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EQUIVALENT LOCAL MARTINGALE MEASURES AND MARKET INCOMPLETENESS IN A CONTINUOUS TIME SEMI-MARKOV REGIME SWITCHING MODEL

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Equivalent local martingale measures and market incompleteness in a continuous time semi-Markov regime switching model

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Abstract

We present a continuous time semi-Markov regime switching model with discrete state-space. The notion of equivalent local martingale measures is studied in this setting. We go on to prove that this market is incomplete. We then extend our market by adding some assets. In this setting, we derive a condition for uniqueness of the equivalent local martingale measure. Furthermore, we derive a condition for completeness of the market. We show that these two conditions are in fact equivalent but only because we work with a finite state space.

1 Introduction

Finding accurate and tractable models to describe the behavior of financial assets is a very active topic of research in modern quantitative finance. Many models have been put forward. Let us cite, amongst others, stochastic volatility models, jump-diffusion models, Lévy models and regime switching models.

Regime switching models are based on the simple and intuitive idea that the economic environment is subject to regular changes and that this should somehow be reflected in the model. The most simple example of a change in the economic environment is that of the successive periods of expansion and recession in the economy. In our view, at the modelling level, this change in market conditions should be reflected as switch in the value of the parameters influencing the model. This is the idea behind regime switching models.

Much of the focus of the litterature on regime switching models in continuous time has been concentrated on homogeneous Markov switching. Let us cite amongst others

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(4; 10; 14; 15; 17; 23). However, many authors have expressed doubts about the validity of such models. In (29), the authors argue against homogeneous Markov switching models on the grounds that it forces duration distributions in each state to be exponentially distributed. Furthermore, many papers on economic cycles have rejected the Markov switching model (see (7; 8; 9; 16)). Finally, some authors express doubts and present results that reject any model possessing the Markov property (see (21)). For more about the Markov property (or rather the lack of it) in financial markets consult (12; 13).

This has lead some authors to consider semi-Markov regime switching models. These models keep the intuitive properties of homogeneous Markov processes while allowing for more flexibility in the duration distributions. What is also interesting is that homogeneous Markov models are just a special case of semi-Markov switching models. The capacity of these models to explain some of the observed features of financial data is presented in (3). Various applications to finance can be found in (18; 19; 29; 30).

A topic of great interest in all financial models applied to quantitative finance is the notion of local equivalent martingale measures and its implications for pricing and hedging financial derivatives. There are two essential ideas. The first is the existence of equivalent local martingale measures. This is linked to absence of arbitrage ((6)). The second idea is the notion of uniqueness of the equivalent local martingale measure. This is related to market completeness i.e. the possibility of building a self-financing replicating portfolio. A rule-of-thumb is that market completeness is equivalent to uniqueness of the equivalent local martingale measure. This is not always true. The interested reader will find more on this topic in (1; 2).

The study of market (in)-completeness and its link to equivalent local martingale measures has received little attention in the regime switching context. Many authors simply rely on the intuition that given there are more sources of risk than assets on the market, the market is incomplete (see (10; 18; 30)). The aim of this paper is to formalize the notion of market incompleteness in the context of finite state space semi-Markov regime switching models.

In this paper, we first define a framework that will allow to view semi-Markov processes as marked point processes. This will enable us to use a suitable version of the Girsanov theorem and this will allow us to characterize equivalent local martingale measures. We will show that there is an infinite number of equivalent local martingale measures.

We prove that the model is incomplete as it is impossible to build a self-financing replicating portfolio. We then add some further assets to the market and determine a condition for this new market to possess a unique equivalent local martingale measure. We also study conditions under which this new market is complete. It turns out the conditions for market completeness and uniqueness of the martingale measure are closely

related and break down to being equivalent in our model **but only because we work with a finite state space**. This is in line with the findings in (1).

The paper is structured as follows. The first chapter provides a general mathematical setting for our model. We move on to the study of martingale measures and market incompleteness. After this we deal with market extension and its impact on uniqueness of the martingale measure. Finally, we study market completeness fo the extended market and show the equivalence between this and uniqueness of the martingale measure.

2 Semi-Markov regime switching model

We consider a financial market defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume the existence of a brownian motion B. We suppose that the financial market is defined for all times $t \in [0, T]$.

Define the set $E \subset \mathbb{R}$ by $E = \{1, ..., m\}$ for a fixed $m \in \mathbb{N}$. Define $\mathcal{E}$ as the sigma-algebra of all the parts of E. So $(E, \mathcal{E})$ is a measurable space.

For each $n \in \mathbb{N}$, let (X_n, T_n) be a pair of random variables taking values in $E \times \mathbb{R}^+$.

Remark 1. We can interpret the sequence $(X_n)_{n\geq 0}$ as being the successive states of a system and $(T_n)_{n\geq 0}$ as the switching times of the system.

Furthermore, we suppose that the process $(X, T) = \{X_n, T_n; n \geq 0\}$ is a homogeneous Markov renewal process with state space E. The associated semi-Markov kernel is denoted by $Q_{ij}(t)$ i.e. the probability that, given the current state i, it will stay in this state for a time less than or equal to t before moving to state j. We denote by P the transition matrix of the embedded Markov chain.

Remark 2. It is clear that given the number of states is finite, the number of jumps in a finite time interval is almost surely finite (for a proof see (26)).

Assumption 1. *We impose the following conditions:*

- *No fictitious transitions are allowed i.e.* $P_{ii} = 0$.
- *No instantaneous transitions are allowed i.e.* $Q_{ij}(0) = 0$.
- *All states in E "communicate" at all times i.e.* $Q_{ij}(t) > 0$,$\forall t > 0$.

Let us define ν_t by

$$\nu_t := \sup(n \geq 0 : T_n \leq t)$$

with $n \in \mathbb{N}$ and $t \in \mathbb{R}^+$ and Y_t as

$$Y_t := X_{\nu_t}$$

Then, the process Y is a semi-Markov process with kernel Q.

Assumption 2. *We suppose that Y is independent of W.*

We define $\mathcal{F}_t$ to be the completed filtration generated by Y_t and B_t i.e. $\mathcal{F}_t = \sigma(N, Y_s,\ W_s,\ s \leq t, N \in \mathcal{N})$ where $\mathcal{N}$ is the collection of all $\mathbb{P}$-null sets of $\mathcal{F}$.

We introduce processes μ_t and σ_t. We suppose that process μ_t can take any of the following values $(\mu^1, ..., \mu^m)$ and σ_t can take any of the following $\sigma = (\sigma^1, ..., \sigma^m)$. The processes are assumed to be driven by Y_t. We impose the following structure

$$\begin{aligned} \mu_t &= \mu(Y_t) = \sum_{k=1}^{m} \mu^{(k)} \mathbb{1}_{\{Y_t = k\}} \\ \sigma_t &= \sigma(Y_t) = \sum_{k=1}^{m} \sigma^{(k)} \mathbb{1}_{\{Y_t = k\}} \end{aligned}$$

Let us introduce process R_t

$$dR_t = (\mu_t - \frac{\sigma_t^2}{2})dt + \sigma_t dB_t \tag{1}$$

with $G_0 = 0$.

Proposition 3. *Equation 1 is well defined.*

Proof. For equation 1 to be well defined it is sufficient that μ_t and σ_t be progressively measurable and that for all $T < \infty$:

$$\int_0^T \sigma_t^2 dt \leq \infty \ \ \mathbb{P} - a.s.$$
$$\int_0^T |\mu_t - \frac{\sigma_t^2}{2}| dt \leq \infty \ \ \mathbb{P} - a.s.$$

Given that all values of μ and σ are supposed to be finite, the last inequalities are trivially verified. A sufficient condition for progressive measurability is to be right-continuous and adapted. Right-continuity follows by our assumptions on Y and by definition of μ and σ. Given our filtration $\mathcal{F}_t$, the processes are also adapted. The result follows.

□

The market is supposed to be composed of two assets. One asset S^0 with constant rate of return r. This imposes the following dynamics:

$$dS_t^0 = rS_t^0 dt$$

Or, to put it differently, $S_t^0 = \exp(rt)$.

A second asset S^1 ("the underlying") is defined by:

$$S_t^1 = S_0^1 \exp(R_t) \tag{2}$$

Or, to put it differently:

$$dS_t^1 = S_t^1(\mu_t dt + \sigma_t dB_t)$$

The use of Itô's formula guarantees that equation 2 is well defined.

3 Martingale Measures

This section will characterize equivalent local martingale measures.

Lemma 4. *Y_t is a semimartingale.*

Proof. Following (26), Y has only a finite number of jumps in any finite interval. All jumps of Y are bounded and so Y is of finite variation. Furthermore, given our filtration, Y is clearly adapted. By construction, it is also clear that Y is a càdlàg process. This implies that Y is a semimartingale as an adapted càdlàg finite variation process. □

We denote the set of all possible jumps of Y by Z and the elements of Z by z_i. Given there are m states, there are $m(m-1)$ possible jumps. Let Z_n denote the size of the nth jump of Y i.e. $Z_n := Y_{T_n} - Y_{T_{n-1}} = X_n - X_{n-1}$. The jump measure of Y is the random integer-valued measure μ on $(0, \infty) \times Z$ defined by

$$\mu = \sum_{n=1}^{\infty} \delta_{(T_n, Z_n)}$$

We can write

$$Y_t = Y_0 + \int_0^t \int_Z z\mu(ds, dz)$$

This can be written as

$$Y_t = Y_0 + \sum_{i=1}^{m(m-1)} z_i N_t(z_i)$$

where

$$N_t(z_i) = \sum_{n \geq 1} \mathbb{1}_{\{T_n \leq t\}} \mathbb{1}_{\{Z_n = z_i\}}$$

The compensator associated with the jump measure μ is given by

$$\nu(ds, dz) = \lambda_s(dz)ds \tag{3}$$

Where the intensity $\lambda_s(dz)$ is defined as (for more details, see (25)):

$$\lambda_s(dz) = \sum_{i=1}^{m(m-1)} \sum_{n\geq 0} \mathbb{1}_{\{T_n<s\leq T_{n+1}\}} \frac{P_{X_n,X_n+z_i} g(X_n, X_n+z_i, t-T_n)}{1-\sum_{z_i} Q_{X_n,X_n+z_i}(t-T_n)} \delta_{\{z_i\}}(dz)$$

where $g(X_n, X_n+z_i, t-T_n)$ is the density of the waiting time distribution between state X_n and X_n+z_i calculated at time $t-T_n$.

Furthermore, the intensity associated with the point process $N_t(z_i)$ is simply $\lambda_t(z_i)$. Remember our process Y_t is closely linked to the multivariate point process $N_t = (N_t(z_1), ..., N_t(z_{m(m-1)}))$ that has (multivariate) intensity $\lambda_t = (\lambda_t(z_1), ..., \lambda_t(z_{m(m-1)}))$. It may be interesting to recall that for every z_i and every predictable process $H_s(z_i)$, the process $\int_0^t H_s(z_i)(dN_s(z_i) - \lambda_s(z_i)ds)$ is an $\mathcal{F}_t$- local martingale.

Because of our assumptions about process Y, the intensity $\lambda_t(z_i)$ is strictly positive for every $t \leq T$ and for every z_i. We will see that this plays an important role when we study the completeness of the market.

One can now discuss the existence of local martingale measures. This is done by following this suitable version of the Girsanov theorem (see (1)).

Theorem 5. *Let $\mathcal{P}$ denote the predictable σ-algebra. Let θ be a progressively measurable process such that*

$$\int_0^t \theta_s^2 ds < \infty$$

Consider the multivariate point process N_t previously defined with $(\mathbb{P}, \mathcal{F}_t)$-intensity λ_t. Consider a $\mathcal{P}$-measurable multivariate process $\psi = (\psi_t(z_1), ..., \psi_t(z_{m(m-1)}))$ such that $\mathbb{P}$-a.s. and for $t \in [0, T]$

$$\sum_{i=1}^{m(m-1)} \int_0^t \psi_s(z_i)\lambda_s(z_i)ds < \infty$$

Define the process L by:

$$L_t = \exp\left\{-\frac{1}{2}\int_0^t \theta_s^2 ds + \int_0^t \theta_s dB_s\right\} \prod_{i=1}^{m(m-1)} \left[\exp\left\{\int_0^t (1-\psi_s(z_i))\lambda_s(z_i)ds\right\} \prod_{n=1}^{N_t(z_i)} \psi_{T_n}(z_i)\right] \tag{4}$$

And suppose that for all finite t:

$$\mathbb{E}^{\mathbb{P}}[L_t] = 1$$

Then, there exists a probability measure $\mathbb{Q}$ on $\mathcal{F}$ equivalent to $\mathbb{P}$ with

$$d\mathbb{Q} = L_t d\mathbb{P} \tag{5}$$

such that if $B_t^{\mathbb{Q}}$ is defined as

$$dB_t^{\mathbb{Q}} = dB_t - \theta_t dt$$

then $B_t^{\mathbb{Q}}$ is a $\mathbb{Q}$-brownian motion and the multivariate point process N_t has $\mathbb{Q}$-intensity given by

$$(\psi_t(z_1)\lambda_t(z_1), ..., \psi_t(z_{m(m-1)})\lambda_t(z_{m(m-1)})) \tag{6}$$

Proof. For a proof see (22).

□

This theorem will help us prove that there is an infinite number of equivalent local martingale measures.

Theorem 6. *The number of equivalent local martingale measures is infinite.*

Proof. Choose

$$\theta_t = \frac{r - \mu_t}{\sigma_t}$$

and suppose it satisfies the hypotheses of theorem 5 and furthermore let ψ_t be any process that satisfies the hypotheses of theorem 5. The number of such processes is infinite. Define L_t as in (4) and $\mathbb{Q}$ as in (5). The measure so defined is an equivalent local martingale measure. The result follows.

□

4 Market incompleteness

The number of equivalent local martingale measures leads us to believe that the market is incomplete in the sense that it is impossible to replicate a contingent claim by using a self-financing strategy based only S^0 and S^1. To prove this we will need the next theorem (for a proof see (22)).

Theorem 7. *Any $\mathcal{F}_t$-local martingale M_t has the representation*

$$M_t = M_0 + \int_0^t \eta_s dB_s + \sum_{i=1}^{m(m-1)} \int_0^t H_s(z_i)(dN_s(z_i) - \lambda_s(z_i)ds)$$

where η_t is predictable and square integrable and $(H_t(z_1), ..., H_t(z_{m(m-1)}))$ is an $\mathcal{F}_t$-predictable process with $H_t(z_i)$ integrable with respect to $\lambda_t(z_i)$.

In our market model, a strategy is a pair of stochastic processes (ϕ^0, ϕ^1) such that ϕ_t^0 represents the number of units of the riskless asset held at time t and ϕ_t^1 represents the number of shares of the risky asset held at time t. The processes ϕ^0 and ϕ^1 are supposed to be predictable. The value V_t of the portfolio associated to this strategy is $V_t = \phi_t^0 S_t^0 + \phi_t^1 S_t^1$. The discounted value $\tilde{V}_t$ of the portfolio is equal to $\tilde{V}_t = e^{-rt} V_t$. It is well known that a portfolio is self-financing iff $\tilde{V}_t = V_0 + \int_0^t \phi_s^1 d\tilde{S}_s^1$ for $t \in [0, T]$. A contingent claim $U(S_T^1)$ is a square integrable $\mathcal{F}_T$-measurable random variable. A portfolio is said to be replicating if $V_T = U(S_T^1)$. We will now show that in our setting there is no self-financing replicating portfolio.

Theorem 8. *The semi-Markov regime switching market is incomplete in the sense that not every contingent claim is replicable by a self-financing strategy.*

Proof. We will suppose there exists a self-financing replicating strategy and show that this leads to an impossibility.

Given we have a self-financing replicating portfolio, there exist a process ϕ_t^1 such that $\forall t \in [0, T]$:

$$\tilde{V}_t = V_0 + \int_0^t \phi_s^1 \tilde{S}_s^1 \sigma(Y_s) dB_s^{\mathbb{Q}}$$

with $V_T = U(S_T^1)$.

This makes $\tilde{V}_t$ a local martingale. So by theorem 7, there exists some processes such that $\forall t \in [0, T]$, $\tilde{V}_t$ can be written as:

$$\tilde{V}_t = V_0 + \int_0^t \eta_s dB_s^{\mathbb{Q}} + \sum_{i=1}^{m(m-1)} \int_0^t H_s(z_i)(dN_s(z_i) - \psi_s(z_i)\lambda_s(z_i)ds) \tag{7}$$

This means that ϕ_t^1 must be such that the following equality holds $\forall t \in [0, T]$:

$$V_0 + \int_0^t \phi_s^1 \tilde{S}_s^1 \sigma(Y_s) dB_s^{\mathbb{Q}} = V_0 + \int_0^t \eta_s dB_s^{\mathbb{Q}} + \sum_{i=1}^{m(m-1)} \int_0^t H_s(z_i)(dN_s(z_i) - \psi_s(z_i)\lambda_s(z_i)ds)$$

Of course, this is not possible. The result follows. □

5 Extending the market

The aim of this subsection is twofold. First, we want to give conditions under which we can obtain a unique equivalent local martingale measure. This will entail adding assets that depend on the multivariate point process in the market. Secondly, we want to give conditions under which our extended market is complete. What will come out quite clearly is that both conditions are closely related and that, because of our finite state space, they are in fact equivalent. But this might not be true in general.

5.1 Uniqueness of the equivalent local martingale measure

Suppose our semi-Markov process has m possible states. Consider $m(m-1)$ assets of price C^i, $1 \leq i \leq m(m-1)$ with the following dynamics

$$dC_t^i = C_{t^-}^i(\alpha_t^i dt + \sum_{j=1}^{m(m-1)} \gamma_t^i(z_j) dN_t(z_j))$$

where α^i and $\gamma^i(z_j)$ are predictable processes.

Let G be given by:

$$G = \begin{bmatrix} \gamma_t^1(z_1) & \cdots & \gamma_t^1(z_{m(m-1)}) \\ \vdots & \ddots & \vdots \\ \gamma_t^{m(m-1)}(z_1) & \cdots & \gamma_t^{m(m-1)}(z_{m(m-1)})) \end{bmatrix} \tag{8}$$

Theorem 9. *If G is injective then the market composed of $(S^0, S^1, C^1, ..., C^{m(m-1)})$ has a unique equivalent local martingale measure.*

Proof. We want to apply theorem 5. Define

$$\theta_t = \frac{r - \mu(Y_t)}{\sigma(Y_t)}$$

This condition alone is not sufficient to obtain a unique equivalent local martingale measure. ψ_t still has to be specified and this will be done by imposing that the discounted prices of **all** assets are local martingales under $\mathbb{Q}$. Given ψ_t and applying Girsanov's theorem we obtain that our new assets have the following dynamics (for $1 \leq i \leq m(m-1)$) under $\mathbb{Q}$:

$$dC_t^i = C_{t^-}^i((\alpha_t^i + \sum_{j=1}^{m(m-1)} \gamma_t^i(z_j)\psi_t(z_j)\lambda_t(z_j))dt + \sum_{j=1}^{m(m-1)} \gamma_t^i(z_j)(dN_t(z_j) - \psi_t(z_j)\lambda_t(z_j)dt))$$

So a sufficient condition to obtain existence of an equivalent local martingale measure is that there exists a multivariate process $\psi_t \geq 0$ such that for every $1 \leq i \leq m(m-1)$

and $\forall t \in [0,T]$:

$$\alpha_t^i + \sum_{j=1}^{m(m-1)} \gamma_t^i(z_j)\psi_t(z_j)\lambda_t(z_j) = r$$

Define A as the following $m(m-1) \times m(m-1)$matrix:

$$\begin{bmatrix} \gamma_t^1(z_1)\lambda_t(z_1) & \cdots & \gamma_t^1(z_{m(m-1)})\lambda_t(z_{m(m-1)}) \\ \vdots & \ddots & \vdots \\ \gamma_t^{m(m-1)}(z_1)\lambda_t(z_1) & \cdots & \gamma_t^{m(m-1)}(z_{m(m-1)})\lambda_t(z_{m(m-1)}) \end{bmatrix}$$

If we define the vector ψ as the column vector of dimension $m(m-1) \times 1$ having the $\psi_t(z_j)$ as components and vector R as the $m(m-1) \times 1$ vector having $r - \alpha_t^i$ as components. Then a sufficient condition to obtain the existence of an equivalent local martingale measure is that $\forall t \in [0,T]$ there exists a solution to the system:

$$A\psi = R \tag{9}$$

In order for this solution to be unique, we need that A (seen as an operator) be injective.

Let us notice that A can be rewritten in the form $A = G.L$ with L given by:

$$\begin{bmatrix} \lambda_t(z_1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_t(z_{m(m-1)}) \end{bmatrix} \tag{10}$$

Because of our decomposition, when L is injective, having A injective is equivalent to having G injective. Because all the $\lambda_t(z_i)$'s are strictly positive, it is clear that $rank(L) = m(m-1)$ and so L is injective. So, in order to have a unique equivalent local martingale measure, we need G to be injective.

□

5.2 Market completeness

Working under this unique martingale measure, we will now investigate the conditions under which the market is complete in the sense that every contingent claim is replicable by a self-financing strategy involving only the risk-free asset and the assets available on the market.

A strategy is now a multidimensional predictable process $(\phi_t^0, \phi_t^1, \beta_t^1, ..., \beta_t^{m(m-1)})$ where ϕ^0 and ϕ^1 are as defined previously and β^i represents the number of units of asset C^i held at time t.

The value V_t of the portfolio associated to this strategy is $V_t = \phi_t^0 S_t^0 + \phi_t^1 S_t^1 + \sum_{i=1}^{m(m-1)} \beta_t^i C_t^i$. The discounted value $\tilde{V}_t$ of the portfolio is equal to $\tilde{V}_t = e^{-rt} V_t$. The portfolio is self-financing iff $\tilde{V}_t = V_0 + \int_0^t \phi_s^1 d\tilde{S}_s^1 + \sum_{i=1}^{m(m-1)} \int_0^t \beta_s^i d\tilde{C}_s^i$.

We are going to show that in this market, there exist a self-financing replicating portfolio and that the price of the option is then uniquely determined by P_t.

Theorem 10. *If G^T is surjective*[1] *then the market composed of assets $(S^0, S^1, C^1, ..., C^{m(m-1)})$ is complete.*

Proof. Define the square integrable martingale $M_t = \mathbb{E}^{\mathbb{Q}}[e^{-rT} U(S_T^1) | \mathcal{F}_t]$. By theorem 7, for every $t \in [0, T]$ we can write:

$$M_t = M_0 + \int_0^t \eta_s dB_s^{\mathbb{Q}} + \sum_{i=1}^{m(m-1)} \int_0^T H_s(z_i)(dN_s(z_i) - \phi_s(z_i)\lambda_s(z_i)ds)$$

If we choose $\phi_t^0 = M_t - \phi_t^1 \tilde{S}_t^1 - \sum_{i=1}^{m(m-1)} \beta_t^i \tilde{C}_t^i$, $\phi_t^1 = \frac{\eta_t}{\tilde{S}_t^1 \sigma(Y_t)}$ and the β^i such that for every i and every $t \in [0, T]$:

$$\sum_{j=1}^{m(m-1)} \beta_t^j \tilde{C}_{t^-}^j \gamma_t^j(z_i) = H_t(z_i) \tag{11}$$

then we have a self-financing strategy. Furthermore, it is then clear that $V_T = U(S_T^1)$ because $\tilde{V}_t = M_t$.

We will now show that the condition for equation 11 to hold true is that G^T be surjective. Indeed, equation 11 can be written as:

$$W\beta = H \tag{12}$$

with H the colum vector of size $m(m-1)$ whose components are given by $H_t(z_i)$, β the colum vector of size $m(m-1)$ whose components are given by β_t^j and where W is the $m(m-1) \times m(m-1)$ matrix defined as follows:

$$\begin{bmatrix} \gamma_t^1(z_1)\tilde{C}_{t^-}^1 & \cdots & \gamma_t^{m(m-1)}(z_1)\tilde{C}_{t^-}^{m(m-1)} \\ \vdots & \ddots & \vdots \\ \gamma_t^1(z_{m(m-1)})\tilde{C}_{t^-}^1 & \cdots & \gamma_t^{m(m-1)}(z_{m(m-1)})\tilde{C}_{t^-}^{m(m-1)} \end{bmatrix}$$

So for a given H, we want to find β such that (12) holds for every $t \in [0, T]$. We want W to be surjective.

[1] Where G^T denotes the transpose of G.

But again we can rewrite W as $G^T.C$ with G defined as previously and C defined as:

$$\begin{bmatrix} \tilde{C}^1_{t^-} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \tilde{C}^{m(m-1)}_{t^-} \end{bmatrix}$$

Given the $\tilde{C}^i_{t^-}$ are strictly positive (being the price of an asset), $rank(C) = m(m-1)$ and so C is surjective. Because of our decomposition, when C is surjective, having W surjective is equivalent to having G^T surjective. And so, in order for the market to be complete we need to have G^T surjective. □

Remark 3. As is the case in jump-diffusion models, our results show us that in order to obtain a complete market in our framework, we need to add assets that span the whole spectrum of possible jumps of the semi-Markov process.

To summarize our results, in order to obtain uniqueness of the martingale measure, we need that G be injective. In order to obtain market completeness, we need G^T to be surjective.

Corollary 11. *Uniqueness of the martingale measure is equivalent to market completeness.*

Proof. In a finite dimensional setting, G injective is equivalent to G^T surjective. Indeed, in our case, G injective is equivalent to $rank(G) = m(m-1)$ and G^T surjective is equivalent to $rank(G^T) = m(m-1)$ and so both conditions are equal.

□

So, in a finite-dimensional setting, our market model is complete iff it admits a unique equivalent martingale measure.

Remark 4. In an infinite dimensional setting, G injective is not equivalent to G^T surjective and so these results may not hold. For more on this fascinating topic, see (1).

References

[1] Björk T., Kabanov Y.M., Runggaldier W.J. ; "Bond market structure in the presence of marked point processes", Mathematical Finance, **7** (2), (1997): 211-223

[2] Björk T. ; *Arbitrage theory in continuous time*; Oxford university press, Oxford, 1998

[3] Bulla J., Bulla I.; "Stylized facts of financial time series and hidden semi-Markov models", Computational Statistics & Data Analysis, **51**(4), (2006): 2192-2209

[4] Chourdakis K. ; "Switching Lvy models in continuous time: finite distributions and option pricing", Working paper, CCFEA, University of Essex, (2006)

[5] Çinlar E. ; *Introduction to stochastic processes*, Prentice-Hall, Englewood Cliffs, New-Jersey, 1975

[6] Delbaen F., Schachermayer W.; "A general version of the fundamental theorem of asset pricing"; Math. Ann., **300**, (1994): 463-520

[7] Diebold F., Rudebusch G.; "A nonparametric investigation of duration dependence in the American business cycle", Journal of Political Economy, **98**, (1990): 596- 616

[8] Diebold F., Rudebusch G.; "Measuring the business cycles: a modern perspective", The Review of Economics and Statistics, **78**(1), (1996): 67-77

[9] Diebold F., Rudebusch G., Sichel D.; "Further evidence on business cycle duration dependence", Business cycles, Indicators and Forecasting, in J. H. Stock and M. W. Watson (eds.), Chicago; University of Chicago for NBER, 1993: 255-284

[10] Di Masi G.B., Kabanov Y.M., Runggaldier W.J.; "Mean-variance hedging of options on stocks with Markov volatilities", Theory of Probability and Its Applications, **39**, (1994): 211-222

[11] Durland J., McCurdy T.; "Duration dependent transitions in a Markov model of U.S. GNP growth", Journal of Business and Economics Statistics, **12**, (1994): 279-288

[12] Easly D., O'Hara M.; "Price, trade size and information in securities markets", Journal of Financial Economics, **19**, (1987): 69-90

[13] Easly D., O'Hara M.; "Time and the process of security price adjustment", Journal of Finance, **47**, (1992): 577-605

[14] Elliott R.J., van der Hoek J.; "An application of hidden Markov models to asset allocation problems", Finance and Stochastics, **3**, (1997): 229-238

[15] Elliott R.J., Chan L., Siu T.K.; "Option pricing and Esscher transform under regime switching", Annals of Finance, **1** (4): 423-432

[16] Filardo A.; "Business cycle phases and their transitional dynamics"; Journal of Business and Economics Statistics, **12**, (1994): 299-308.

[17] Fuh C.D., Wang R.H., Cheng J.C. ; "Option pricing in a Black-Scholes model with Markov switching", Technical Report, Institute of Statistical Science, Academia Sinica, **C-2002-10**, (2002)

[18] Ghosh M.K., Goswami A.; "Risk minimizing option pricing in semi-Markov modulated market" ; SIAM Journal on Control and Optimization, **48**(3), (2009): 1519-1541

[19] Ghosh M.K., Goswami A., Kumar S.; "Portfolio optimization in a semi-Markov modulated market"; Applied Mathematics and Optimization, **60**(2), (2009): 275-296

[20] Gihman I.I., Skorohod A.V.; *The theory of stochastic processes II*, Springer-Verlag, Berlin, 1975

[21] Hong Y., Li H.; "Nonparametric specification testing for continuous-time models with applications to term structure of interest rates", Review of Financial Studies, **18**: 37-84

[22] Jacod J., Shiryaev A.N.; *Limit theorems for stochastic processes*, Springer-Verlag, Berlin, 1987

[23] Kijima M., Yoshida T.; "A simple option pricing model with Markovian volatilities", Journal of the Operations Research Society of Japan , **36**(3), (1993): 149-166

[24] Landén C.; "Bond pricing in a hidden Markov model of the short rate"; Finance and Stochastics, **4**, (2000): 371-389

[25] Prigent J.-L.; "Option pricing with a general marked point process"; Mathematics of Operations Research, **26** (1), (2001): 50-66

[26] Pyke R.; "Markov renewal processes: definitions and preliminary properties"; The Annals of Mathematical Statistics, **32** (4), (1961): 1231-1242

[27] Rogers L.C.G, Jobert A.; "Option pricing with Markov-modulated dynamics"; SIAM Journal on Control and Optimization, **44**, (2006): 2063-2078

[28] Runggaldier W.J. ; "Jump-diffusion models"; *Handbook of heavy tailed distributions in finance* , S.T.Rachev, ed.; **Handbooks in finance, Book 1**, (W.Ziemba Series ed.); (Elsevier/North-Holland, 2003): 170-207

[29] Silvestrov D., Stenberg F. ; "A pricing process with stochastic volatility controlled by a semi-Markov process"; Communications in statistics: theory and methods, **33** (3), (2004): 591-608

[30] Svishchuk A.V. ; "Hedging of options under mean-square criterion and semi-Markov volatility"; Ukrainian Journal of Mathematics, **47** (7), (1995): 1119-1127

[31] Wu S., Zeng Y.; "The term structure of interest rates under regime shifts and jumps"; Economics Letters, **93**, (2006): 215-221

[32] Yao D.D., Zhang Q., Zhou X.Y. ; "A regime switching model for european options"; *Stochastic Processes, Optimization, and Control Theory Applications in Financial Engineering, Queueing Networks and Manufacturing Systems*, H. M. Yan, G. Yin, and Q. Zhang, Eds.; Springer, New York, 2006: 281-300