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# A SHORT NOTE ON CONTINUOUS-TIME MARKOV AND SEMI-MARKOV PROCESSES

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# A short note on continuous-time Markov and semi-Markov processes

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**Abstract**

In this paper, we present the some features of jump Markov processes (both homogeneous and non-homogeneous) and semi-Markov processes. We also insist on these processes from the viewpoint of marked point processes. This allows us to highlight differences and convergence points between these processes. Specifically, we recall and show that a Markov process can have a duration distribution that is not exponential. We also show that the form of the compensator associated to these processes can allow us to differentiate between them in terms of the Markov property. Finally, we briefly discuss which process to use when it comes to modelling.

## 1 Introduction

Homogeneous Markov processes have been widely studied and many texts exist that present them in various ways (see (1) amongst others). Non-homogeneous Markov processes and semi-Markov processes, although very general, are in constrast less well known than their homogeneous Markov counterparts. Interesting references in this domain include (1), (3) and (5).

This paper aims at recalling a few notions and properties of these processes and emphasize their differences and advantages. We will consider continuous-time finite state-space processes. The processes under review will be jump type processes i.e. processes that spend a certain amount of time in a state before moving to another state and so on. We will discuss the distribution of duration times in a state for all processes under consideration. We will also emphasize the impact of the Markov property (and the lack of it in the case of semi-Markov processes) and see how this is related to the "form" of the intensity when you consider these processes from the point of view of marked point processes.

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The paper is divided as follows, the first part will be a very short reminder on marked point processes. We then go on to discuss Markov processes both homogeneous and non-homogeneous. The next section is devoted to semi-Markov processes and their properties. Finally, we conclude by summarizing our results.

## 2 Marked point processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $(E, \mathcal{E})$ a measurable space. Let $T$ be an infinite set. Let $\mathcal{F}_t$ be a filtration on the probability space. This section will follow the results presented in (4).

**Remark 1.** In what follows, we will always suppose that $T = \mathbb{R}^+$ and that $E = \{1, ..., m\}$ with $m$ finite. It is possible to consider more general state-spaces but we don't aim for full generality.

**Definition 1.** A point process is an increasing sequence of positive random variables $(T_n)_{n\geq 1}$ taking values in $T$. A marked point process $(T_n, X_n)_{n\geq 1}$ is a point process to which we associate to every $T_n$, a random variable $X_n$ taking values in $E$ .

**Remark 2.** In what follows, we will suppose that $\sup(T_n) = \infty$. Such processes are called regular. For conditions insuring regularity see (1). We also define $T_0 = 0$.

**Remark 3.** In this paper, the reader should interpret $X_n$ as being the state of a system and $T_n$ the times at which the system switches state.

Let us fix some notation by defining $S_n := T_n - T_{n-1}$ as the duration in each state.

It is well know that a marked point process is related to the discrete random counting measure defined on $T \times E$ and given by

$$\mu(\omega, dt, dx) = \sum_{n\geq 1} \delta_{(T_n(\omega),\, X_n(\omega))}(dt, dx)$$

Let $G_n(dt, dx)$ be the distribution of $(S_{n+1}, X_{n+1})$ conditional on $\mathcal{F}_{T_n}$. It can be shown that (see (4)) the compensator of the measure $\mu$ is given by

$$\nu(\omega, dt, dx) = \sum_{n\geq 0} \frac{G_n(dt - T_n, dx)}{G_n([t - T_n, \infty], E)} \mathbb{1}_{\{T_n < t \leq T_{n+1}\}}$$

**Remark 4.** For more on compensators and their link with the Doob-Meyer decomposition see (4).

**Definition 2.** Suppose that $\nu(\omega, dt, dx)$ can be written as $\nu(\omega, dt, dx) = \sum_{n\geq 0} 1\!\!1_{\{T_n < t \leq T_{n+1}\}} \lambda_t(\omega, dx)dt$. Then, $\lambda_t(\omega, dx)$ is called the intensity process. Roughly speaking, the intensity describes the propensity of a process to jump at time $t$ **given the whole history up to** $t$.

It is known that the compensator characterizes the distribution of a marked point process. In view of this, we will use it as a tool to compare between three different marked point processes namely homogeneous Markov processes, non-homogeneous Markov processes and semi-Markov processes. The characteristics of these processes will be directly apparent from the form of the compensator.

# 3 Markov processes

**Definition 3.** A stochastic process $X_t$ is a Markov process with respect to filtration $\mathcal{F}_t$ if $X_t$ is adapted to the filtration and, given $s > 0$, $X_{t+s} \perp\!\!\!\perp \mathcal{F}_t \mid X_t$ i.e. $X_{t+s}$ is independent of $\mathcal{F}_t$ given $X_t$.

**Remark 5.** Intuitively, this means that the future is independent of the past given the present.

**Remark 6.** In all that follows, unless otherwise stated, we will always assume that the processes under consideration are cadlag. In fact, we will only consider so-called jump-Markov processes with finite state-space (think of set $E$ as defined previously). Roughly, the idea is that the process starts in a state, stays there for a certain length of time, then jumps to another state, stays for a certain length of time and so on. We denote by $X_n$ the successive states visited and $T_n$ the jump times.

An oft seen way of expressing the Markov property in our context is given by

$$\mathbb{P}[X_{t+s} = j \mid \mathcal{F}_t] = \mathbb{P}[X_{t+s} = j \mid X_t]$$

The conditional probability above is often denoted as

$$\mathbb{P}[X_{t+s} = j \mid X_t] = p_{t,\,s}(X_t, j)$$

The function $p_{t,\,s}(X_t, j)$ is known as the transition function of the Markov process. If this function is independent of $t$, the Markov process will be called a homogeneous Markov process.

Another way of specifying a Markov process is through its so-called infinitesimal transition rates. We will see that this is closely linked to the compensator defined in the previous section. The infinitesimal transition rates are defined as follows:

**Definition 4.** The infinitesimal transition rate $q_t(ij)$ is defined as follows:

$$\begin{aligned} q_t(ij) &= \lim_{\epsilon\to 0} \frac{\mathbb{P}\{X_{t+\epsilon} = j \mid X_t = i\}}{\epsilon} = \left[\frac{dp_{t,\,s}(i,j)}{ds}\right]_{s=t} \qquad j \neq i \\ q_t(i) &:= q_t(ii) = -\sum_{j\neq i} q_t(ij) \end{aligned}$$

**Remark 7.** Of course, in the case of a homogeneous Markov process, we get infinitesimal transition rates $q(ij)$ and $q(i)$ independent of $t$.

**Remark 8.** The interpretation of the infinitesimal transition rate is the change in the probability that the process quits state $i$ at time $t$ to enter state $j$ at time $t + dt$.

The following result gives an idea of how to link the infinitesimal transition rates to the pathwise behavior of the jump Markov process.

**Theorem 1.** *Let $T_s$ represent the first jump time after $s$. Then, there exists a probability distribution $\Pi_t$ such that*

$$\mathbb{P}(s < T_s \leq t,\ X_{T_s} = j | X_s = i) = \int_s^t \Pi_v(ij) q_v(i) \exp\left(-\int_s^v q_w(i) dw\right) dv$$

*Proof.* For a proof, see (3).

□

In view of the last result, we can give some indication on the waiting time distribution in a state.

**Theorem 2.** *The distribution of $T_{n+1} - T_n$ conditional on $\mathcal{F}_{T_n}$ is given by*

$$\mathbb{P}(T_{n+1} - T_n \leq t | \mathcal{F}_{T_n}) = 1 - \exp\left(-\int_{T_n}^{T_n+t} q_s(X_n) ds\right)$$

*Proof.* For a proof see (3).

□

**Remark 9.** The waiting time distribution is not exponential except in the case of homogeneous Markov processes where the infinitesimal transition rates are independent of $t$. This corrects many authors who claim that a Markov process necessarily has exponential waiting time distributions. This is only true of homogeneous Markov processes. However, the waiting time distribution can only depend on the current state and not on the arrival state after the jump.

With $X_n$ representing the successive states of the jump Markov process and $T_n$, the jump times, we can build a marked point process $(X_n, T_n)$. We can give an expression of the compensator of this marked point process. The following result gives more details.

**Theorem 3.** *Let $\Pi_t(ij)$ be the probability distributions of theorem 1. Then the compensator associated to the marked point process (and so to the jump Markov process) is given by*

$$\nu(\omega, dt, j) = \sum_{n\geq 0} 1\!\!1_{\{T_n<t\leq T_{n+1}\}}\Pi_t(X_n, j)q_t(X_n)dt$$

*Proof.* The proof is an application of theorem 1 to the definition of a compensator. □

**Remark 10.** In the homogeneous case, the compensator is given by

$$\nu(\omega, dt, j) = \sum_{n\geq 0} 1\!\!1_{\{T_n<t\leq T_{n+1}\}}\Pi(X_n, j)q(X_n)dt$$

We observe that the intensity is independent of time.

We observe that the intensity of the compensator is either time-dependent in the non-homogeneous Markov process case or time-independent in the homogeneous Markov case. This is means that the propensity to jump at time $t$ depends only on what happens at time $t$ and not on the past before $t$. Just as in the Markov property, the future of the compensator is independent of the past given the present. What is interesting is the ability to read this property directly from the form of the compensator. This will allow us to see the difference between Markov processes and semi-Markov processes as will be shown further in the text.

# 4 Semi-Markov processes

For each $n \in \mathbb{N}$ let $(X_n, T_n)$ be a pair of random variables taking values in $E \times \mathbb{T}$.

**Definition 5.** The stochastic process $\{X_n, T_n; n \geq 0\}$ is a time-homogeneous Markov renewal process if

$$\mathbb{P}[X_{n+1} = j, T_{n+1} - T_n \leq t|X_0, ..., X_n; T_0, ...T_n] = \mathbb{P}[X_{n+1} = j, T_{n+1} - T_n \leq t|X_n = i] = Q_{ij}(t)$$

for all $n \geq 0$, $i, j \in E$ and $t \in \mathbb{R}^+$. The family of probabilities $Q = \{Q_{ij}(t) : i, j \in E, t \in \mathbb{R}^+\}$ is called a semi-markov kernel over $E$.

**Remark 11.** In what follows, we will suppose that all states communicate at all times.

One can show that $X_n$ is a Markov chain. The transition matrix of $X_n$ is $P_{ij}$ and it is given by $P_{ij} = Q_{ij}(\infty)$. Let us define

$$F_{ij}(t) = \frac{Q_{ij}(t)}{P_{ij}} \qquad i, j \in E,\ t \in \mathbb{R}^+ \tag{1}$$

Then $F_{ij}(t)$ has the following interpretation:

$$F_{ij}(t) = \mathbb{P}[T_{n+1} - T_n \leq t | X_n = i, X_{n+1} = j]$$

**Definition 6.** Let $\nu_t$ be given by

$$\nu_t = \sup(n \geq 0 : T_n \leq t)$$

with $n \in \mathbb{N}$ and $t \in \mathbb{R}^+$. Define $Y$ as

$$Y_t = X_{\nu_t}$$

The process $Y$ is called a semi-markov process with kernel $Q$.

**Remark 12.** It can easily be shown that given the number of states is finite, the number of jumps in a finite time interval is almost surely finite (for a proof see (5)).

The definition of a semi-Markov process implies that the duration distribution can be very general. It can depend on both the present and future state. Let us note that if this distribution depends only on the current state and is exponential, then we fall back on the homogeneous Markov case.

**Proposition 4.** *The semi-Markov process $Y$ does not -in general- satisfy the Markov property of definition 3 except at times of jump $T_n$.*

*Proof.* See (2) □

**Remark 13.** The fact that the duration distribution is not exponential is not the sole reason why process $Y$ doesn't satisfy the Markov property, indeed the duration of non-homogeneous Markov processes is not exponential but non-homogeneous processes satisfy the Markov property. We will see that it is really on the dependence on the past that semi-Markov processes differ from Markov processes.

Let $(X_n, T_n)$ represent the state and jump times of the semi-Markov process. From this, we build a marked point process. The next theorem will give an explicit representation of the compensator associated to this marked point process.

**Theorem 5.** *Let $f_{ij}(t)$ represent the density associated to the distribution function $F_{ij}(t)$. Then, the compensator of the semi-Markov process is given by*

$$\nu(\omega, dt, j) = \sum_{n\geq 0} 1\!\!1_{\{T_n<t\leq T_{n+1}\}} \frac{P_{X_n,\, j} f_{X_n,\, j}(t-T_n)}{1-\sum_j Q_{X_n,\, j}(t-T_n)} dt$$

*Proof.* This is a direct application of the definition of compensator and of semi-Markov processes.

□

As we can see, in the case of semi-Markov processes, the intensity of the compensator depends not only on $t$ but on $t - T_n$, the time elapsed since the last transition. This means that the compensator depends on the past and not only through the present. This explains the fact that semi-Markov processes don't satisfy the markov property.

**Remark 14.** Proposition 4 becomes easier to understand in view of theorem 5. Actually, it can also be shown that process $(Y_t, t - T_{N_t})$ (process $Y$ and the process giving the time elapsed since the last jump) form a Markov process. Again, this is easy to understand in view of theorem 5.

# 5 Conclusions

This section will summarize the main results. We defined the Markov property as the independence from the past given the present. There are two main type of jump processes satisfying this property: homogeneous Markov processes and non-homogeneous Markov processes. We have shown that non-homogeneous Markov processes are characterized by the a time-dependent intensity whereas homogeneous Markov processes have a time-independent intensity. We have also shown that homogeneous Markov processes have a duration time with exponential distribution and that this is not the case for non-homogeneous Markov processes. This contradicts a frequent belief that non-exponential distribution means non-Markov process. As for semi-Markov processes, their intensity depends on the past (and not only through the present i.e. it depends on the time elapsed since the last jump) and this is linked to the fact that semi-Markov processes cannot satisfy the Markov property. In this type of process, the waiting time distribution is completely general and can also depend on the future state to be visited.

When building a model using a jump-process, we should always keep in mind the situation being modelled. If we want our model to satisfy the Markov property, we will choose between homogeneous and non-homogeneous Markov processes. This choice will depend on whether we want the duration distribution to be exponential or not. If we don't want the model to satisfy the Markov property, then we can choose a semi-Markov model for its simplicity and generality.

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## References

[1] Çinlar E. ; *Introduction to stochastic processes*, Prentice-Hall, Englewood Cliffs, New-Jersey, 1975

[2] Çinlar E. ; "Markov renewal processes: a survey"; Discussion paper, Northwestern university, Evanston, Illinois, **53**, (1973)

[3] Gikhman I.I., Skorohod A.V. ; *Introduction to the theory of random processes*, W.B. Saunders Company, Philadelphia, 1969

[4] Jacod J. ; "Multivariate point processes: predictable projection, Radon-Nikodym derivatives, representation of martingales "; Probability Theory and Related Fields, **31** (3), (1975): 235-253

[5] Pyke R.; "Markov renewal processes: definitions and preliminary properties"; The Annals of Mathematical Statistics, **32** (4), (1961): 1231-1242