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JACKKNIFE EMPIRICAL LIKELIHOOD METHOD FOR COPULAS

PENG, L., QI, Y. and I. VAN KEILEGOM

Jackknife Empirical Likelihood Method for Copulas

Liang PENG[1], Yongcheng QI[2] and Ingrid VAN KEILEGOM[3]

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Abstract

Copulas are used to depict dependence among several random variables. Both parametric and non-parametric estimation methods have been studied in the literature. Moreover, profile empirical likelihood methods based on either empirical copula estimation or smoothed copula estimation have been proposed to construct confidence intervals of a copula. In this paper, a jackknife empirical likelihood method is proposed to reduce the computation with respect to the existing profile empirical likelihood methods.

KEY WORDS: Copulas, empirical likelihood method, jackknife

1 Introduction

Dependence among variables plays an important role in understanding and interpreting multivariate data series in economics, finance, insurance and other fields in social sciences. Although some commonly used dependence measures such as Pearson's correlation coefficient, Kendall's tau and Spearman's rho are useful in decribing dependence, they can not completely capture the dependence structure among variables. Instead, as a function independent of marginals, copulas become more or less a standard tool in risk management (see McNeil, Frey and Embrechts (2005)).

Suppose $(X_1, Y_1), \cdots, (X_n, Y_n)$ are independent and identically distributed random vectors with a joint distribution function F. The copula of F is defined as $C(x, y) = F(F_1^-(x), F_2^-(y))$, where $F_1(x) = F(x, \infty)$, $F_2(y) = F(\infty, y)$ and $(\cdot)^-$ denotes the generalized inverse function of $(\cdot)$. We

[1]School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA. Email: peng@math.gatech.edu

[2]Department of Mathematics ad Statistics, University of Minnesota Duluth, 1117 University Drive, Duluth, MN 55812, USA. Email: yqi@d.umn.edu

[3]Université catholique de Louvain, Institute of Statistics, Voie du Roman Pays 20, B-1348 Louvain-la-Neuve, Belgium. Email: ingrid.vankeilegom@uclouvain.be

refer to Nelsen (1998) and Joe (1997) for an overview of copulas. A wide range of applications of copulas can be found in the literature of economics, econometrics and finance. For example, Zimmer and Trivedi (2006) used copulas to study self-selection and interdependence between health insurance and health care demand among married couples; Frees and Wang (2006) employed copula to insurance pricing; Van de Goorbergh, Genest and Werker (2005) applied dynamic copulas to option pricing; Cameron et al. (2004) modeled counted data by copulas; Hennesy and Lapan (2002) used copulas to study portfolio allocations; Junkers and May (2005) proposed to use transformed copulas to study the aggregate financial risk on a portfolio level; Smith (2003) employed copulas to model data with selectivity bias; Chen and Yan (2006) used copulas to model errors of multivariate nonlinear time series.

For estimating a copula, both nonparametric and parametric methods have been proposed in the literature. A simple nonparametric estimator is the so-called empirical copula :

$$C_n(x,y) = \frac{1}{n}\sum_{i=1}^{n} I(F_{n1}(X_i) \le x, F_{n2}(Y_i) \le y), \tag{1.1}$$

where

$$F_{n1}(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \le x) \quad \text{and} \quad F_{n2}(y) = \frac{1}{n}\sum_{i=1}^{n} I(Y_i \le y)$$

are the two marginal empirical distribution functions; see Deheuvels (1979). For a detailed study of the asymptotic limit of the empirical copula estimate, we refer to Fermanian, Radulovic and Wegkamp (2004). Smoothing estimation of a copula was investigated by Fermanian and Scaillet (2003) and Chen and Huang (2007).

Recently, Chen, Peng and Zhao (2009) and Molanes Lopez, Van Keilegom and Veraverbeke (2009) proposed empirical likelihood methods, based on smoothed copula estimation or empirical copula estimation, to construct confidence intervals for a copula. The basic idea is to introduce linking variables $s = F_1^-(x)$ and $t = F_2^-(y)$ and then apply profile empirical likelihood methods to the following constraints :

$$F(s,t) = \theta, \quad F_1(s) = x, \quad F_2(t) = y,$$

where $\theta = C(x,y)$ is the quantity which we like to construct a confidence interval for. The important reason for introducing the linking variables is to take the unknown marginals in (1.1) into account so as to obtain a chi-square limit. However, the constraints due to the linking variables cause computational burden in the empirical likelihood methods, especially for a high dimensional copula. Since we are only interested in a confidence interval for θ, one may ask whether it is possible to have an empirical likelihood method with only one constraint and which has a chi-square limit.

Recently, Jing, Yuan and Zhou (2009) proposed to apply empirical likelihood methods to jackknife pseudo samples for a U-statistic so as to avoid a non-linear minimization problem caused by direct application of the empirical likelihood method. In this paper, we show that the idea in Jing, Yuan and Zhou (2009) can be employed to remove unnecessary constraints in the setup of copulas. However, one has to work with smoothed copula estimation. That is, a smoothed jackknife empirical likelihood method is needed. This new method has a great computational advantage over the existing profile empirical likelihood methods for constructing confidence intervals for a copula especially for a high dimensional copula.

We organize this paper as follows. The new methodology is given in Section 2. Section 3 presents some simulation results. All proofs are put in Section 4.

2 Methodology

Define

$$F_{n1,i}(x) = \frac{1}{n-1}\sum_{j\neq i} I(X_j \le x), \quad F_{n2,i}(y) = \frac{1}{n-1}\sum_{j\neq i} I(Y_j \le y),$$

$$C_{n,i}(x,y) = \frac{1}{n-1}\sum_{j\neq i} I(F_{n1,i}(X_j) \le x, F_{n2,i}(Y_j) \le y).$$

As in Jing, Yuan and Zhou (2009), the jackknife pseudo sample may be defined as

$$V_i(x,y) = nC_n(x,y) - (n-1)C_{n,i}(x,y), \quad i = 1, \cdots, n.$$

Based on the above pseudo sample, one can define the empirical likelihood function as

$$L_n(x,y;\theta) = \sup\{\Pi_{i=1}^n p_i : p_1 > 0, \cdots, p_n > 0, \sum_{i=1}^n p_i = 1, \sum_{i=1}^n p_i V_i(x,y) = \theta\},$$

where $\theta = C(x,y)$ is the quantity for which we like to construct a confidence interval. Unfortunately, the above procedure does not work although only one constraint is involved. The reason is as follows.

It is known that a key feature of the empirical likelihood method is the automatic standardization, see Owen (2001) for an overview on empirical likelihood methods. In order to have the above procedure work, one needs to show that $\frac{1}{n}\sum_{i=1}^n \{V_i(x,y) - C(x,y)\}^2$ is a consistent estimator of $nVar(C_n(x,y))$, i.e.,

$$\nu_n(x,y) = \frac{1}{n}\sum_{i=1}^n \{V_i(x,y) - \frac{1}{n}\sum_{j=1}^n V_j(x,y)\}^2$$

is a consistent estimator of $nVar(C_n(x,y))$. Note that $\nu_n(x,y)$ is the jackknife variance estimate, see Shao and Tu (1995) for details on jackknife methods. Unfortunately, a brief simulation study, not reported here, shows that $\nu_n(x,y)$ is an inconsistent estimator of $nVar(C_n(x,y))$. At first sight, one may argue that the definition of copula involves quantiles and the jackknife variance estimate for a quantile is inconsistent, see Martin (1990). This is not the exact reason as we show that a smooth version of $\nu_n(x,y)$ works with the non-smoothed marginal empirical distributions $F_{n1,i}, F_{n2,i}, F_{n1}, F_{n2}$ remained.

Let k be a density function and put $K(x) = \int_{-\infty}^x k(y)\,dy$. Then we smooth $C_n(x,y)$ and $C_{n,i}(x,y)$ by

$$\begin{cases} \hat{C}_n(x,y) = \frac{1}{n}\sum_{j=1}^n K(\frac{x-F_{n1}(X_j)}{h})K(\frac{y-F_{n2}(Y_j)}{h}) \\ \hat{C}_{n,i}(x,y) = \frac{1}{n-1}\sum_{j\neq i} K(\frac{x-F_{n1,i}(X_j)}{h})K(\frac{y-F_{n2,i}(Y_j)}{h}), \end{cases} \tag{2.1}$$

where $h = h(n) > 0$ is a bandwidth, and define the smoothed jackknife pseudo sample as

$$\hat{V}_i(x,y) = n\hat{C}_n(x,y) - (n-1)\hat{C}_{n,i}(x,y).$$

Therefore, the jackknife variance estimate of $nVar(\hat{C}_n(x,y))$, based on the pseudo sample of $\hat{V}_i(x,y)'s$ is

$$\hat{\nu}_n(x,y) = \frac{1}{n}\sum_{i=1}^n\{\hat{V}_i(x,y) - \frac{1}{n}\sum_{j=1}^n \hat{V}_j(x,y)\}^2.$$

Note that the asymptotic variance of $\sqrt{n}\{\hat{C}_n(x,y) - C(x,y)\}$ is

$$\begin{aligned} \sigma^2(x,y) \;=\;& C(x,y)\{1-C(x,y)\} + x(1-x)\{\tfrac{\partial}{\partial x}C(x,y)\}^2 \\ & +y(1-y)\{\tfrac{\partial}{\partial y}C(x,y)\}^2 - 2C(x,y)(1-x)\tfrac{\partial}{\partial x}C(x,y) \\ & -2C(x,y)(1-y)\tfrac{\partial}{\partial y}C(x,y) + 2\{C(x,y)-xy\}\{\tfrac{\partial}{\partial x}C(x,y)\}\{\tfrac{\partial}{\partial y}C(x,y)\}, \end{aligned} \tag{2.2}$$

which is the same as that of $\sqrt{n}\{C_n(x,y) - C(x,y)\}$, see Fermanian, Radulovic and Marten (2004).

Our first result shows that $\hat{\nu}_n(x,y)$ is consistent.

Theorem 1. Assume that k is a symmetric density with support $[-1,1]$ and that $h = h(n) \to 0$, $nh^2 \to \infty$ and $nh^4 \to 0$ as $n \to \infty$. Further assume that $C(x,y)$ has continuous first derivatives. Then,

$$\hat{\nu}_n(x,y)/\sigma^2(x,y) \xrightarrow{p} 1.$$

Based on Theorem 1, one can expect that an application of the empirical likelihood method to the above smoothed jackknife pseudo sample of $\hat{V}_i(x,y)'s$ works. Now we define the empirical likelihood function as

$$\hat{L}_n(x,y;\theta) = \sup\{\Pi_{i=1}^n p_i : p_1 > 0, \cdots, p_n > 0, \sum_{i=1}^n p_i = 1, \sum_{i=1}^n p_i\hat{V}_i(x,y) = \theta\}.$$

An application of the Lagrange multiplier method gives

$$p_i = \frac{1}{n}\frac{1}{1+\lambda(\hat{V}_i(x,y)-\theta)},$$

where $\lambda = \lambda(x, y; \theta)$ satisfies

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\hat{V}_i(x,y)-\theta}{1+\lambda(\hat{V}_i(x,y)-\theta)} = 0. \tag{2.3}$$

Hence the log empirical likelihood ratio becomes

$$\hat{l}_n(x, y; \theta) = -2\log\{n^{-n}\hat{L}_n(x, y; \theta)\} = 2\sum_{i=1}^{n}\log\{1+\lambda(\hat{V}_i(x,y)-\theta)\}.$$

Theorem 2. Under the conditions of Theorem 1, we have

$$\hat{l}_n(x, y; C(x, y)) \xrightarrow{d} \chi^2(1)$$

as $n \to \infty$.

Based on Theorem 2, a confidence interval of level γ for $C(x, y)$ is given by

$$I_\gamma(x, y; h) = \{\theta : \hat{l}_n(x, y; \theta) \le \chi^2_{1,\gamma}\},$$

where $\chi^2_{1,\gamma}$ is the γ quantile of $\chi^2(1)$.

3 Simulation study

In this section we investigate the finite sample behavior of the proposed jackknife empirical likelihood method and compare it with the bootstrap confidence interval based on the empirical copula. We employ the same setting as in Chen, Peng and Zhao (2009) and Molanes Lopez, Van Keilegom and Veraverbeke (2009) so that the comparison with the smoothed and the empirical profile method is obtained without repeating the procedure in the above two papers.

We draw 1,000 random samples of size $n = 200$ and 400 from the following mixture copula :

$$C(x, y; \theta_1, \theta_2, \lambda) = \lambda\{x^{-\theta_1} + y^{-\theta_1} - 1\}^{-1/\theta_1} + (1-\lambda)\exp\{-((-\log x)^{\theta_2} + (-\log y)^{\theta_2})^{1/\theta_2}\}$$

with marginals being a standard normal, where $\theta_1 > 0$, $\theta_2 > 1$ and $\lambda \in [0, 1]$. Note that the above mixture copula becomes the Clayton copula and the Gumbel copula when $\lambda = 0$ and 1, respectively. In particular, we consider

$$(\theta_1, \theta_2, \lambda) = (2, 3, 0.0), (2, 3, 0.5), (2, 3, 1.0), \quad (x, y) = (0.25, 0.25), (0.5, 0.5), (0.75, 0.75)$$

and confidence levels $\gamma = 0.9, 0.95$. For computing the jackknife empirical likelihood based confidence interval, we employ the kernel $k(x) = \frac{3}{4}(1 - x^2)I(|x| \le 1)$ and the bandwidth

$$h = 0.2n^{-1/3}, \quad 0.5n^{-1/3}, \quad 0.8n^{-1/3},$$

since the optimal rate of the bandwidth in smoothing distribution estimation is of order $n^{-1/3}$. The bootstrap confidence intervals are obtained by using $1,000$ bootstrap samples.

In Tables 1 and 2 we report the empirical coverage probabilities for the jackknife empirical likelihood confidence intervals $I_\gamma(x, y; 0.2n^{-1/3})$, $I_\gamma(x, y; 0.5n^{-1/3})$ and $I_\gamma(x, y; 0.8n^{-1/3})$ and the bootstrap confidence interval $I_\gamma^*(x, y)$ based on $C_n(x, y)$. These two tables show that, in most cases, the jackknife empirical likelihood method has better coverage accuracy than the bootstrap method based on the empirical copula estimator. Also, the choice of $h = 0.5n^{-1/3}$ gives good results in general. Comparing with Tables 1 and 2 in Chen, Peng and Zhao (2009) and Table 1 in Molanes Lopez, Van Keilegom and Veraverbeke (2009), we found that the jackknife empirical likelihood method is comparable to the profile empirical likelihood methods in the above two papers. However, as explained in the introduction, the computation of the proposed jackknife empirical likelihood method is much less intensive than the profile empirical likelihood method.

4 Proofs

Lemma 1. Under the conditions of Theorem 1, we have

$$\sqrt{n}\{C_n(x, y) - C(x, y)\} \xrightarrow{D} W(x, y) := B(x, y) - \frac{\partial}{\partial x}C(x, y)B(x, 1) - \frac{\partial}{\partial y}C(x, y)B(1, y),$$

where $B(x, y)$ is a Gaussian process with zero mean and covariance

$$E\{B(x_1, y_1)B(x_2, y_2)\} = C(x_1 \wedge x_2, y_1 \wedge y_2) - C(x_1, y_1)C(x_2, y_2).$$

Further, for any fixed $x, y \in (0, 1)$,

$$\sqrt{n}\{\hat{C}_n(x, y) - C(x, y)\} \xrightarrow{d} N(0, \sigma^2(x, y)),$$

where $\sigma^2(x, y)$ is defined in (2.2), and

$$\frac{\partial \hat{C}_n(x, y)}{\partial x} \xrightarrow{p} \frac{\partial C(x, y)}{\partial x}, \quad \frac{\partial \hat{C}_n(x, y)}{\partial y} \xrightarrow{p} \frac{\partial C(x, y)}{\partial y}. \tag{4.1}$$

Proof. See Fermanian, Radulovic and Wegkamp (2004).

Lemma 2. Under the conditions of Theorem 1, we have

$$\sqrt{n}\{\frac{1}{n}\sum_{i=1}^n \hat{V}_i(x, y) - C(x, y)\} \xrightarrow{d} N(0, \sigma^2(x, y))$$

for any fixed $x, y \in (0, 1)$.

Proof. Write

$$\begin{aligned}\hat{V}_i(x,y) &= \textstyle\sum_{j=1}^n \{K(\frac{x-F_{n1}(X_j)}{h})K(\frac{y-F_{n2}(Y_j)}{h}) - K(\frac{x-F_{n1,i}(X_j)}{h})K(\frac{y-F_{n2,i}(Y_j)}{h})\} \\ &\quad +K(\frac{x-F_{n1,i}(X_i)}{h})K(\frac{y-F_{n2,i}(Y_i)}{h}) \\ &= \hat{V}_{i,1}(x,y)+\hat{V}_{i,2}(x,y).\end{aligned} \tag{4.2}$$

Since

$$\begin{cases} \sup_{1\le i\le n}|F_{n1,i}(x)-F_{n1}(x)| = \sup_{1\le i\le n}|\frac{1}{n-1}F_{n1}(x)-\frac{1}{n-1}I(X_i\le x)| \le n^{-1} \\ \sup_{1\le i\le n}|F_{n2,i}(y)-F_{n2}(y)| = \sup_{1\le i\le n}|\frac{1}{n-1}F_{n2}(y)-\frac{1}{n-1}I(Y_i\le y)| \le n^{-1} \\ \sum_{i=1}^n\{F_{n1,i}(x)-F_{n1}(x)\} = \sum_{i=1}^n\{F_{n2,i}(y)-F_{n2}(y)\} = 0, \end{cases} \tag{4.3}$$

it follows from the mean-value theorem that for fixed $x, y \in (0,1)$

$$\begin{aligned} & \tfrac{1}{n}\textstyle\sum_{i=1}^n \hat{V}_{i,1}(x,y) \\ =\ & \tfrac{1}{n}\textstyle\sum_{i=1}^n\sum_{j=1}^n \frac{F_{n1,i}(X_j)-F_{n1}(X_j)}{h}k(\frac{x-F_{n1}(X_j)}{h})K(\frac{y-F_{n2}(Y_j)}{h}) \\ & +\tfrac{1}{n}\textstyle\sum_{i=1}^n\sum_{j=1}^n \frac{F_{n2,i}(Y_j)-F_{n2}(Y_j)}{h}K(\frac{x-F_{n1}(X_j)}{h})k(\frac{y-F_{n2}(Y_j)}{h}) \\ & +\tfrac{1}{n}\textstyle\sum_{i=1}^n\sum_{j=1}^n\{\frac{1}{2}(\frac{F_{n1,i}(X_j)-F_{n1}(X_j)}{h})^2k'(\frac{x-\xi_{n1,i,j}}{h})K(\frac{y-\xi_{n2,i,j}}{h}) \\ & +\tfrac{1}{2}(\frac{F_{n2,i}(Y_j)-F_{n2}(Y_j)}{h})^2K(\frac{x-\xi_{n1,i,j}}{h})k'(\frac{y-\xi_{n2,i,j}}{h}) \\ & +\frac{F_{n1,i}(X_j)-F_{n1}(X_j)}{h}\frac{F_{n2,i}(Y_j)-F_{n2}(Y_j)}{h}k(\frac{x-\xi_{n1,i,j}}{h})k(\frac{y-\xi_{n2,i,j}}{h})\} \\ =\ & \tfrac{1}{n}\textstyle\sum_{i=1}^n\sum_{j=1}^n\{\frac{1}{2}(\frac{F_{n1,i}(X_j)-F_{n1}(X_j)}{h})^2k'(\frac{x-\xi_{n1,i,j}}{h})K(\frac{y-\xi_{n2,i,j}}{h}) \\ & +\tfrac{1}{2}(\frac{F_{n2,i}(Y_j)-F_{n2}(Y_j)}{h})^2K(\frac{x-\xi_{n1,i,j}}{h})k'(\frac{y-\xi_{n2,i,j}}{h}) \\ & +\frac{F_{n1,i}(X_j)-F_{n1}(X_j)}{h}\frac{F_{n2,i}(Y_j)-F_{n2}(Y_j)}{h}k(\frac{x-\xi_{n1,i,j}}{h})k(\frac{y-\xi_{n2,i,j}}{h})\}, \end{aligned} \tag{4.4}$$

where $\xi_{n1,i,j}$ is between $F_{n1,i}(X_j)$ and $F_{n1}(X_j)$ and $\xi_{n2,i,j}$ is between $F_{n2,i}(Y_j)$ and $F_{n2}(Y_j)$. By (4.3), we have

$$\begin{cases} \sup_{1\le i\le n} I(|\frac{x-\xi_{n1,i,j}}{h}|\le 1) \le I(x-h-n^{-1}\le F_{n1}(X_j)\le x+h+n^{-1}) \\ \sup_{1\le i\le n} I(|\frac{y-\xi_{n2,i,j}}{h}|\le 1) \le I(x-h-n^{-1}\le F_{n2}(Y_j)\le x+h+n^{-1}). \end{cases} \tag{4.5}$$

Hence, (4.3) and (4.5) imply that for some $M_1 > 0$

$$|\frac{1}{n}\sum_{i=1}^n \hat{V}_{i,1}(x,y)| \le \frac{M_1}{n^3h^2}\sum_{i=1}^n\sum_{j=1}^n\{I(|\frac{x-\xi_{n1,i,j}}{h}|\le 1)+I(|\frac{y-\xi_{n2,i,j}}{h}|\le 1)\} \le \frac{2M_1}{nh}, \tag{4.6}$$

which is of order $o(1/\sqrt{n})$. Since

$$K(\frac{x-F_{n1,i}(X_i)}{h})K(\frac{y-F_{n2,i}(Y_i)}{h}) = K(\frac{\frac{n-1}{n}x-\frac{1}{n}+F_{n1}(X_i)}{(n-1)h/n})K(\frac{\frac{n-1}{n}y+\frac{1}{n}-F_{n2}(Y_i)}{(n-1)h/n}),$$

Lemma 1 implies that

$$\sqrt{n}\{\frac{1}{n}\sum_{i=1}^n \hat{V}_{i,2}(x,y) - C(x,y)\} \xrightarrow{d} N(0,\sigma^2(x,y)).$$

Hence, the lemma follows from (4.6) and (4.4).

Lemma 3. Under the conditions of Theorem 1, we have

$$\frac{1}{n}\sum_{i=1}^{n}\{\hat{V}_i(x,y)-C(x,y)\}^2 \xrightarrow{p} \sigma^2(x,y)$$

for fixed $x, y \in (0,1)$.

Proof. It is straightforward to check that

$$\begin{cases} \frac{1}{n}\sum_{i=1}^{n}\{F_{nj,i}(x_1)-F_{nj}(x_1)\}\{F_{nj,i}(x_2)-F_{nj}(x_2)\} = \frac{1}{(n-1)^2}\{F_{nj}(x_1\wedge x_2)-F_{nj}(x_1)F_{nj}(x_2)\} \\ \frac{1}{n}\sum_{i=1}^{n}\{F_{n1,i}(x_1)-F_{n1}(x_1)\}\{F_{n2,i}(x_2)-F_{n2}(x_2)\} = \frac{1}{(n-1)^2}\{F_{n}(x_1,x_2)-F_{n1}(x_1)F_{n2}(x_2)\} \end{cases} \tag{4.7}$$

for $j=1,2$ and $x_1, x_2 \in R$. Let $\hat{V}_{i,1}(x,y)$ and $\hat{V}_{i,2}(x,y)$ be defined as in decomposition (4.2). Then, as in the proof of (4.6), we can show that

$$\begin{aligned} &\frac{1}{n}\sum_{i=1}^{n}\hat{V}_{i,1}^2(x,y) \\ =\ &\frac{1}{n}\sum_{i=1}^{n}\{\sum_{j=1}^{n}\frac{F_{n1,i}(X_j)-F_{n1}(X_j)}{h}k(\frac{x-F_{n1}(X_j)}{h})K(\frac{y-F_{n2}(Y_j)}{h}) \\ &+\sum_{j=1}^{n}\frac{F_{n2,i}(Y_j)-F_{n2}(Y_j)}{h}K(\frac{x-F_{n1}(X_j)}{h})k(\frac{y-F_{n2}(Y_j)}{h})\}^2+o_p(1), \end{aligned}$$

where by (4.7), we have that

$$\begin{aligned} &\frac{1}{n}\sum_{i=1}^{n}\hat{V}_{i,1}^2(x,y) \\ =\ &\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{1}{(n-1)^2h^2}\{F_{n1}(X_j\wedge X_k)-F_{n1}(X_j)F_{n1}(X_k)\}\times \\ &k(\frac{x-F_{n1}(X_j)}{h})K(\frac{y-F_{n2}(Y_j)}{h})k(\frac{x-F_{n1}(X_k)}{h})K(\frac{y-F_{n2}(Y_k)}{h}) \\ &+\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{1}{(n-1)^2h^2}\{F_{n2}(Y_j\wedge Y_k)-F_{n2}(Y_j)F_{n2}(Y_k)\}\times \\ &K(\frac{x-F_{n1}(X_j)}{h})k(\frac{y-F_{n2}(Y_j)}{h})K(\frac{x-F_{n1}(X_k)}{h})k(\frac{y-F_{n2}(Y_k)}{h}) \\ &+2\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{1}{(n-1)^2h^2}\{F_{n}(X_j,Y_k)-F_{n1}(X_j)F_{n2}(Y_k)\}\times \\ &k(\frac{x-F_{n1}(X_j)}{h})K(\frac{y-F_{n2}(Y_j)}{h})K(\frac{x-F_{n1}(X_k)}{h})k(\frac{y-F_{n2}(Y_k)}{h})+o_p(1) \\ =\ &\{x-x^2\}\{\frac{\partial}{\partial x}C(x,y)\}^2+\{y-y^2\}\{\frac{\partial}{\partial y}C(x,y)\}^2 \\ &+2\{C(x,y)-xy\}\{\frac{\partial}{\partial x}C(x,y)\}\{\frac{\partial}{\partial y}C(x,y)\}+o_p(1). \end{aligned} \tag{4.8}$$

Note that in the last step we have used (4.1). Similarly, we can show that

$$\begin{aligned} &\frac{1}{n}\sum_{i=1}^{n}\hat{V}_{i,1}(x,y)\hat{V}_{i,2}(x,y) \\ =\ &\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}K(\frac{x-F_{n1}(X_i)}{h})K(\frac{y-F_{n2}(Y_i)}{h})\frac{F_{n1,i}(X_j)-F_{n1}(X_j)}{h}k(\frac{x-F_{n1}(X_j)}{h})K(\frac{y-F_{n2}(Y_j)}{h}) \\ &+\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}K(\frac{x-F_{n1}(X_i)}{h})K(\frac{y-F_{n2}(Y_i)}{h})\frac{F_{n2,i}(Y_j)-F_{n2}(Y_j)}{h}K(\frac{x-F_{n1}(X_j)}{h})k(\frac{y-F_{n2}(Y_j)}{h})+o_p(1) \\ =\ &\sum_{j=1}^{n}\frac{1}{n}\sum_{i=1}^{n}K(\frac{x-F_{n1}(X_i)}{h})K(\frac{y-F_{n2}(Y_i)}{h})\frac{1}{(n-1)h}\{F_{n1}(X_j)-I(X_i\le X_j)\}k(\frac{x-F_{n1}(X_j)}{h})K(\frac{y-F_{n2}(Y_j)}{h}) \\ &+\sum_{j=1}^{n}\frac{1}{n}\sum_{i=1}^{n}K(\frac{x-F_{n1}(X_i)}{h})K(\frac{y-F_{n2}(Y_i)}{h})\frac{1}{(n-1)h}\{F_{n2}(Y_j)-I(Y_i\le Y_j)\}\times \\ &K(\frac{x-F_{n1}(X_j)}{h})k(\frac{y-F_{n2}(Y_j)}{h})+o_p(1) \end{aligned} \tag{4.9}$$

$$\begin{aligned}
&= \tfrac{1}{(n-1)h}\sum_{j=1}^n\{\tfrac{1}{n}\sum_{i=1}^n K(\tfrac{x-F_{n1}(X_i)}{h})K(\tfrac{y-F_{n2}(Y_i)}{h})\}F_{n1}(X_j)k(\tfrac{x-F_{n1}(X_j)}{h})K(\tfrac{y-F_{n2}(Y_j)}{h}) \\
&\quad -\tfrac{1}{(n-1)h}\sum_{j=1}^n\{\tfrac{1}{n}\sum_{i=1}^n K(\tfrac{x-F_{n1}(X_i)}{h})K(\tfrac{y-F_{n2}(Y_i)}{h})I(F_{n1}(X_i)\le F_{n1}(X_j))\}k(\tfrac{x-F_{n1}(X_j)}{h})K(\tfrac{y-F_{n2}(Y_j)}{h}) \\
&\quad +\tfrac{1}{(n-1)h}\sum_{j=1}^n\{\tfrac{1}{n}\sum_{i=1}^n K(\tfrac{x-F_{n1}(X_i)}{h})K(\tfrac{y-F_{n2}(Y_i)}{h})\}F_{n2}(Y_j)K(\tfrac{x-F_{n1}(X_j)}{h})k(\tfrac{y-F_{n2}(Y_j)}{h}) \\
&\quad -\tfrac{1}{(n-1)h}\sum_{j=1}^n\{\tfrac{1}{n}\sum_{i=1}^n K(\tfrac{x-F_{n1}(X_i)}{h})K(\tfrac{y-F_{n2}(Y_i)}{h})I(F_{n2}(Y_i)\le F_{n2}(Y_j))\}\times \\
&\quad K(\tfrac{x-F_{n1}(X_j)}{h})k(\tfrac{y-F_{n2}(Y_j)}{h})+o_p(1) \\
&= C(x,y)x\tfrac{\partial}{\partial x}C(x,y)-C(x,y)\tfrac{\partial}{\partial x}C(x,y) \\
&\quad +C(x,y)y\tfrac{\partial}{\partial y}C(x,y)-C(x,y)\tfrac{\partial}{\partial y}C(x,y)+o_p(1).
\end{aligned} \tag{4.10}$$

It follows from (4.3) that

$$\max_{1\le i\le n}\sup_x |F_{nj,i}(x)-F_j(x)| \xrightarrow{p} 0 \tag{4.11}$$

for $j=1,2$. By Lemma 1 and (4.11), we have

$$\frac{1}{n}\sum_{i=1}^n \hat{V}_{i,2}^2(x,y) \xrightarrow{p} C(x,y). \tag{4.12}$$

Hence, the lemma follows from (4.8), (4.9), (4.12) and the fact that

$$\frac{1}{n}\sum_{i=1}^n \hat{V}_i(x,y) \xrightarrow{p} C(x,y),$$

which is implied by Lemma 2.

Proof of Theorem 1. This follows from Lemmas 2 and 3.

Proof of Theorem 2. It follows from the mean-value theorem that

$$\begin{aligned}
\hat{V}_{i,1}(x,y) &= \sum_{j=1}^n\{\tfrac{F_{n1,i}(X_j)-F_{n1}(X_j)}{h}k(\tfrac{x-\xi_{n1,i,j}}{h})K(\tfrac{y-\xi_{n2,i,j}}{h}) \\
&\quad +\tfrac{F_{n2,i}(Y_j)-F_{n2}(Y_j)}{h}K(\tfrac{x-\xi_{n1,i,j}}{h})k(\tfrac{y-\xi_{n2,i,j}}{h})\},
\end{aligned}$$

where $\xi_{n1,i,j}$ lies between $F_{n1,i}(X_j)$ and $F_{n1}(X_j)$, $\xi_{n2,i,j}$ lies between $F_{n2,i}(Y_j)$ and $F_{n2}(Y_j)$, and $\hat{V}_{i,1}(x,Y)$ is defined in the proof of Lemma 2. By (4.3) and (4.5), we can show that $\max_{1\le i\le n}|\hat{V}_{i,1}(x,y)|$ is bounded, and so is $\max_{1\le i\le n}|\hat{V}_i(x,y)|$. By standard arguments of the empirical likelihood method (see Owen (1988)), we can show that

$$\hat{l}_n(x,y;C(x,y)) = \{\sum_{i=1}^n \hat{V}_i(x,y)-C(x,y)\}^2/\sum_{i=1}^n\{\hat{V}_i(x,y)-C(x,y)\}^2+o_p(1) \xrightarrow{d} \chi^2(1)$$

as $n\to\infty$. The details are omitted here.

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Table 1: Empirical coverage probabilities for the Jackknife empirical likelihood based confidence interval $I_\alpha(x, y; h)$ and the bootstrap confidence interval $I^*_\alpha(x, y)$ based on $C_n(x, y)$ with sample size $n = 200$. Bandwidths are chosen as $h_1 = 0.2n^{-1/3}$, $h_2 = 0.5n^{-1/3}$ and $h_3 = 0.8n^{-1/3}$.

(λ, x, y)	$I_{0.90}(x, y; h_1)$ $I_{0.95}(x, y; h_1)$	$I_{0.90}(x, y; h_2)$ $I_{0.95}(x, y; h_2)$	$I_{0.90}(x, y; h_3)$ $I_{0.95}(x, y; h_3)$	$I^*_{0.90}(x, y)$ $I^*_{0.95}(x, y)$
$(0.0, 0.25, 0.25)$	0.876 0.936	0.903 0.945	0.815 0.897	0.860 0.910
$(0.0, 0.50, 0.50)$	0.864 0.919	0.893 0.942	0.815 0.904	0.870 0.933
$(0.0, 0.75, 0.75)$	0.876 0.939	0.876 0.938	0.594 0.744	0.879 0.926
$(0.5, 0.25, 0.25)$	0.867 0.921	0.890 0.940	0.877 0.936	0.834 0.909
$(0.5, 0.50, 0.50)$	0.838 0.903	0.873 0.931	0.910 0.954	0.778 0.859
$(0.5, 0.75, 0.75)$	0.865 0.921	0.874 0.927	0.789 0.874	0.885 0.934
$(1.0, 0.25, 0.25)$	0.862 0.929	0.907 0.947	0.846 0.918	0.826 0.880
$(1.0, 0.50, 0.50)$	0.848 0.907	0.880 0.938	0.884 0.941	0.842 0.904
$(1.0, 0.75, 0.75)$	0.825 0.905	0.814 0.908	0.768 0.868	0.849 0.913

Table 2: Empirical coverage probabilities for the Jackknife empirical likelihood based confidence interval $I_\alpha(x, y; h)$ and the bootstrap confidence interval $I^*_\alpha(x, y)$ based on $C_n(x, y)$ with sample size $n = 400$. Bandwidths are chosen as $h_1 = 0.2n^{-1/3}$, $h_2 = 0.5n^{-1/3}$ and $h_3 = 0.8n^{-1/3}$.

(λ, x, y)	$I_{0.90}(x, y; h_1)$ $I_{0.95}(x, y; h_1)$	$I_{0.90}(x, y; h_2)$ $I_{0.95}(x, y; h_2)$	$I_{0.90}(x, y; h_3)$ $I_{0.95}(x, y; h_3)$	$I^*_{0.90}(x, y)$ $I^*_{0.95}(x, y)$
$(0.0, 0.25, 0.25)$	0.888 0.937	0.890 0.942	0.832 0.900	0.861 0.923
$(0.0, 0.50, 0.50)$	0.889 0.941	0.904 0.945	0.857 0.917	0.897 0.933
$(0.0, 0.75, 0.75)$	0.880 0.934	0.876 0.934	0.666 0.800	0.869 0.927
$(0.5, 0.25, 0.25)$	0.874 0.926	0.881 0.932	0.887 0.942	0.826 0.895
$(0.5, 0.50, 0.50)$	0.793 0.865	0.807 0.890	0.853 0.910	0.765 0.844
$(0.5, 0.75, 0.75)$	0.854 0.929	0.865 0.915	0.765 0.871	0.883 0.946
$(1.0, 0.25, 0.25)$	0.875 0.932	0.881 0.939	0.875 0.929	0.835 0.889
$(1.0, 0.50, 0.50)$	0.877 0.927	0.887 0.940	0.896 0.941	0.861 0.920
$(1.0, 0.75, 0.75)$	0.771 0.855	0.754 0.843	0.701 0.809	0.817 0.897