth independent risk is a decreasing function of n. Diamond (1984) was the first to provide conditions under which adding risks is beneficial, that is, the conditions for adding independent risk to reduce insurer's risk aversion, which are the conditions when Samuelson (1963)'s "fallacy of large numbers" is not a fallacy.

According to Diamond (1984), adding independent risks provides true diversification if the incremental compensating premium for adding the second risk to the portfolio is lower than for adding the first risk. Consider independent (but not necessarily identically distributed) net losses X and Y, and define $\pi[X]$, $\pi[Y]$, and $\pi[X+Y]$ as the solutions of (2.1) for net losses X, Y, and $X+Y$, respectively. Adding risks reduces the incremental compensating premium if

$$\pi[X+Y] < \pi[X] + \pi[Y]. \tag{2.2}$$

This means that the functional $\pi[\cdot]$ is subadditive. In order to show that this favors adding risks, let us consider the increase in the compensating premium caused by the addition of a new policy X_{n+1} in a portfolio $\sum_{i=1}^{n} X_i$ of size n, that is, $\pi[\sum_{i=1}^{n+1} X_i] - \pi[\sum_{i=1}^{n} X_i]$. If we take $X = \sum_{i=1}^{n} X_i$ and $Y = X_{n+1}$ then (2.2) becomes

$$\pi\left[\sum_{i=1}^{n+1} X_i\right] - \pi\left[\sum_{i=1}^{n} X_i\right] < \pi[X_{n+1}]. \tag{2.3}$$

Hence, it is less expensive to cover X_{n+1} if the policy is included in an existing portfolio of n independent risks $X_1, X_2, \ldots, X_n$, compared to the coverage of X_{n+1} in isolation. This property is called the "acceptance property" of the insurer's utility function.[1]

Let us come back to (2.2) and define the indirect utility function $v(\cdot)$ as

$$v(y) = \mathbb{E}[u(\kappa + X + y)].$$

This is also a utility function as it inherits non-decreasingness and concavity from u. The expected utility of the insurer bearing $X+Y$ is $\mathbb{E}[v(Y)]$. The condition (2.2) holds if u is more risk averse than v, that is,

$$-\frac{v^{(2)}(0)}{v^{(1)}(0)} = -\frac{\mathbb{E}[u^{(2)}(\kappa+X)]}{\mathbb{E}[u^{(1)}(\kappa+X)]} < -\frac{u^{(2)}(\kappa)}{u^{(1)}(\kappa)}. \tag{2.4}$$

[1] When the acceptance property holds, adding a new contract is beneficial whatever the number of policies. A weaker requirement is that this might occur only for sufficiently large portfolios. This leads to the concept of "eventual acceptance", introduced by Ross (1999) and further studied by Hammarlid (2005). The eventual acceptance can be defined as the property for which there exists a finite n (i.e. there exists a sufficiently large portfolio) such that, given the sequence $X_1, X_2, \ldots, X_n$, the collective $X_1 + \ldots + X_n$ will eventually be accepted. In this work we do not consider this alternative concept and we focus on Diamond's condition, where the requirement is for portfolios of every dimension.

Sufficient conditions for u to be more risk averse than v can be obtained by finding sufficient conditions for (2.4) to hold. As pointed out by Diamond (1984), a set of sufficient conditions obtained by Jensen inequality is given by $u^{(3)} \geq 0$ and $u^{(4)} \geq 0$ (with at least one strict inequality).

Note that these sufficient conditions are determined only by imposing restrictions on the utility function and do not depend on the distribution of X. This means that if these conditions are satisfied then the acceptance property holds whatever the distribution of X provided $\pi[X]$ is well-defined.

If $u^{(3)} \leq 0$ and $u^{(4)} \leq 0$ (with at least one strict inequality) then the inequality in (2.4) is reverted and the functional $\pi[\cdot]$ is superadditive. In this case, defining $\pi[X+Y]$, $\pi[X]$, and $\pi[Y]$ as the solution of the indifference equation (2.1), the superadditivity of $\pi[\cdot]$ means that

$$\pi[X+Y] > \pi[X] + \pi[Y]. \tag{2.5}$$

Thus, utility functions of this type are not eligible for describing insurers' preferences. Indeed, inequality (2.3) is reversed in this case. Including a new risk X_{n+1} in a portfolio of size n is, thus, more expensive than covering X_{n+1} in isolation. This clearly prevents the formation of insurance portfolios.

2.3 Inclusion of a safety loading

The previous results are obtained under the assumption $\mathbb{E}[X] = 0$, that is, when the premium is the pure premium. As stated by Diamond (1984), if X is a gamble that agents accept voluntarily this assumption must be replaced by $\mathbb{E}[X] > 0$. In an insurance context, this means that some safety loading has been included in the premium. With the expected value premium principle, this means that X is now equal to $(1+\eta)\mathbb{E}[L] - L$ where $\eta > 0$ is the safety loading coefficient. Thus, $\mathbb{E}[X] = \eta\mathbb{E}[L] > 0$.

In this case a set of sufficient conditions for the acceptance property is $u^{(4)} \geq 0$ and decreasing absolute risk aversion (DARA). Note that DARA (meaning that the index of absolute risk aversion $a(\cdot)$ defined as $a(x) = -\frac{u^{(2)}(x)}{u^{(1)}(x)}$ is decreasing in x) is a stronger condition than $u^{(3)} \geq 0$, since $u^{(3)} \geq 0$ is necessary but not sufficient for it.

3 New findings on the shape of insurers' utility functions

3.1 Commonly used utility functions do not satisfy the acceptance property

The results summarised in Section 2 showed some features of the utility function ensuring that the acceptance property is satisfied. However, it can be also shown that the presence of these features is not so common. Many negative results can even be obtained in this field.

First, Eeckhoudt & Gollier (2001) proved that the compensating premium is superadditive in the number of independent and identically distributed risks if the utility function is proper. Properness was defined by Pratt & Zeckhauser (1987). Properness answers

the following question: if an individual considering two independent undesirable risks is required to take one of them, should he continue to find the other undesirable?

The preceding analysis shows that a utility function inducing superadditive compensating premiums prevents the formation of insurance portfolios because incremental compensating premiums become larger as the size of the portfolio increases. This leads to the following result.

Proposition 3.1. *If the utility function exhibits proper risk aversion then the acceptance property* (2.2) *is not satisfied.*

Proper risk aversion is difficult to characterize and it is difficult to determine whether a particular utility function satisfies this condition. A necessary condition for properness is DARA and a sufficient condition is, in general, DARA plus decreasing absolute prudence (DAP, meaning that the index of absolute prudence $p(\cdot)$ defined as $p(x) = -\frac{u^{(3)}(x)}{u^{(2)}(x)}$ is decreasing in x). In many cases, DAP alone is sufficient for properness. This holds since DAP often implies DARA. Indeed, both Kimball (1993) and Maggi, Magnani & Menegatti (2006) prove that global DAP implies global DARA. Specifically, under the assumption $u^{(1)}(+\infty) = 0^+$, each local minimum (maximum) of the function $x \mapsto a(x) = -\frac{u^{(2)}(x)}{u^{(1)}(x)}$ is followed (or coincides with) at least one local minimum (maximum) of the function $x \mapsto p(x) = -\frac{u^{(3)}(x)}{u^{(2)}(x)}$. This means that global DAP implies global DARA under this assumption. Furthermore, Maggi, Magnani & Menegatti (2006) also prove a similar implication for local analysis in different cases. Assume, for instance, that $a(\cdot)$ has one maximum in x_0 and $p(\cdot)$ has one maximum in x_1 (where $x_1 > x_0$). Now, for $x > x_1$ DAP implies DARA.

Let us now list some commonly used utility functions which violate the acceptance property.

Proposition 3.2. *The acceptance property* (2.2) *is not satisfied for the following utility functions:*

(i) a power utility function;
(ii) a logarithimic utility function;
(iii) a HARA utility function of the form $u(x) = a(b + xc^{-1})^{1-c}$ *with* $c > 0$*;*
(iv) an exponential utility function;
(v) a quadratic utility function.

Proof. Utility functions in cases (i), (ii) and (iii) exhibit both DARA and DAP and are thus proper. By Proposition 3.1 this implies that they do not satisfy (2.2).

For case (iv), given the exponential utility function of the form $u(x) = -a^{-1}\exp(-ax)$ it is easy to see that for this function we have

$$-\frac{\mathbb{E}[u^{(2)}(\kappa + X)]}{\mathbb{E}[u^{(1)}(\kappa + X)]} = -\frac{\mathbb{E}[-a\exp(-a(\kappa + X))]}{\mathbb{E}[\exp(-a(\kappa + X))]} = a$$

and

$$-\frac{u^{(2)}(\kappa)}{u^{(1)}(\kappa)} = -\frac{-a\exp[-a\kappa]}{\exp[-a\kappa]} = a$$

This implies

$$-\frac{\mathbb{E}[u^{(2)}(\kappa+X)]}{\mathbb{E}[u^{(1)}(\kappa+X)]} = -\frac{u^{(2)}(\kappa)}{u^{(1)}(\kappa)}.$$

excluding (2.4) and thus (2.2).

For case (v), given the quadratic utility function $u(x) = ax - bx^2$, we get

$$-\frac{\mathbb{E}[u^{(2)}(\kappa+X)]}{\mathbb{E}[u^{(1)}(\kappa+X)]} = \frac{2b}{a-2b(\kappa+\mathbb{E}[X])} = \frac{2b}{a-2b\kappa} = -\frac{u^{(2)}(\kappa)}{u^{(1)}(\kappa)}$$

which excludes (2.4) and also (2.2). □

3.2 Domain of definition of insurers' utility functions

Another negative general result can be obtained with refence to utility functions defined on an unbounded domain. Considering utility functions defined over the domain $\mathbb{R}^+ = [0, +\infty)$, Menegatti (2001) established that, once the signs of the first and second derivatives are fixed to be respectively positive and negative, the sign of the fourth derivative cannot be negative for all $x \in \mathbb{R}^+$. This means that decision-makers who are non-satiated and risk averse cannot exhibit $u^{(4)} \geq 0$ for all $x \in \mathbb{R}^+$. This result implies the following one.

Proposition 3.3. *No increasing and concave utility function $u(\cdot)$ defined over $\mathbb{R}^+$ can satisfy Diamond's sufficient conditions for the acceptance property for all $x \in \mathbb{R}^+$.*

The result above holds only under the assumption that the utility function u is defined over the domain $[0, +\infty)$. A different conclusion can be obtained if the function is defined over a domain which is bounded above, i.e when $x \in [0, x_1]$. In order to illustrate the case of a bounded domain, we start from the sufficient conditions $u^{(4)} \geq 0$ and $u^{(3)} \geq 0$, obtained when $\mathbb{E}[X] = 0$, and we examine the following example.

Example 3.4. Let us consider the following utility function defined on a bounded domain $[0, x_1]$:

$$u(x) = bx - cx^2 + dx^3 + x^a \tag{3.1}$$

with $x \in [0, x_1]$, $1 < a < 2$ and $b, c, d > 0$. An appropriate choice for contants b, c and d ensures that $u^{(1)} > 0$, $u^{(2)} < 0$ and $u^{(3)} > 0$ while it is easy to see that $u^{(4)} > 0$. This utility function, thus, satisfies Diamond's sufficient conditions.

3.3 The case of a safety loading

A more complicated picture is obtained when we consider the stronger conditions $u^{(4)} \geq 0$ together with DARA, obtained when we assume $\mathbb{E}[X] > 0$. In this case we get the following general result.

Proposition 3.5. *If the increasing and concave utility function u is defined over the domain $[0, x_1]$ and if it exhibits DARA then a necessary condition for $u^{(4)}$ to be positive is that*

$$\lim_{x \to x_1} a(x) < \lim_{x \to x_1} p(x).$$

Proof. The proof of this result is simple. First, note that

$$\frac{da(x)}{dx} = a(x)\Big(a(x) - p(x)\Big).$$

Hence,

$$\lim_{x\to\ x_1} a(x) > \lim_{x\to\ x_1} p(x) \Rightarrow \lim_{x\to\ x_1} \frac{da(x)}{dx} > 0,$$

violating DARA. Second, note that

$$\frac{dp(x)}{dx} = \frac{-u^{(4)}(x)u^{(2)}(x) + (u^{(2)}(x))^2}{(u^{(3)}(x))^2}$$

implying that $u^{(4)}$ positive is a sufficient condition for increasing absolute prudence. Maggi, Magnani & Menegatti (2006) prove that if $\lim_{x\to\ x_1} a(x) = \lim_{x\to\ x_1} p(x)$ then increasing absolute prudence implies increasing absolute risk aversion. This excludes DARA in this second case. □

Example 3.6. Given this general conclusion let us now go back to the utility function in Example 3.4. Indeed it is easy to see that for a sufficiently large b, we have

$$(b - 2cx + 3dx^2 + ax^{a-1})(6d + a(a-1)(a-2)x^{a-3}) > (-2c + 6dx + a(a-1)x^{a-2})^2$$

This means that $u^{(1)}(x)u^{(3)}(x) > (u^{(2)})^2$ for all x in the domain. Since

$$\frac{da(x)}{dx} = \frac{-u^{(3)}(x)u^{(1)}(x) + (u^{(2)}(x))^2}{(u^{(1)}(x))^2},$$

this implies DARA for all x in the domain. This shows that the utility function in Example 3.4 satisfies the sufficient conditions for (2.2) in the case $\mathbb{E}[X] > 0$, too.

4 Discussion

4.1 Implications of previous results for the choice of insurers' utility

Section 2 indicates that insurers' preferences should imply that adding risk reduces the incremental compensating risk premium requested by the insurer. However, the results in Section 3 show that it is not easy to find increasing and concave utility functions satisfying this requirement.

Proposition 3.2 shows that the most commonly used utility functions (power, logarithmic, exponential, quadratic and many HARA functions) violate (2.3), preventing the formation of insurance portfolios, and are thus not eligible to describe the preferences of insurance companies. This negative result has two main implications. First, it casts doubt on the conclusions drawn in the papers postulating these kinds of utility for guiding insurers' choices. Second, it clearly indicates the same problem must be avoided in future analyses.

Quadratic and completely monotonic utility functions have been widely used in the insurance literature. In his study of reinsurance agreements, Borch (1974, Part III) demonstrated the particular role played by exponential, power and logarithmic utilities. Also, quadratic utilities are studied in considerable detail since they allow for more explicit results. See also Borch (1990, Chapter 8). Bowers et al. (1997, Chapter 1) discuss expected utility applied to insurance. The quadratic, exponential, power and logarithmic utilities are used throughout the examples. The same occurs in the review paper by Gerber & Pafumi (1999) or in the textbooks by Rotar (2007, Chapter 3) and by Kaas et al. (2008, Chapter 1), for instance.

In many empirical works, completely monotonic utility functions are routinely applied to derive numerical results. For instance, Hainaut & Devolder (2007) analyze the dividend policy and the asset allocation of a pension fund in a financial market composed of three assets: cash, stocks and a rolling bond. The choice of the utility function has a huge impact on asset liability management. In the CRRA case, dividends, value function and optimal investment policy are a function of the fund equity (the higher is the equity, the higher are dividends and positions in risky assets). For CARA utility functions, the optimal asset allocation is totally independent from liabilities which turns out to be unrealistic. Dividends are always positive for CRRA functions whereas a contribution can be required from shareholders for CARA utilities. Hainaut & Devolder (2007) conclude that an exponential utility function should not be used for ALM purposes. The present paper reinforces this conclusion by showing that completely monotonic utilities do not positively value the aggregation of independent risks.

The picture of Proposition 3.2 is completed by the results in Proposition 3.3 implying that the sufficient conditions for the acceptance property (2.2) cannot be satisfied for the whole domain of the utility function if this domain is unbounded. It is important to note that this conclusion is weaker than the previous one. It does not mean in fact that we necessarily have a superadditive compensating premium (as in the case of the utility functions mentioned in Proposition 3.2). It means, however, that for an unbounded domain we cannot find a utility function which always allows for aggregating independent risks, i.e. which allows aggregating independent risks for every distribution of X. Proposition 3.2 finally does not exclude that a utility function satisfying (2.4) can be obtained for specific distributions of X. However this is just a partial limitation of this problem since, in real world cases, it can be difficult to clearly identify the exact distribution of a risk. Moreover, insurance companies most often cover many different types of risks, with different underlying distribution functions.

A more satisfying framework is obtained when the domain of the utility function is bounded. Indeed, in this case, as shown by Example 3.4, it is possible to find utility functions compatible with the idea of adding risks. With reference to the restriction of the domain of the insurer's utility function from $\mathbb{R}^+$ to $(0, x_1]$ it should however be emphasized that this is not a very strong assumption. As insurance companies cover potential losses, the most favorable case is that no claim originates from the portfolio. The terminal wealth of the insurance company then equals the initial capital plus the annual premium income. This amount provides a natural upper bound on the domain of u.

4.2 Exponential utilities and ruin probabilities

The results in Section 3 show some possible limitations in the use of exponential utilities in describing insurer's preferences. However, this use could be examined from a different standpoint too.

Exponential utilities have often been used in the actuarial literature. These utilities are also closely related to the "fallacy of large numbers", equating the decreasingness in the insolvency probability to less risk. An insurer with $n = 10,000$ policies may be less likely to become insolvent than an insurer with $n = 1,000$ policies but it may also generate a much larger loss. Also, the variance which is a classical measure of risk, grows linearly with the size of the portfolio. It is thus not obvious, as it may first seem to an actuary, that the group of n contracts can be accepted if a single one is rejected. For instance, the probability of insolvency may not decrease fast enough compared to the negative tail of this utility function, for the collective to be accepted.

Samuelson (1963) established that if a utility function u rejects a risk X at all wealth levels, then it will also reject any collective $\sum_{i=1}^{n} X_i$ when the X_i's are independent copies of X. This leads to a real world paradox since the probability of insolvency is often found to be decreasing in n. Ross (1999) pointed out that the application of the result of Samuelson (1963) is nevertheless limited since the only utilities rejecting the same risk at all wealth levels are the linear and the exponential utility functions. In the latter case, (2.1) admits the explicit solution

$$\pi[X] = \frac{1}{a} \ln L_X(a) \tag{4.1}$$

where $L_X(\cdot)$ is the Laplace transform of X, that is, $L_X(t) = \mathbb{E}[\exp(-tX)]$. We thus see that the compensating premium does not depend on the capital owned by the insurance company. In this case, the analysis of Samuelson (1963) applies. Hence, if the premium charged to a policyholder generating a net loss X is smaller than the compensating premium $\pi[X]$ in (4.1) then no collective $\sum_{i=1}^{n} X_i$ made of independent copies of X will be accepted by an insurer with an exponential utility, no matter the number n of policies. Since (4.1) is additive for independent losses, it is easily seen that such an insurer agrees to increase the size of the portfolio as long as the premium charged to each contract is at least equal to (4.1). In this case, there is no net diversification benefit.

Even if the kind of behavior expressed by the exponential utility is very particular, it is worth to mention that (4.1) possesses an appealing interpretation in terms of ruin theory as shown, e.g., in Kaas et al. (2008, Chapter 5). Specifically, let us decompose the annual gain of an insurance company into the premium income p and the total claim amount S_n, that is,

$$W_n = W_{n-1} + p - S_n, \quad n = 1, 2, \ldots \tag{4.2}$$

Here, S_n is the sum of all the claims filed by the policies comprised in the portfolio during year n, and we follow the path $W_1, W_2, W_3, \ldots$ of the insurer's wealth over time, at the end of year 1, 2, 3... starting with an initial capital $W_0 = \kappa$. Ruin occurs if $U_n < 0$ for some n. We assume that the annual total claims S_n, $n = 1, 2, \ldots$, are independent and distributed as S. The following question then arises: how large should the initial capital κ and the premium p be for ruin not to occur with high probability? The probability of ruin is bounded from above by $\exp(-\rho\kappa)$ where ρ denotes the adjustment coefficient, i.e. the root

of the equation $\exp(\rho p) = m_S(\rho)$ where $m_S(\cdot)$ is the moment generating function of S, that is, $m_S(t) = \mathbb{E}[\exp(tS)]$. If we set the upper bound equal to ε, then $\rho = \frac{|\ln \varepsilon|}{\kappa}$. Hence, we get a ruin probability bounded by ε by choosing the premium p as

$$p = \frac{1}{\rho} \ln m_S(\rho), \quad \text{where } \rho = \frac{1}{\kappa} |\ln \varepsilon|. \tag{4.3}$$

Formally, this premium is the exponential premium (4.1) for S, with parameter ρ. However, (4.3) is not derived from expected utility theory but from ruin theory and requires the existence of the adjustment coefficient.

4.3 Conclusion

Insurance companies like to add independent (and identically distributed) risks in their portfolio in order to reduce the probability of insolvency. This attitude reveals implicit properties of their utility function that partially conflict with those widely admitted for the utility function of a risk averse decision-maker. As a result, the utility functions (e.g., exponential, logarithmic, power, etc.) that are so often used to analyze an insurer's risk management decisions may not be appropriate.

In this paper, we have analyzed in details the reasons for the opposition between the "acceptance property" and the standard properties of the utility function. We have also suggested an insurer's utility function that accomodates both approaches. Finally, it is shown that the sufficient conditions for adding risks that can be found in the literature need to be refined by restricting the domain of definition of the insurer's utility function.

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