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**FIRST-ORDER MORTALITY RATES
AND SAFE-SIDE ACTUARIAL CALCULATIONS
IN LIFE INSURANCE**

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FIRST-ORDER MORTALITY RATES AND
SAFE-SIDE ACTUARIAL CALCULATIONS IN LIFE
INSURANCE

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Abstract

In this paper, we discuss how to define conservative biometric bases in life insurance. The first approach is based on cumulative hazard (or survival probabilities), the second one on the hazard itself, and the third one on the hazard ascent. The second case has been studied in the literature and the sum-at-risk plays a central role in defining safe-side requirements. The two other cases appear to be new and concepts related to sum-at-risk are defined.

KEY WORDS: variations in the technical basis; calculating on the safe side; Solvency II; first-order basis; second-order basis; sum at risk.

1 Introduction and motivation

The calculation of premiums and reserves on the safe side has always attracted a lot of interest in life insurance. Life insurance calculations are performed either with first-order technical bases or with second-order technical bases. First-order bases include a safety margin whereas second-order ones do not contain any margin and are assumed to be close to reality.

Practical experience shows that mortality rates can change significantly within one decade. Typically, we are in a situation as exemplified by Figure 1.1. The real mortality rate differs from the estimated one (black solid line) because of, for example, an unforeseen catastrophe (upper dashed line) or a longevity effect (lower dashed line). By applying statistical methods on data of the past, we can usually narrow future uncertainties down to a confidence band (grey area with grey solid curves as bounds). Premiums and reserves should now be chosen in such a way that they are on the safe side with respect to all kinds of mortality scenarios that are within that confidence band.

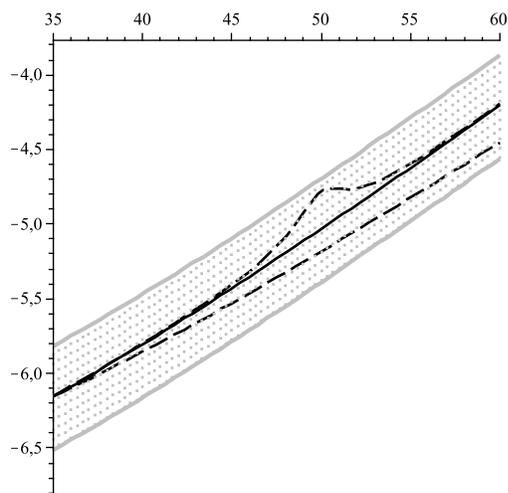


Figure 1.1: Log mortality rates: best estimate (black solid curve), alternative scenarios (black dashed curves), and confidence band (grey area with grey solid curves as bounds)

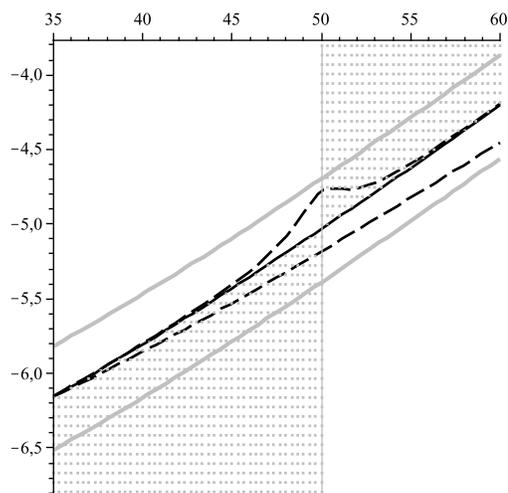


Figure 1.2: Log mortality rates: best estimate (black solid curve) with confidence bounds (grey solid curves), alternative scenarios (black dashed curves), and safe side area with respect to the best estimate (grey area)

So far, the literature offers three concepts for the construction of first-order mortality scenarios. First, there is the method based on the sum-at-risk, which was developed by Lidstone (1905), Norberg (1985), Hoem (1988), Ramlau-Hansen (1988), and Linnemann (1993). The sum-at-risk quantifies the financial consequence of a death occurring at time t , in which case the insurer has to pay the death benefit and the reserve is released. For a given first-order mortality rate with corresponding sum-at-risk, these authors showed that premiums and reserves are on the safe side if the second-order mortality rate is smaller there where the sum-at-risk is positive and if the second-order mortality rate is greater there where the sum-at-risk is negative. This is exemplified in Figure 1.2. Assume that the sum-at-risk

– here calculated on the basis of the best estimate (black solid line) – is positive until the policyholder reaches age 50 and negative afterwards. Think, for example, of a combination of a pure endowment insurance and a temporary life insurance. The sum-at-risk method yields now that premiums and reserves are on the safe side with respect to any second-order mortality rate within the grey area. Unfortunately, we can not say anything about our two alternative scenarios (dashed lines), because they are not completely within the grey area.

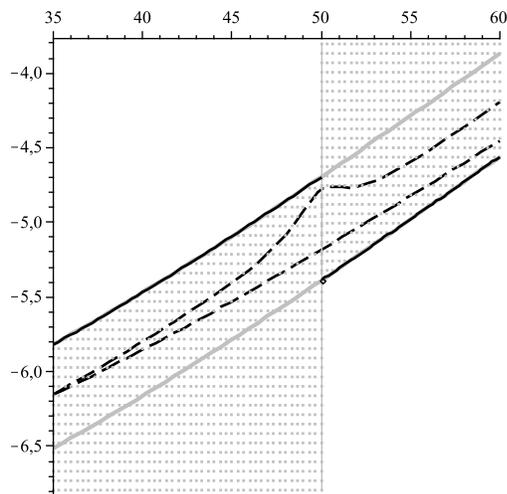


Figure 1.3: Log mortality rates: first-order basis (black solid curve), confidence bounds (grey solid curves), alternative scenarios (black dashed curves), and **desired** safe side area with respect to the first-order basis (grey area)

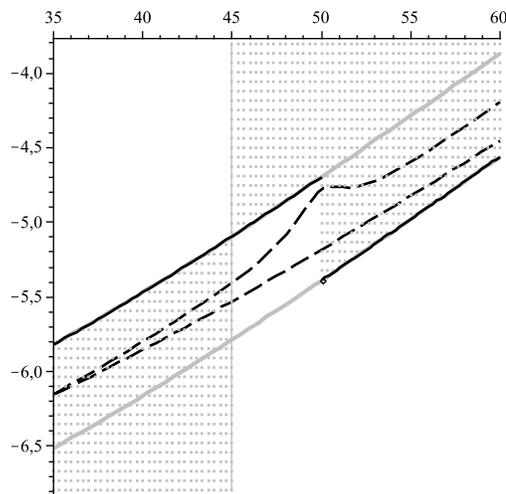


Figure 1.4: Log mortality rates: first-order basis (black solid curve), confidence bounds (grey solid curves), alternative scenarios (black dashed curves), and **true** safe side area with respect to the first-order basis (grey area)

The worst second-order basis with respect to our best estimate first-order basis is shown in Figure 1.3. It is on the upper and lower bound of the confidence band there where the sum-at-risk is positive and negative, respectively. Now we take that worst-scenario as our new first-order basis. Seemingly, we expanded the safe-side area in a way we wished for. The safe side area presumably contains now the whole confidence band, illustrated in Figure 1.3. Unfortunately, Figure 1.3 is in general wrong, because changing the first-order basis changes at the same time the sum-at-risk. The effect is shown in Figure 1.4. The switching point between positive and negative sums-at-risk moved in our example to age 45, and the grey area illustrates the actual safe-side area according to the sum-at-risk method with respect to our new first-order basis. We see that between age 45 and age 50 our confidence band is even completely outside the safe side area, which means that for all scenarios within the confidence band the sum-at-risk method can not decide whether they are on the safe side or not. To put it into a nutshell, the sum-at-risk method does not yield a first-order basis that is definitely on the safe-side with respect to all scenarios within a confidence band.

The second method to be found in the literature is based on derivatives. References using such an approach include Dienst (1995), Bowers et al. (1997), Kalashnikov and Norberg (2003), Christiansen and Helwich (2008), or Christiansen (2008a, 2008b). The problem

is here that differentiation in general is a local concept. Strictly speaking, we can only study infinitesimal changes of the mortality rate. We get good approximations for realistic changes of the mortality rate if the confidence band for the second-order basis is not too wide, but still the approximation error is generally difficult to control. Thus, the method based on derivatives works only for narrow confidence bands and yields not exact but only approximative results.

A third method for the construction of first-order mortality scenarios is given in Christiansen (2009). Based on Thiele's integral equation, another integral equation is developed whose solution yields the maximal prospective reserve with respect to all cumulative mortality intensities whose ascent is within some confidence band. In contrast to the first and the second method, the third method yields a first-order basis that is definitely on the safe-side with respect to a confidence band, and the results are always exact regardless of the width of the confidence band. However, by bounding the ascent of the cumulative mortality intensity and not the cumulative mortality intensity itself, it may happen that we exclude mortality scenarios that can occur in reality. On the other hand, the method of Christiansen (2009) includes scenarios that might be seen as rather unrealistic, for example, scenarios where the mortality intensity is not always rising with increasing age.

In the present paper we describe three approaches for the calculation of a first-order basis, that all yield scenarios that are definitely on the safe side with respect to a confidence band, that all offer exact results regardless of the width of the confidence band, and that mainly differ in the sets of mortality scenarios that are included and excluded. Specifically,

- (a) in approach 1, we allow for any cumulative hazard rate within a lower and an upper bound.
- (b) in approach 2, we allow for cumulative hazard rates whose ascent is within a lower and an upper bound. In case of differentiability, that is equivalent to have a lower and an upper bound for the hazard rate.
- (c) in approach 3, we allow for cumulative hazard rates whose acceleration is within a lower and an upper bound. In case of twice differentiability, that is equivalent to have a lower and an upper bound for the ascent of the hazard rate.

The second approach is based on the method of Christiansen (2009). The first and third approaches seem to be new in the literature. Suppose that confidence bands for (a), (b), and (c) are given. Then approach (a) includes the biggest set of mortality scenarios. In return we obtain premiums and reserves that have a strong safety loading, but the first-order basis is not necessarily a true cumulative hazard rate itself. Approach (b) makes stronger restrictions and includes less mortality scenarios than (a), thus the first-order basis is always a true cumulative hazard rate, and premiums and reserves now have a smaller safety loading. Approach (c) makes the strongest restrictions on the set of admissible mortality scenarios, now hazard rates are never decreasing, and in return we obtain the smallest safety loading for premiums and reserves. It is not obvious which of the restrictions of (a) to (c) on the set of admissible mortality scenarios are really satisfied in reality. Therefore, we present and compare in this paper all three approaches and let it to the practitioner to decide which a-priori assumptions he is willing to accept. The following tabular gives a condensed overview:

	construction of a first-order basis on the safe side with respect to a confidence band	cumulative hazard rates of second-order within the confidence band are ...
method based on sum-at-risk	no	
method based on derivatives	approximately , confidence band for cumulative hazard rate	arbitrary
method (a), section 3	yes , confidence band for cumulative hazard rate	arbitrary
method (b), section 4	yes , confidence band for ascent of cumulative hazard rate	increasing
method (c), section 5	yes , confidence band for acceleration of cumulative hazard rate	increasing with increasing speed

2 Basic modeling

Consider a life insurance policy that is issued at time 0. We write x for the age of the policyholder at the beginning of the contract period, T for his or her total lifetime, and ω_x for the limiting age for individuals with age x at contract time zero.

The cash-flows of the contract are described by the following functions:

1. The lump sum $c(t)$ is payable upon death at time t . We assume that the function c has bounded variation on $[0, \omega_x]$ and is left-continuous (left-continuity ensures that when the death benefit corresponds to the reserve or to the part of a loan still to be reimbursed, the payment at the time of death is not taken into account).
2. The functions $B(t)$ and $\Pi(t)$ give the accumulated annuity benefits and premiums in case of survival up to t . We assume that B and Π have bounded variation on $[0, \omega_x]$ and are right-continuous.

We write $v(s, t)$ for the value at time s of a unit payable at time $t > s$ and assume that it has a representation of the form

$$v(s, t) = e^{-\int_{(s,t]} \varphi(u) du}$$

with φ being the *interest intensity*.

The *cumulative mortality intensity* (or cumulative hazard rate) is defined by

$$\Lambda_x(t) := -\ln P(T > x + t | T > x).$$

We assume that Λ_x is continuous. In order to distinguish between different cohorts, we do *not* further simplify this notation to $\Lambda_x(t) = \Lambda(x + t)$. If Λ_x is differentiable, we can also define a *mortality intensity* (or hazard rate),

$$\lambda_x(t) := \frac{d}{dt} \Lambda_x(t).$$

If Λ_x is even twice differentiable, we define

$$\alpha_x(t) := \frac{d}{dt}\lambda_x(t) = \frac{d^2}{dt^2}\Lambda_x(t)$$

and denote it as *mortality intensity ascent* (or hazard rate ascent).

3 Worst-case if the cumulative hazard rate is bounded

The prospective reserve at time s is obtained as the expected present value of future benefits minus the expected present value of future premiums, that is,

$$\begin{aligned} V(s) &:= \mathbb{E} \left[\int_{(s, T-x)} v(s, t) d(B - \Pi)(t) + v(s, T-x) c(T-x) \Big| T-x > s \right] \\ &= \int_{(s, \omega_x]} e^{\Lambda_x(s) - \Lambda_x(t)} v(s, t) d(B - \Pi)(t) - \int_{(s, \omega_x]} v(s, t) c(t) e^{\Lambda_x(s)} de^{-\Lambda_x(t)}. \end{aligned}$$

Now we regard $V(s)$ as a mapping of the conditional survival function

$$[s, \omega_x] \ni t \mapsto e^{\Lambda_x(s) - \Lambda_x(t)} = P(T > x + t \mid T > x + s).$$

What happens to the prospective reserve if the conditional survival function is shifted by an amount of $Q(\cdot)$ to $\exp\{\Lambda_x(s) - \Lambda_x(\cdot)\} + Q(\cdot)$? In the following we assume that $Q(\cdot)$ is right-continuous, has bounded variation on $[s, \omega_x]$, and is equal to zero at s and ω_x . Using the linearity of $V(s)$ with respect to the conditional survival function and applying Fubini's Theorem, we get in obvious notation

$$\begin{aligned} &V(s, e^{\Lambda_x(s) - \Lambda_x(\cdot)} + Q(\cdot)) - V(s, e^{\Lambda_x(s) - \Lambda_x(\cdot)}) \\ &= \int_{(s, \omega_x]} Q(t) v(s, t) d(B - \Pi)(t) - \int_{(s, \omega_x]} v(s, t) c(t) dQ(t) \\ &= \int \mathbf{1}_{(s, \omega_x]}(u) \left(\int_{[u, \omega_x]} v(s, t) d(B - \Pi)(t) - v(s, u) c(u) \right) dQ(u) \\ &=: \int_{(s, \omega_x]} S_s(u) dQ(u) \end{aligned} \tag{3.1}$$

where

$$S_s(u) := v(s, u) \left(\int_{[u, \omega_x]} v(u, t) d(B - \Pi)(t) - c(u) \right) \tag{3.2}$$

can be seen as *cumulative survival cost at time s for survival at and after u* . To motivate that definition, look at the example where $Q = \varepsilon \mathbf{1}_{[t_0, \omega_x]}$ for some fixed $t_0 > s$ and an $\varepsilon > 0$. For a homogeneous portfolio that means that we have from time t_0 on throughout $(100\varepsilon)\%$ more policyholders that are still alive. According to (3.1), the effect of shift $Q = \varepsilon \mathbf{1}_{[t_0, \omega_x]}$ on the prospective reserve $V(s)$ is $\varepsilon S_s(t_0)$. Coming back to the homogeneous portfolio, $\varepsilon S_s(t_0)$ is the increase of the discounted cost per policy due to increasing the survival rate on $[t_0, \omega_x]$

by ε . We can get another interesting interpretation of function S_s after applying partial integration on the last term of (3.1), which gives

$$\begin{aligned} V(s, e^{\Lambda_x(s)-\Lambda_x(\cdot)} + Q(\cdot)) - V(s, e^{\Lambda_x(s)-\Lambda_x(\cdot)}) &= \int_{(s, \omega_x]} S_s(u) dQ(u) \\ &= \int_{(s, \omega_x)} -Q(u) dS_s(u). \end{aligned} \quad (3.3)$$

(Note that Q is right-continuous, S_s is left-continuous, and that we assumed that $Q(s) = Q(\omega_x) = 0$.) Now we see that $-dS_s(u)$ describes the effect that the increase $Q(u)$ of the survival function at time u has on the prospective reserve $V(s)$. Therefore, we denote $-dS_s(u)$ as *survival cost at time s for survival at time u* , and by differentiating (3.2) we can show that

$$-dS_s(u) = v(s, u)(dB(u) - d\Pi(u) - \varphi(u)c(u)du + dc(u)) =: v(s, u)dS(u) \quad (3.4)$$

for all $u \geq s$, where $dS(u)$ is denoted as *survival cost for survival at time u* . This representation allows for an intuitive interpretation: By infinitesimally delaying the death of the policyholder at time u , additional benefits of $dB(u)$ fall due, additional premiums of $d\Pi(u)$ are paid, the insurer gets a discounting advantage for the death benefit of $\varphi(u)c(u)du$, and the contractual liabilities concerning death change by $dc(u)$.

Now we assume that the conditional survival function $\exp\{\Lambda_x(s) - \Lambda_x(\cdot)\}$ has a lower and an upper bound,

$$e^{U_x(s)-U_x(t)} \leq e^{\Lambda_x(s)-\Lambda_x(t)} \leq e^{L_x(s)-L_x(t)}, \quad t \in [s, \omega_x], \quad (3.5)$$

where the bounds shall be continuous survival functions with respect to t . Instead of studying shifts of the survival function of the form $\exp\{\Lambda_x(s) - \Lambda_x(\cdot)\} + Q(\cdot)$ within the boundaries (3.5), we will study shifts of the form $\Lambda_x + H$ and use the equivalent bounds

$$L_x(t) - L_x(s) \leq \Lambda_x(t) - \Lambda_x(s) \leq U_x(t) - U_x(s), \quad t \in [s, \omega_x], \quad (3.6)$$

where the bounds have to be continuous cumulative hazard rates with limiting age ω_x . From now on we see the prospective reserve $V(s) = V(s, \Lambda_x)$ as a mapping of the cumulative hazard rate Λ_x .

3.1 Construction of a worst-case scenario

We are interested in the maximal value that the prospective reserve $V(s)$ can take if the conditional survival function may be chosen arbitrarily within the bounds (3.5) or, equivalently, if the cumulative mortality intensity may be chosen arbitrarily within the bounds (3.6). In other words, we are looking for the worst-case prospective reserve (or at least an upper bound for it) from the perspective of the insurer. Let Y and Z be random variables with survival functions $e^{\Lambda_x(s)-\Lambda_x(\cdot)}$ and $e^{\Lambda_x(s)+H(s)-\Lambda_x(\cdot)-H(\cdot)}$. If S_s is non-increasing or, equivalently, $dS(t)$ is never negative, then the fact that

$$V(s, \Lambda_x) = \int_{(s, \omega_x]} S_s(t) de^{\Lambda_x(s)-\Lambda_x(\cdot)}(t) = \mathbb{E}(-S_s(Y)),$$

leads to

$$P(Y > t) \geq P(Z > t) \text{ for all } t \Rightarrow V(s, \Lambda_x) = \mathbb{E}(-S_s(Y)) \geq \mathbb{E}(-S_s(Z)) = V(s, \Lambda_x + H)$$

or equivalently

$$H(s) - H(t) \leq 0 \text{ for all } t \in [s, \omega_x] \Rightarrow V(s, \Lambda_x) = \mathbb{E}(-S_s(Y)) \geq \mathbb{E}(-S_s(Z)) = V(s, \Lambda_x + H).$$

The same relation holds for the prospective reserves if S_s is non-decreasing and $H(s) - H(t) \geq 0$ for all $t \in [s, \omega_x]$. Thus, we get that $\Lambda_x = L_x$ maximizes the prospective reserve if S_s is non-increasing and $\Lambda_x = U_x$ maximizes the prospective reserve if S_s is non-decreasing. In other words, the lower bound L_x and the upper bound U_x are worst-case scenarios if the survival cost dS is throughout non-negative and throughout non-positive, respectively. This result can be generalized to cases where dS may change its sign, as shown next.

Property 3.1. *Let dS be the survival cost according to (3.4). Then, for all continuous functions H with bounded variation on $[s, \omega_x]$, we have*

$$\text{sign}(H(s) - H(t)) = \text{sign}(dS(t)) \text{ for all } t > s \quad \Longrightarrow \quad V(s, \Lambda_x + H) \geq V(s, \Lambda_x) \quad (3.7)$$

and

$$\text{sign}(H(s) - H(t)) = -\text{sign}(dS(t)) \text{ for all } t > s \quad \Longrightarrow \quad V(s, \Lambda_x + H) \leq V(s, \Lambda_x). \quad (3.8)$$

Proof. Because of (3.3), the difference

$$V(s, \Lambda_x + H) - V(s, \Lambda_x) = \int_{(s, \omega_x)} e^{\Lambda_x(s) - \Lambda_x(t)} (e^{H(s) - H(t)} - 1) v(s, t) dS(t)$$

is always non-negative and non-positive under conditions (3.7) and (3.8), respectively. \square

Property 3.1 allows us to calculate an upper bound for the prospective reserve:

Proposition 3.2. *Let dS be the survival cost according to (3.4). Then $\bar{\Lambda}_x$ defined by*

$$\bar{\Lambda}_x(t) - \bar{\Lambda}_x(s) := \begin{cases} L_x(t) - L_x(s) & : dS(t) > 0 \\ U_x(t) - U_x(s) & : dS(t) < 0 \\ \frac{1}{2}L_x(t) - \frac{1}{2}L_x(s) + \frac{1}{2}U_x(t) - \frac{1}{2}U_x(s) & : dS(t) = 0 \end{cases} \quad (3.9)$$

with arbitrary but fixed initial value $\bar{\Lambda}_x(s)$ satisfies $V(s, \bar{\Lambda}_x) \geq V(s, \Lambda_x)$ for all cumulative mortality intensities Λ_x that are within the bounds (3.6).

Proof. Apply Property 3.1, and note that in the proof of Property 3.1 the function $e^{\Lambda_x(s) - \Lambda_x(t)} + Q$ is not necessarily a survival function but only has to be right-continuous and of bounded variation on $[s, \omega_x]$. \square

We denote $\bar{\Lambda}_x$ as *worst-case scenario with respect to the bounds (3.6)*. Note that $\bar{\Lambda}_x(\cdot)$ is not necessarily monotone and, hence, not always a true survival function. The worst-case scenario can in fact be arbitrarily defined on $\{t | dS(t) = 0\}$ without losing the maximality property.

Remark 3.3 (Time invariance). What happens to the worst-case scenario of $V(s)$ if time s is moving forward? The worst-case scenario according to Proposition 3.2 depends only on the sign of dS which does not depend on s . That means that if we once calculated $\bar{\Lambda}_x$ at the beginning of the contract period $s = 0$, it remains to be a worst-case scenario during the whole contract time.

However, the approach presented in this section has a significant disadvantage. In many examples the worst-case scenario $\bar{\Lambda}_x$ is not monotone and, hence, not a true cumulative hazard rate anymore. This implies that the upper bound for the prospective reserve is in fact not sharp. This is why in Section 4 we bound the increase of Λ_x instead of Λ_x itself.

4 Worst-case if the ascent of the cumulative hazard rate is bounded

In contrast to (3.6), we assume now that the ascent of the cumulative hazard rate is bounded,

$$dL_x(t) \leq d\Lambda_x(t) \leq dU_x(t), \quad t \in [s, \omega_x], \quad (4.1)$$

where L_x and U_x are continuous and increasing functions with bounded variation on $[s, \omega_x]$. In case of differentiability, that is equivalent to

$$l_x(t) \leq \lambda_x(t) \leq u_x(t), \quad t \in [s, \omega_x], \quad (4.2)$$

where l_x and u_x are the derivatives of L_x and U_x . The monotony of L_x implies that Λ_x is monotone and, hence, is always a true cumulative hazard rate.

The prospective reserve at time s can be written as

$$V(s) = \int_{(s, \omega_x]} e^{\Lambda_x(s) - \Lambda_x(t)} v(s, t) d(B - \Pi)(t) + \int_{(s, \omega_x]} v(s, t) c(t) e^{\Lambda_x(s) - \Lambda_x(t)} d\Lambda_x(t). \quad (4.3)$$

Alternatively, we can see the prospective reserve as the unique solution of Thiele's integral equation

$$V(s) = (B - \Pi)(\omega_x) - (B - \Pi)(s) - \int_{(s, \omega_x]} V(t-) \varphi(t) dt + \int_{(s, \omega_x]} R(t) d\Lambda_x(t) \quad (4.4)$$

with initial value $V(\omega_x) = 0$, where $R(s) := c(s) - V(s) - \Delta(B - \Pi)(s)$ is the so-called *sum-at-risk for occurrence of dead at time s* .

What happens to the prospective reserve if Λ_x is shifted by an amount of H to $\Lambda_x + H$? By generalizing the ideas of Lidstone (1905), Norberg (1985), Hoem (1988), Ramlau-Hansen (1988), and Linnemann (1993) to a model with a *cumulative* mortality intensity, we obtain the following result, which is the basis for the 'sum-at-risk method'.

Property 4.1. *Let $R(s, \Lambda_x)$ be the sum-at-risk that corresponds to Λ_x . If the shifted cumulative mortality intensity $\Lambda_x + H$ is still a continuous cumulative hazard rate, then we have*

$$\text{sign}(dH(t)) = \text{sign}(R(t, \Lambda_x)) \text{ for all } t > s \implies V(s, \Lambda_x + H) \geq V(s, \Lambda_x) \quad (4.5)$$

and

$$\text{sign}(dH(t)) = -\text{sign}(R(t, \Lambda_x)) \text{ for all } t > s \implies V(s, \Lambda_x + H) \leq V(s, \Lambda_x). \quad (4.6)$$

Proof. Let $W(s) := V(s, \Lambda_x + H) - V(s, \Lambda_x)$ be the difference between the prospective reserves at time s . By replacing $V(s, \Lambda_x + H)$ and $V(s, \Lambda_x)$ with the right hand side of (4.4), we get an integral equation for W ,

$$\begin{aligned} W(s) = & - \int_{(s, \omega_x]} \left(V(t-, \Lambda_x + H) - V(t-, \Lambda_x) \right) \varphi(t) dt \\ & + \int_{(s, \omega_x]} \left(R(t, \Lambda_x + H) d(\Lambda_x + H)(t) - R(t, \Lambda_x) d\Lambda_x(t) \right) \end{aligned}$$

with initial value $W(\omega_x) = 0$. With defining C by

$$C(\omega_x) - C(s) = \int_{(s, \omega_x]} R(t, \Lambda_x) dH(t),$$

we can write the above integral equation for W in the form

$$W(s) = C(\omega_x) - C(s) - \int_{(s, \omega_x]} W(t-) \varphi(t) dt + \int_{(s, \omega_x]} -W(t) d(\Lambda_x + H)(t).$$

We can interpret this integral equation as a Thiele integral equation for a policy with no death benefits and accumulated annuity benefits and premiums of C . (In the actuarial literature, $C(s)$ is interpreted as the accumulated surplus at time s .) Hence, the integral equation has the solution (see (4.3))

$$\begin{aligned} W(s) &= \int_{(s, \omega_x]} e^{\Lambda_x(s) + H(s) - \Lambda_x(t) - H(t)} v(s, t) dC(t) \\ &= \int_{(s, \omega_x]} e^{\Lambda_x(s) + H(s) - \Lambda_x(t) - H(t)} v(s, t) R(t, \Lambda_x) dH(t). \end{aligned} \quad (4.7)$$

Under the conditions of (4.5) and (4.6) we obtain $W(s) \geq 0$ and $W(s) \leq 0$ and, hence, $V(s, \Lambda_x + H) - V(s, \Lambda_x) = W(s) \geq 0$ and $V(s, \Lambda_x + H) - V(s, \Lambda_x) = W(s) \leq 0$. \square

Defining

$$R_s(t) := e^{\Lambda_x(s) + H(s) - \Lambda_x(t) - H(t)} v(s, t) R(t, \Lambda_x) \quad (4.8)$$

as the *sum-at-risk at time s for occurrence of death at time t* , we get from (4.7) an expression similar to (3.3):

$$V(s, \Lambda_x + H) - V(s, \Lambda_x) = \int_{(s, \omega_x]} R_s(t) dH(t). \quad (4.9)$$

While the survival cost $-dS_s(t)$ describes the effect that a $Q(t)$ shift of the survival function has on $V(s)$, it is here $R_s(t)$ that quantifies the effect that a $dH(t)$ shift of the ascent of the

cumulative mortality intensity has on $V(s)$. Property 4.1 is similar to Property 3.1. While $\text{sign}(dS(t))$ describes the direction of the effect that a $H(t)$ shift of the cumulative mortality intensity at time t has on $V(s)$, it is here $\text{sign}(R(t))$ that quantifies the direction of the effect that a $dH(t)$ shift of the ascent of the cumulative mortality intensity at time t has on $V(s)$. It is then tempting to believe that we can find a worst-case scenario analogously to Proposition 3.2 by letting $d\Lambda_x$ be equal to dL_x and dU_x there where $R(t, \Lambda_x)$ is negative and positive, respectively. As already indicated in the introduction of the present paper, this idea does not work. The problem is here that quantity (4.8) depends on shift H , whereas $-dS_s$ did not depend on shift Q . Therefore the worst-case problem is more complicated here.

4.1 Construction of a worst-case scenario

Property 4.1 guarantees that the valuation basis Λ_x is on the safe side with respect to all alternative mortality scenarios $\Lambda_x + H$ that meet condition (4.6). This safe side area usually does not contain the whole confidence band (4.1). (See also the explanations in the introduction of the present paper.) But if we had a mortality scenario $\bar{\Lambda}_x$ that satisfies

$$d\bar{\Lambda}_x(t) = \begin{cases} dL_x(t) & : R(t, \bar{\Lambda}_x) < 0 \\ dU_x(t) & : R(t, \bar{\Lambda}_x) > 0 \end{cases}, \quad (4.10)$$

then Property 4.1 would yield a safe side area for $\bar{\Lambda}_x$ that indeed contains the whole confidence band (4.1), because all possible shifts H meet (4.6). The natural questions are therefore:

- Does such a special scenario $\bar{\Lambda}_x$ always exist?
- If so, how do we find $\bar{\Lambda}_x$?

Answers to that questions can be found in Christiansen (2009). By replacing the cumulative mortality intensity in Thiele's integral equation (4.4) with the right hand side of (4.10), we get a new integral equation that does not directly depend on $\bar{\Lambda}_x$ anymore,

$$\begin{aligned} \bar{V}(s) = & (B - \Pi)(\omega_x) - (B - \Pi)(s) - \int_{(s, \omega_x]} \bar{V}(t-) \varphi(t) dt \\ & + \int_{(s, \omega_x]} \frac{\bar{R}(t) - |\bar{R}(t)|}{2} dL_x(t) + \int_{(s, \omega_x]} \frac{\bar{R}(t) + |\bar{R}(t)|}{2} dU_x(t) \end{aligned} \quad (4.11)$$

with initial value $\bar{V}(\omega_x) = 0$, where we use the short notation $\bar{V}(s) := V(s, \bar{\lambda}_x)$ and $\bar{R}(s) := R(s, \bar{\lambda}_x)$. Christiansen (2009) showed that the integral equation (4.11) has a unique solution in

$$\left\{ V : [0, \omega_x] \rightarrow \mathbb{R} \mid V \text{ is right-continuous and has bounded variation, } V(\omega_x) = 0 \right\}.$$

If we once have a solution \bar{V} for (4.11), then we can construct a worst-case mortality scenario as follows.

Property 4.2. Let \bar{V} be the unique solution of integral equation (4.11) with corresponding sum-at-risk \bar{R} . Then $\bar{\Lambda}_x$ defined by

$$d\bar{\Lambda}_x(t) = \begin{cases} dL_x(t) & : \bar{R}(t) < 0 \\ dU_x(t) & : \bar{R}(t) > 0 \\ d(\frac{1}{2}L_x + \frac{1}{2}U_x)(t) & : \bar{R}(t) = 0 \end{cases} \quad (4.12)$$

and an arbitrary but fixed initial value $\bar{\Lambda}_x(0)$ is a cumulative mortality intensity with $V(s, \bar{\Lambda}_x) \geq V(s, \Lambda_x)$ for all $s \in [0, \omega_x]$ and all Λ_x that satisfy (4.1).

Proof. Christiansen (2009) showed that $\bar{\Lambda}_x$ is indeed a cumulative mortality intensity. In the same way that we derived (4.11) from (4.10), we can verify that $V(\cdot, \bar{\Lambda}_x)$ is equal to the unique solution \bar{V} of (4.11). Thus, we also have $\bar{R} = R(\cdot, \bar{\Lambda}_x)$, which means that (4.12) satisfies (4.10). By applying Property 4.1 now for each $s \in [0, \omega_x]$, we get the maximality of $V(s, \bar{\Lambda}_x)$ for all $s \in [0, \omega_x]$. \square

We denote $\bar{\Lambda}_x$ according to (4.12) as *worst-case scenario with respect to* (4.1). Note that $d\bar{\Lambda}_x(t)$ can in fact be arbitrarily defined on $\{t : \bar{R}(t) = 0\}$ without losing the maximality property.

Remark 4.3 (Time invariance & characterization of the worst-case). Note that the worst-case scenario $\bar{\Lambda}_x$ maximizes not only the prospective reserve at some fixed time s , but also at any other time $t \in [0, \omega_x]$. That means that if we once calculated $\bar{\Lambda}_x$ at the beginning of the contract period, it remains to be a worst-case scenario during the whole contract time. This implies that $R(t, \bar{\Lambda}_x) = c(t) - V(t, \bar{\Lambda}_x) - \Delta(B - \Pi)(t)$ is minimal for all t , and consequently

$$\text{sign}(R(t, \bar{\Lambda}_x)) = \inf_{\Lambda_x} \text{sign}(R(t, \Lambda_x))$$

for all t . By interpreting a positive sum-at-risk as occurrence character and a negative sum-at-risk as survival character, we get that the worst-case scenario is always that scenario that has the biggest share of survival character during the contract period.

The worst-case method in this section fixes the problem of the previous section that the worst-case scenario $\bar{\Lambda}_x$ is in general no cumulative hazard rate anymore. However, we still get unrealistic scenarios where the mortality intensity jumps between extremes and where mortality rates can also fall with increasing age. Such scenarios make sense in risk management if one is interested not in usual but in extreme developments of mortality.

Still, we can ask the question if it is possible to calculate worst-case scenarios which additionally have the following two properties: (a) they never fall with increasing age and (b) they have no extreme jumps. An answer to that question is given in Section 5.

4.2 Alternative construction of a worst-case scenario

Earlier in this section we discussed that, in contrast to (3.3), formula (4.9) does not yield a construction method for a worst-case scenario because the integrand depends on shift H . The 'method based on derivatives' (see the introduction of this paper) gets rid of that dependence on H by just allowing for local shifts H . Christiansen (2008a) shows that

$$\left| V(s, \Lambda_x + H) - V(s, \Lambda_x) - \int_{(s, \omega_x]} e^{\Lambda_x(s) - \Lambda_x(t)} v(s, t) R(t, \Lambda_x) dH(t) \right| = o(\|H\|), \quad (4.13)$$

where $\|H\|$ is the total variation of H on $[0, \omega_x]$. Thus, given that $o(\|H\|)$ is negligible, the prospective reserve $V(s, \Lambda_x + H)$ can be maximized by choosing $d(\Lambda_x + H)$ equal to dL_x and dU_x there where $R(t, \Lambda_x)$ is negative and positive, respectively. This is the same scenario as the one suggested by the sum-at-risk method. The difference to the true worst-case scenario rises with $o(\|H\|)$. Christiansen (2008a) interpreted the integrand of (4.13) as some of form of generalized gradient

$$(\nabla_{\Lambda_x} V)(t) := e^{\Lambda_x(s) - \Lambda_x(t)} v(s, t) R(t, \Lambda_x)$$

which gives us a new idea for the construction of a worst-case scenario. In the same way that gradient ascent methods are used to find local maxima of differentiable functions on \mathbb{R}^n , we could do iterated small steps in direction of the generalized gradient $\nabla_{\Lambda_x} V$ in order to find a maximizing mortality scenario:

1. Choose a starting mortality scenario $\Lambda_x^{(0)}$.
2. Calculate a new scenario by using the iteration

$$d\Lambda_x^{(n+1)}(t) := d\Lambda_x^{(n)}(t) + K (\nabla_{\Lambda_x^{(n)}} V)(t) dt.$$

If the right hand side is below dL_x or above dU_x , we cut $d\Lambda_x^{(n+1)}$ off at dL_x or dU_x , respectively. Here, $K > 0$ is some step size that has to be chosen.

3. Repeat step 2 until $|V(s, \Lambda_x^{(n+1)}) - V(s, \Lambda_x^{(n)})|$ is below some error tolerance.

In order to increase the speed of convergence, we could try to increase K to infinity. As the sign of $(\nabla_{\Lambda_x^{(n)}} V)(t)$ is equal to the sign of $R(t, \Lambda_x^{(n)})$, and since we cut $d\Lambda_x^{(n+1)}$ off at dL_x and dU_x , we obtain the following algorithm:

1. Choose a starting mortality scenario $\Lambda_x^{(0)}$.
2. Calculate a new scenario by using the iteration

$$d\Lambda_x^{(n+1)} := \begin{cases} dL_x(t) & : R(t, \Lambda_x^{(n)}) < 0 \\ dU_x(t) & : R(t, \Lambda_x^{(n)}) > 0 \\ d\Lambda_x^{(n)}(t) & : R(t, \Lambda_x^{(n)}) = 0 \end{cases} . \quad (4.14)$$

3. Repeat step 2 until $|V(s, \Lambda_x^{(n+1)}) - V(s, \Lambda_x^{(n)})|$ is below some error tolerance.

For $n = 0$, step 2 yields the same scenario as the one that is suggested by the sum-at-risk method or the method based on derivatives in order to maximize the prospective reserve $V(s)$. Hence, the second algorithm is just an iteration of the sum-at-risk method or the method based on derivatives. The question is whether that algorithm converges to the true worst-case. The following result provides the answer.

Proposition 4.4. *Let $\Lambda_x^{(0)}, \Lambda_x^{(1)}, \Lambda_x^{(2)}, \dots$ be a series of cumulative mortality intensities calculated by iterating (4.14). Then*

$$\lim_{n \rightarrow \infty} V(s, \Lambda_x^{(n)}) = V(s, \bar{\Lambda}_x),$$

where $\bar{\Lambda}_x$ is the worst-case scenario according to (4.12).

Note that Proposition 4.4 states the convergence of the prospective reserves $V(s, \Lambda_x^{(n)})$ and not the convergence of the scenarios $\Lambda_x^{(n)}$. The latter do not necessarily converge on $\{t : R(t, \bar{\Lambda}_x) = 0\}$.

Proof. We only give a sketch of the proof here. The series $V(s, \Lambda_x^{(0)}), V(s, \Lambda_x^{(1)}), V(s, \Lambda_x^{(2)}), \dots$ is equivalent to the series that we obtain if we change (4.11) into an iteration equation by adding the superscript ' $(n+1)$ ' on the left hand side and the superscripts ' (n) ' on the right hand side. Christiansen (2009) defined such a series and showed that it converges to the unique solution of (4.11) in order to proof Property 4.2. Thus, following that proof of Christiansen (2009), we can verify that our iteration method converges. \square

We see that the sum-at-risk method yields an approximation of the true worst-case, and we can improve this approximation by just iterating the sum-at-risk method.

5 Worst-case if the acceleration of the cumulative hazard rate is bounded

Here we generally assume that λ_x exists. In contrast to (4.2) we do not bound the mortality intensity but the mortality intensity ascent, in other words, the speed with which the mortality increases with respect to age. Specifically, assume that the inequalities

$$d\lambda_x(t) \leq d\lambda_x(t) \leq du_x(t), \quad t \in [s, \omega_x], \quad (5.1)$$

are valid where l_x and u_x are hazard rates, and let $\lambda_x(s)$ be an arbitrary but fixed starting value at present (time s). If Λ_x is twice differentiable, (5.1) is equivalent to bounding the mortality intensity ascent α_x on $[s, \omega_x]$. If we choose a lower bound $d\lambda_x$ that is positive, then the mortality intensity is always monotonic increasing on $[s, \omega_x]$,

$$\lambda_x(t) - \lambda_x(s) \geq \int_{(s,t]} d\lambda_x(u).$$

We now regard the prospective reserve at time s as a mapping of the mortality intensity λ_x and are looking for a scenario within the bounds (5.1) and with fixed starting value $\lambda_x(s)$ that maximizes $V(s, \lambda_x)$. Let λ_x be some starting point that is shifted by a function h that is right-continuous and has bounded variation on $[s, \omega_x]$. By applying Fubini's Theorem, (4.9) can be transformed to

$$V(s, \lambda_x + h) - V(s, \lambda_x) = \int_{(s, \omega_x]} R_s(t) \left(\int_{(s,t]} dh(u) \right) dt = \int_{(s, \omega_x]} \left(\int_{[u, \omega_x]} R_s(t) dt \right) dh(u).$$

With writing H for the cumulative version of h , we denote

$$CR_s(u) := \int_{[u, \omega_x]} R_s(t) dt = \int_{[u, \omega_x]} e^{\Lambda_x(s) - \Lambda_x(t) + H(s) - H(t)} v(s, t) R(t, \Lambda_x) dt$$

as *cumulative sum-at-risk at time s for occurrence of death at and after u* . This gives an expression similar to (3.3) and (4.9), that is,

$$V(s, \lambda_x + h) - V(s, \lambda_x) = \int_{(s, \omega_x]} CR_s(u) dh(u). \quad (5.2)$$

The cumulative sum-at-risk $CR_s(u)$ describes the effect that a change $dh(u)$ of the ascent of the mortality intensity (the acceleration of the cumulative mortality intensity) has on $V(s)$. Analogously to Property 3.1 and Property 4.1 we get the following result:

Property 5.1. *If the shifted mortality intensity $\lambda_x + h$ is a still a regular hazard rate, then*

$$\text{sign}(dh(t)) = \text{sign}(CR_s(t)) \text{ for all } t > s \implies V(s, \lambda_x + h) \geq V(s, \lambda_x) \quad (5.3)$$

and

$$\text{sign}(dh(t)) = -\text{sign}(CR_s(t)) \text{ for all } t > s \implies V(s, \lambda_x + h) \leq V(s, \lambda_x). \quad (5.4)$$

For the proof just apply (5.2). In contrast to $dS(t)$ in Property 3.1 and $R(t)$ in Property 4.1, the sign of $CR_s(t)$ depends on shift h . Thus we can not just transform the ideas of the previous sections in order to find a maximizing scenario. Again, we can get rid of the dependence on h by allowing just for local shifts. Applying Fubini's Theorem, the integral in (4.13) can be transformed to

$$\begin{aligned} & \int_{(s, \omega_x]} e^{\Lambda_x(s) - \Lambda_x(t)} v(s, t) R(t, \Lambda_x) \left(\int_{(s, t]} dh(u) \right) dt \\ &= \int_{(s, \omega_x]} (\nabla_{\lambda_x} V)(u) dh(u), \end{aligned} \quad (5.5)$$

where $\nabla_{\lambda_x} V$ is interpreted as some form of generalized gradient defined by

$$(\nabla_{\lambda_x} V)(u) := \int_{[u, \omega_x]} e^{\Lambda_x(s) - \Lambda_x(t)} v(s, t) R(t, \Lambda_x) dt.$$

If $o(\|H\|)$ in (4.13) is negligible, then the prospective reserve $V(s, \lambda_x + h)$ can be maximized by choosing $d(\lambda_x + h)$ equal to dl_x and du_x there where $(\nabla_{\lambda_x} V)$ is negative and positive, respectively. If the confidence band (5.1) is not very narrow, $o(\|H\|)$ might not be negligible. But the first-order Taylor expansion (4.13) in the version of (5.5) allows at least to formulate a characteristic of global maxima.

Property 5.2. *Let $\bar{\lambda}_x$ be a scenario within the bounds (5.1) that maximizes $V(s, \bar{\lambda}_x)$. Then $\bar{\lambda}_x$ satisfies*

$$d\bar{\lambda}_x(t) = \begin{cases} dl_x(t) & : (\nabla_{\bar{\lambda}_x} V)(t) < 0 \\ du_x(t) & : (\nabla_{\bar{\lambda}_x} V)(t) > 0 \end{cases} \quad (5.6)$$

on $(s, \omega_x]$.

Proof. Assume that $\bar{\lambda}_x$ does not satisfy (5.6). Then

$$\int_{(s, \omega_x] \cap \{(\nabla_{\bar{\lambda}_x} V)(t) < 0\}} (\nabla_{\bar{\lambda}_x} V)(t) d(l_x - \bar{\lambda}_x)(t) + \int_{(s, \omega_x] \cap \{(\nabla_{\bar{\lambda}_x} V)(t) > 0\}} (\nabla_{\bar{\lambda}_x} V)(t) d(u_x - \bar{\lambda}_x)(t)$$

is strictly positive. Let $\tilde{\lambda}_x$ be defined by the right hand side of (5.6). Applying (4.13) in the version of (5.5), we obtain for $\varepsilon > 0$

$$V(s, \bar{\lambda}_x + \varepsilon(\tilde{\lambda}_x - \bar{\lambda}_x)) = V(s, \bar{\lambda}_x) + \underbrace{\varepsilon \int_{(s, \omega_x]} (\nabla_{\bar{\lambda}_x} V)(u) d(\tilde{\lambda}_x - \bar{\lambda}_x)(u)}_{>0} + o(\varepsilon).$$

Now choose ε small enough such that the integral (linear Taylor term) is greater than the absolute value of the remainder $o(\varepsilon)$. Then we have $V(s, \bar{\lambda}_x + \varepsilon(\tilde{\lambda}_x - \bar{\lambda}_x)) > V(s, \bar{\lambda}_x)$, which means that $\bar{\lambda}_x$ is not maximal. \square

If $\bar{\lambda}_x$ is a maximizing scenario, then the characteristic (5.6) that $\bar{\lambda}_x$ is mainly on the bounds is analogous to (3.9) and (4.12) (or (4.10)). However, a worst-case integral equation similar to (4.11) seems to be out of reach here. The crux of (4.11) is that the discontinuities at $R(t) = 0$ of the integrator (4.10) in the last integral in (4.4) are annihilated by the integrand $R(t)$. We do not have such a property for (5.6) since the signs of $CR_s(t)$ and $(\nabla_{\bar{\lambda}_x} V)(t)$ can differ. However, in order to find a worst-case scenario, we can at least design gradient ascent methods similar to the algorithms in Section 4.2:

1. Choose a starting mortality scenario $\lambda_x^{(0)}$.
2. Calculate a new scenario by using the iteration

$$d\lambda_x^{(n+1)}(t) := d\lambda_x^{(n)}(t) + K (\nabla_{\lambda_x^{(n)}} V)(t) dt.$$

If the right hand side is below $d\lambda_x$ or above du_x , we cut $d\lambda_x^{(n+1)}$ off at $d\lambda_x$ or du_x , respectively. Here, $K > 0$ is some step size that has to be chosen.

3. Repeat step 2 until $|V(s, \lambda_x^{(n+1)}) - V(s, \lambda_x^{(n)})|$ is below some error tolerance.

If we increase K to infinity, we obtain the following algorithm:

1. Choose a starting mortality scenario $\lambda_x^{(0)}$.
2. Calculate a new scenario by using the iteration

$$d\lambda_x^{(n+1)} := \begin{cases} d\lambda_x(t) & : (\nabla_{\lambda_x^{(n)}} V)(t) < 0 \\ du_x(t) & : (\nabla_{\lambda_x^{(n)}} V)(t) > 0 \\ d\lambda_x^{(n)}(t) & : (\nabla_{\lambda_x^{(n)}} V)(t) = 0 \end{cases}. \quad (5.7)$$

3. Repeat step 2 until $|V(s, \lambda_x^{(n+1)}) - V(s, \lambda_x^{(n)})|$ is below some error tolerance.

If these algorithms converge, the limit satisfies (5.6). The second algorithm makes better use of the fact that maximizing scenarios are always of the form (5.6).

Remark 5.3 (Bounds for higher order derivatives). In sections 3 and 4 and in this section we looked for worst-case mortality rates with respect to confidence bounds for Λ_x , $d\Lambda_x$, and $d(\Lambda'_x) = d\lambda_x$. The ideas in this section can be generalized to confidence bounds of higher order $d(\Lambda''_x)$, $d(\Lambda'''_x)$, ... by applying Fubini's Theorem in (5.5) not only once but consecutively in order to obtain generalized gradients $(\nabla_{\Lambda''_x} V)$, $(\nabla_{\Lambda'''_x} V)$, etc. With this gradients, one can find characterizations of the maximizing scenario similar to (5.6) and design iteration methods similar to (5.7).

6 Numerical illustrations

6.1 Confidence bands for the underlying hazard

Under the Lee-Carter model, the (central) death rate applying to age x in calendar year t is assumed to be of the form

$$m_x(t|\boldsymbol{\kappa}) = \exp(\alpha_x + \beta_x \kappa_t) \quad (6.1)$$

where the parameters β_x and κ_t are subject to constraints ensuring model identification. Here, the parameters are estimated from the mortality surface available from the Belgian Federal Planning Bureau. The ages considered here range from 30 to $\omega = 115$, and the observation period is 1970-2006. The time index κ_t is viewed as a stochastic process. Box-Jenkins techniques are therefore used to estimate and forecast κ_t within an ARIMA times series model. First, the κ_t 's are differenced, to remove the downward linear trend. Considering the first differences of the time index, the autocorrelation functions and partial autocorrelation functions (which both tail off) clearly suggests that an ARIMA(0,1,0) process is appropriate. Running a Shapiro-Wilk test indicates that the residuals seem to be approximately Normal. The corresponding Jarque-Bera statistics confirms that there is no significant departure from Normality. The random walk with drift model outperforms its competitor on the basis of standard information criteria. So, the κ_t 's obey the dynamics

$$\kappa_t = \kappa_{t-1} + \theta + \xi_t \text{ with iid } \xi_t \sim \mathcal{N}or(0, \sigma^2), \quad (6.2)$$

where θ is known as the drift parameter and $\mathcal{N}or(0, \sigma^2)$ stands for the Normal distribution with mean 0 and variance σ^2 . The estimated parameters are $\hat{\theta} = -1.117411$ and $\hat{\sigma}^2 = 1.226763$. The projected κ_t 's are then obtained from last $\hat{\kappa}_{2006}$ by adding a linear trend with slope $\hat{\theta}$.

Consider the cohort reaching age x_0 in year t_0 . We take as a reference $m_{x_0+k}^{\text{ref}}$ the central forecast produced by the Lee-Carter approach, that is,

$$m_{x_0+k}^{\text{ref}} = \exp(\alpha_{x_0+k} + \beta_{x_0+k}(\kappa_{t_0} + k\theta)).$$

The $m_{x_0+k}^{\text{ref}}$'s thus correspond to the deterministic projected life table produced by the Lee-Carter approach to mortality forecasting. For this cohort, we determine the band $(\pi_{\text{low}} m_{x_0+k}^{\text{ref}}, \pi_{\text{up}} m_{x_0+k}^{\text{ref}})$ such that

$$\Pr[\exp(\alpha_{x_0+k} + \beta_{x_0+k} \kappa_{t_0+k}) \notin (\pi_{\text{low}} m_{x_0+k}^{\text{ref}}, \pi_{\text{up}} m_{x_0+k}^{\text{ref}}) \text{ for some } k = 1, 2, \dots] \leq \epsilon_{\text{mort}}$$

for some probability level ϵ_{mort} small enough.

In order to fix the values of π_{low} and π_{up} , we require that

$$\Pr[\ln \pi_{\text{up}} \geq \beta_{x_0+k}(\kappa_{t_0+k} - (\kappa_{t_0} + k\theta)) \geq \ln \pi_{\text{low}} \text{ for all } k = 1, 2, \dots] \geq 1 - \epsilon_{\text{mort}}.$$

These values can then be determined as a quantile of the random vector

$$\left(\beta_{x_0+1}(\kappa_{t_0+1} - (\kappa_{t_0} + \theta)), \dots, \beta_{\omega}(\kappa_{t_0+\omega-x_0} - (\kappa_{t_0} + (\omega - x_0)\theta)) \right)^T$$

that is multivariate Normal with $\mathbf{0}$ mean and variance-covariance matrix

$$\tilde{\Sigma} = \begin{pmatrix} \sigma^2 \beta_{x_0+1}^2 & \sigma^2 \beta_{x_0+1} \beta_{x_0+2} & \cdots & \sigma^2 \beta_{x_0+1} \beta_{\omega} \\ \sigma^2 \beta_{x_0+1} \beta_{x_0+2} & 2\sigma^2 \beta_{x_0+2}^2 & \cdots & 2\sigma^2 \beta_{x_0+2} \beta_{\omega} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma^2 \beta_{x_0+1} \beta_{\omega} & 2\sigma^2 \beta_{x_0+2} \beta_{\omega} & \cdots & (\omega - x_0) \sigma^2 \beta_{\omega}^2 \end{pmatrix}.$$

Consider the generation aged $x_0 = 30$ in year $t_0 = 2006$. Imposing that the future life table should be in the band $(\pi_{\text{low}} m_{x_0+k}^{\text{ref}}, \pi_{\text{up}} m_{x_0+k}^{\text{ref}})$ with probability at least 99%, we get $\pi_{\text{low}} = 0.9393242$ and $\pi_{\text{up}} = 1.064595$. These values have been found using the `qmvnorm` function of the R package `mvtnorm`.

6.2 Annuity with death benefits

Consider an annuity insurance with additional death benefits. A constant premium is paid yearly in advance from age 30 on till retirement at age 65. From then on a constant annuity benefit of 1 is paid yearly in advance till death. The functions Π and B are thus given by

$$\Pi(t) = \text{const} \sum_{k=0}^{34} \mathbf{1}_{[k, \infty)}(t), \quad B(t) = \sum_{k=35}^{\omega_x} \mathbf{1}_{[k, \infty)}(t).$$

If the policyholder dies before age 65, a death benefit is paid that has the size of the prospective reserve just before retirement. If the policyholder dies after retirement but before age 85, a death benefit is paid that equals the prospective reserve at that time. The function c is thus given by

$$c(t) = V(35-) \cdot \mathbf{1}_{[0, 35)}(t) + V(t-) \cdot \mathbf{1}_{[35, 55)}(t).$$

We assume that the yearly interest rate is at 2.25% and that interest is paid continuously with an intensity of $\varphi(t) = \ln(1.0225)$. For our exemplary calculations we use the mortality intensity $m_{x_0+k}^{\text{ref}}$ derived from (6.1) with parameters estimated from Belgian mortality statistics as best estimate and confidence bands where

- (A) the lower and upper bound are -6.59131% and $+6.4595\%$ below and above the best estimate as obtained from the coefficients π_{low} and π_{up} .
- (B) the lower and upper bound are -25% and $+15\%$ below and above the best estimate as suggested in Consultation Paper no. 49 of the Committee of European Insurance and Occupational Pensions Supervisors (CEIOPS) for the Solvency II project.

The equivalence principle and the best estimate mortality rate yield a constant yearly premium of 0.4133781786. Figure 6.1 shows the death benefit function $c(t)$. Figures 6.2, 6.3, and 6.4 illustrate the cumulative survival cost $S_0(t)$ at time zero for survival at and after time t , the sum-at-risk $R(t)$ for occurrence of death at time t , and the cumulative sum-at-risk $CR_0(t)$ at time zero for occurrence of death at and after time t . All illustrations are based on the best estimate mortality scenario. For the calculation of $CR_0(t)$, we assumed that the shift H is zero, which implies that the cumulative sum-at-risk is equal to the generalized gradient $(\nabla_{\lambda_x} V)(t)$ here. The following tabular shows the prospective reserve $V(0-)$ before

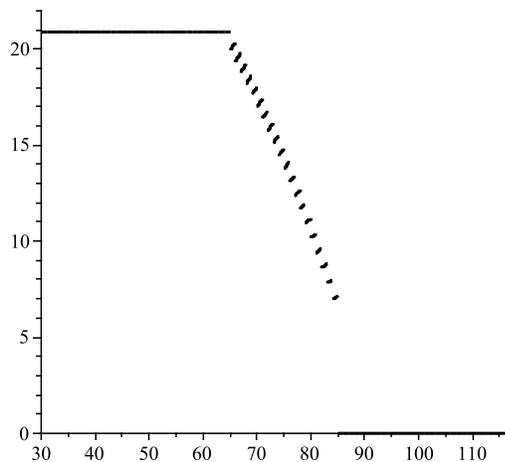


Figure 6.1: Death benefit function $c(t)$

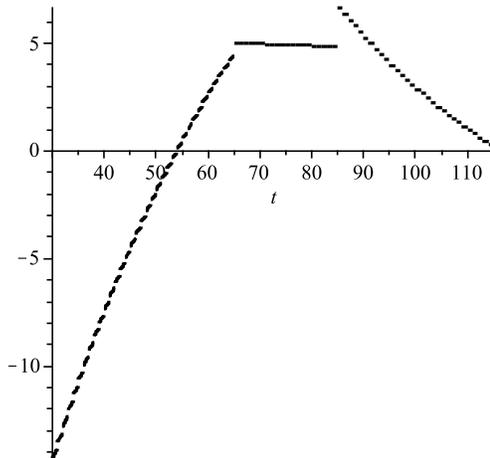


Figure 6.2: Approach I: Cumulative survival cost $S_0(t)$

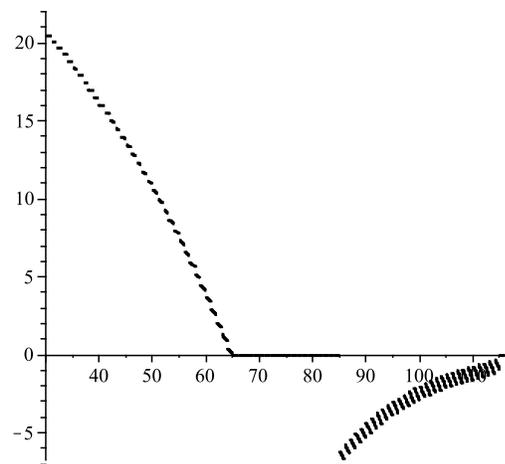


Figure 6.3: Approach II: Sum-at-risk $R(t)$ with respect to the best estimate mortality rate

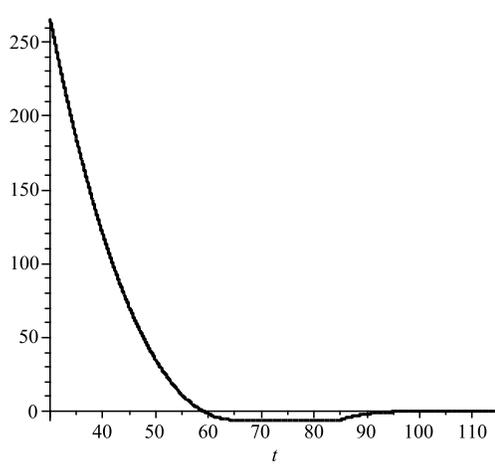


Figure 6.4: Approach III: Cumulative sum-at-risk $CR_0(t)$ with respect to the best estimate mortality rate and zero shift

beginning of the contract with respect to different mortality scenarios:

valuation basis	prospective reserve $V(0-)$ w.r.t. confidence band (A)	prospective reserve $V(0-)$ w.r.t. confidence band (B)
best estimate	0.000000	0.000000
lower bound	0.009744	0.083661
upper bound	-0.003088	0.001069
separated contract	0.299774	0.981273
worst-case method I	0.156119	0.539246
worst-case method II	0.081861	0.298761
worst-case method III	0.058308	0.233918
sum-at-risk method	0.079849	0.265951
2× sum-at-risk method	0.080565	0.276372
3× sum-at-risk method	0.081848	0.298761

'Separated contracts' means that the policy is unbundled into an annuity policy and a temporary life insurance policy, and the two parts are valued on the basis of the lower bound and the upper bound. The results '2× sum-at-risk method' and '3× sum-at-risk method' are obtained by applying the sum-at-risk method iteratively. The rows 'worst-case method I, II, and III' refer to the methods of sections 3, 4, and 5. The bounds are given by

$$L_x(t) - L_x(s) = \pi_{low} (\Lambda_x(t) - \Lambda_x(s)), \quad U_x(t) - U_x(s) = \pi_{up} (\Lambda_x(t) - \Lambda_x(s))$$

for approach I, by

$$l_x(t) = \pi_{low} \lambda_x(t), \quad u_x(t) = \pi_{up} \lambda_x(t)$$

for approach II and the sum-at-risk method, and by

$$\begin{aligned} dl_x(t) &= \Delta l_x(t) = \min\{\pi_{up} \Delta \lambda_x(t), \pi_{low} \Delta \lambda_x(t)\}, \\ du_x(t) &= \Delta u_x(t) = \max\{\pi_{up} \Delta \lambda_x(t), \pi_{low} \Delta \lambda_x(t)\} \end{aligned}$$

for integers $t = 1, \dots, \omega_x$, and $dl_x(t) = du_x(t) = 0$ else for approach III. As starting value $\lambda_x(0)$ we use the best estimate. The following table shows the (numerically calculated) mortality scenarios, which all are at any time t either equal to a bound or equal to the best estimate:

ages where the valuation basis is equal to the upper bound of confidence band (A)	... the best estimate	... the lower bound of confidence band (A)
best estimate		$(30, \omega)$	
worst-case method I	$(30, 65] \cup (84, 85]$		$(65, 84] \cup (85, \omega]$
worst-case method II	$(30, 64.762)$		$(64.762, \omega)$
worst-case method III	$\{31, \dots, 58\}$		$\{59, \dots, \omega\}$
sum-at-risk method	$(0, 65)$	$(65, 85)$	$(85, \omega)$

ages where the valuation basis is equal to the upper bound of confidence band (B)	... the best estimate	... the lower bound of confidence band (B)
best estimate		$(30, \omega)$	
worst-case method I	$(30, 65] \cup (84, 85]$		$(65, 84] \cup (85, \omega]$
worst-case method II	$(30, 63.888)$		$(63.888, \omega)$
worst-case method III	$\{31, \dots, 56\}$		$\{57, \dots, \omega\}$
sum-at-risk method	$(0, 65)$	$(65, 85)$	$(85, \omega)$

Worst-case method I sets the cumulative mortality intensity equal to the upper and lower bound there where the cumulative survival cost S_0 is increasing and decreasing. The shifting times are independent of the confidence band, because S_0 does not depend on the bounds. Note that we have two negative jumps for the cumulative mortality intensity. The sum-at-risk method sets the mortality intensity equal to the upper and lower bound there where the sum-at-risk R (with respect to the best estimate) is positive and negative. We see that the result differs from the true worst-case calculated by worst-case method II. For the sum-at-risk method, the shifting times do not depend on the confidence band, but for worst-case method II they do. For worst-case method III, our choice of the bounds dl_x and du_x allows only for changes of λ_x at integer times. The shift between high and low mortality intensity ascent is for both confidence bands earlier than the shift between high and low mortality intensity in method II, which is a result from the facts that the cumulative sum-at-risk aggregates the future sums-at-risk and that the sum-at-risk is first throughout positive and then throughout negative. While the mortality intensity of method II shows an extreme jump from u_x to l_x near the age of retirement, the mortality intensity of method III evolves gradually with small steps Δl_x or Δu_x .

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