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ON CONCORDANCE MEASURES FOR DISCRETE DATA AND DEPENDENCE PROPERTIES OF POISSON MODEL

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On Concordance Measures for Discrete Data and Dependence Properties of Poisson Model

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Abstract

The purpose of this paper is twofold. First, we study Kendall's tau and Spearman's rho concordance measures for discrete variables. We mainly propose their best bounds using positive dependence properties. Second, we provide useful dependence properties of the bivariate Poisson distribution and show the relationship between parameters of the Poisson distribution and both tau and rho.

Key words and phrases: Concordance order; Fréchet bounds; Kendall's τ; Spearman's ρ; Positive quadrant dependence.

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1 Introduction

The best known dependence property is "lack of dependence", or what is known as stochastic independence. In many application, independence between two random variables is assumed, this can be a strong assumption in the undertaken analysis. Taking into account the dependence structure between the variables lead to appropriate modeling approaches and correct conclusions. To study stochastic dependence, concordance concept and positive dependence are well used tools. This is because many dependence properties can be described by means of the joint distribution of the variables and these measures and properties are often margins free. In this paper we study two concordance measures Kendall's tau (Kruskal ,1958) and Spearman's rho (Lehmann, 1969). These measures have several properties known as Rényi's axioms, for more details see seeRényi (1959). Among these axioms, we focus on the range of the association measure.

Many research has been concerned with the study of tau and rho in the case of continuous variables. Schweizer and Wolff (1981), in one seminal paper, show that the study of concordance measures for continuous random variables can be characterized as the study of copulas (Nelsen 1999). However, for non continuous variables, this interrelationship generally does not hold. There are few papers concerning the discrete version of Kendall's tau and Spearman 's rho. Conti (1993) gives definitions of two approaches of indifference and links them to concordance and discordance properties of the data. Tajar et al. (2001) propose a copula-type representation for random couples with binary margins. They show that appropriate measures of association for binary random variables do not depend on the marginal distribution of the variables under study. Mesfioui and Tajar (2005) and Denuit and Lambert (2005) have shown independently that the range of tau and rho in the discrete case is not the unit interval as in the continuous case. Neslehova (2006) considers an alternative transformation of an arbitrary random variable to a uniform distribution variable in order to study the rank measures for non continuous random variables.

In this paper, we focus on the range of the concordance measures. Aside from identifying the best bounds of tau and rho in the case of discrete random variables, we present some dependence properties of the bivariate Poisson model and discuss their relationship with the concordance measures tau and rho. The paper is organized as follows. The next section provides a method of constructing the ranges of tau and rho for discrete data. Section 3, develops explicit expressions for the best bounds of tau and rho in the discrete Fréchet space with the same marginal. Section 4, discuss some dependence properties of the bivariate Poisson model.

2 Defintions and properties

Following Hoeffding (1948) and Kruskal (1958) and Lehman (1966), Schweizer and Wolff (1981) express Kendall 's tau and Spearman's rho for continuous random vector (X, Y) in terms of the joint distribution $H(x, y)$ of (X, Y) and the margins $F(x)$ for X and $G(y)$ for Y. A general representation for each of τ and ρ has been first proposed by Kowalczyk and Niewiadomska-Bugaj (1998); namely

$$\tau = \mathrm{E}_H[H(X,Y)] + \mathrm{E}_H[H(X^-,Y^-)] + \mathrm{E}_H[H(X^-,Y)] + \mathrm{E}_H[H(X,Y^-)] - 1, \tag{2.1}$$

and

$$\rho = 3\left\{\mathrm{E}_\Pi[H(X^-,Y)] + \mathrm{E}_\Pi[H(X,Y^-)] + \mathrm{E}_\Pi[H(X^-,Y^-)] + \mathrm{E}_\Pi[H(X,Y)] - 1\right\} \tag{2.2}$$

where $\mathrm{H}(x^-, y^-) = \mathrm{P}[X < x, Y < y]$.

Several results in this paper are based on the monotonicity property of Kendall's τ and Spearman's ρ. This property has first been proposed for continuous variables by Yanagimoto and Okamoto (1969) (see also Joe (1997)). Tchen (1980) obtained similar monotonicity property for τ and ρ when the supports of the joint distributions consist in a finite number of atoms. Mesfioui and Tajar (2005) extend various dependence relationships between Kendall's τ and Sperrman's ρ in Capéraà and Genest (1993) and Nelsen (1999), to the discrete case. One key result of their paper is the generalization to any kind of random variables for continuous and/ or discrete variables.

For the remainder of the paper, We recall the property of concordance orderings, defined as follows:

Let (X_1, Y_1) and (X_2, Y_2) be a random vectors with identical marginal and respective *cdf* H_1 and H_2. The random couple (X_2, Y_2) is said to be more concordant than (X_1, Y_1), denoted as $(X_1, Y_1) \prec_c (X_2, Y_2)$, if $H_1(x_1, x_2) \leq H_2(x_1, x_2)$ holds for all $x_1, x_2 \in \mathbb{R}$.

In the following proposition, we propose an alternative proof of the monotonicity property.

Proposition 1. *Let (X_1, Y_1) and (X_2, Y_2) be two random couples with respective distribution function H_1 and H_2 in $\Gamma(F, G)$, the Fréchet space of all distribution functions with fixed marginal F and G. Then,*

$$(X_1, Y_1) \preceq_c (X_2, Y_2) \Rightarrow \tau_{H_1} \leq \tau_{H_2}, \tag{2.3}$$

and

$$(X_1, Y_1) \preceq_c (X_2, Y_2) \Rightarrow \rho_{H_1} \leq \rho_{H_2}. \tag{2.4}$$

■

Proof. Using Fubini's theorem, we note that

$$\begin{aligned}
\mathrm{E}_{H_1}\left[H_2(X,Y)\right] &= \mathrm{E}_{H_2}\left[\bar{H}_1(X^-,Y^-)\right], \\
\mathrm{E}_{H_1}\left[H_2(X^-,Y)\right] &= \mathrm{E}_{H_2}\left[\bar{H}_1(X,Y^-)\right], \\
\mathrm{E}_{H_1}\left[H_2(X,Y^-)\right] &= \mathrm{E}_{H_2}\left[\bar{H}_1(X^-,Y)\right], \\
\mathrm{E}_{H_1}\left[H_2(X^-,Y^-)\right] &= \mathrm{E}_{H_2}\left[\bar{H}_1(X,Y)\right].
\end{aligned}$$

where $\bar{H}_i$ denotes the survival functions associated to H_i, $i = 1, 2$.
Now without loss of generality if we assume that $H_1 \leq H_2$, which is equivalent to $\bar{H}_1 \leq \bar{H}_2$. We then get

$$\begin{aligned}
\mathrm{E}_{H_1}\left[H_1(X,Y)\right] &\leq \mathrm{E}_{H_1}\left[H_2(X,Y)\right] \\
&= \mathrm{E}_{H_2}\left[\bar{H}_1(X^-,Y^-)\right] \\
&\leq \mathrm{E}_{H_2}\left[\bar{H}_2(X^-,Y^-)\right] \\
&= \mathrm{E}_{H_2}\left[H_2(X,Y)\right].
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
\mathrm{E}_{H_1}\left[H_1(X^-,Y)\right] &\leq \mathrm{E}_{H_2}\left[H_2(X^-,Y)\right], \\
\mathrm{E}_{H_1}\left[H_1(X,Y^-)\right] &\leq \mathrm{E}_{H_2}\left[H_2(X,Y^-)\right], \\
\mathrm{E}_{H_1}\left[H_1(X^-,Y^-)\right] &\leq \mathrm{E}_{H_2}\left[H_2(X^-,Y^-)\right].
\end{aligned}$$

Combining the later inequalities with (2.1), we then obtain (2.3). It's easy seen that (2.4) is immediate from (2.2). □

Proposition (1) and the Fréchet bounds allow to construct optimal bounds for τ and ρ as stated in the following corollary.

Corollary 1. *Let (X, Y) be a random couple with distribution function H in $\Gamma(F, G)$. Then,*

$$\tau_{\min} \leq \tau_H \leq \tau_{\max}$$

and

$$\rho_{\min} \leq \tau_H \leq \rho_{\max}$$

where $\tau_{\min}$, $\rho_{\min}$ and $\tau_{\max}$, $\rho_{\max}$ denote the values of Kendall's τ and Spearman's ρ corresponding to the Fréchet lower and upper bounds in $\Gamma(F, G)$, respectively.

■

As stated earlier, the main objective in this paper is to examine the bounds of τ and ρ in the Fréchet space $\Gamma(F,G)$ when F and G are discrete. To do that, let (X,Y) be a discrete random couple with *cdf* $H \in \Gamma(F,G)$. Since Kendall's τ and Spearman's ρ are scale invariants they remain unchanged under strictly increasing transformations of the marginal distributions. We can then suppose, without any loss of generality, that X and Y are valued in $\mathbb{Z}$, the set of all integers. Therefore, we can see from (2.1) and (2.2) that τ and ρ can be written as

$$\tau = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} T_{ij} S_{ij} - 1 \tag{2.5}$$

$$\rho = 3 \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} T_{ij} \left[F(i) - F(i-1)\right] \left[G(j) - G(j-1)\right] - 3 \tag{2.6}$$

where

$$T_{ij} = H(i,j) + H(i-1,j-1) + H(i,j-1) + H(i-1,j) \tag{2.7}$$

$$S_{ij} = H(i,j) - H(i,j-1) - H(i-1,j) + H(i-1,j-1). \tag{2.8}$$

In order to obtain the best bounds $\tau_{\min}$, $\rho_{\min}$ and $\tau_{\max}$, $\rho_{\max}$, respectively the minimum and maximum values corresponding to lower and upper bound of τ and ρ, we replace H in (2.5) and (2.6) by the Fréchet bounds $H_{\min}(i,j) = \max\left[F(i) + G(j) - 1, 0\right]$ and $H_{\max}(i,j) = \min\left[F(i), G(j)\right]$, respectively.

For discrete data, the ranges of τ and ρ are different from the usual unit interval $[-1,1]$. This is a violation of the monotone dependence properties of concordance measures, as stated in Nelsen (1999)). To correct this problem, we propose the corrections.

$$\tau_c = \begin{cases} \frac{\tau}{\tau_{\max}} & \text{if} \quad \tau \geq 0, \\ \\ -\frac{\tau}{\tau_{\min}} & \text{if} \quad \tau < 0 \end{cases} \tag{2.9}$$

and

$$\rho_c = \begin{cases} \frac{\rho}{\rho_{\max}} & \text{if} \quad \rho \geq 0, \\ \\ -\frac{\rho}{\rho_{\min}} & \text{if} \quad \rho < 0. \end{cases} \tag{2.10}$$

The main importance of these corrections is that they allow to interpret the levels of the new measures, τ_c and ρ_c, as percentages. Illustrations of these transformations are proposed in Section 4 with the bivariate Poisson distribution.

3 Explicit bounds of discrete τ and ρ in $\Gamma(F,F)$

The aim of this section is to study the effect of the marginal distributions on the range of τ and ρ for discrete data. Considering the problem in $\Gamma(F,G)$ requires several assumptions on F and G. We will then concentrate on the ranges of τ and ρ in $\Gamma(F,F)$, where F is a discrete distribution function.

The next proposition presents optimal bounds of Spearman's ρ.

Proposition 2. *The best bounds for ρ, in the space, $\Gamma(F,F)$ are given by*

$$\rho_{\max} = 3E\left[1 - F^2(X) - F^2(X-1)\right] \tag{3.1}$$

and

$$\rho_{\min} \quad = \quad 3E\left[\psi(X) + \psi(X-1) - 1\right], \tag{3.2}$$

where

$$\psi(i) = \sum_{j=\phi(i)}^{\infty} \left[F(i) + F(j) - 1\right]\left[F(j+1) - F(j-1)\right]. \tag{3.3}$$

■

Proof. Let $H(i,j) = \min\left[F(i), F(j)\right]$. From (2.7), we observe that

$$T_{ij} = \left[F(i) + 3F(i-1)\right] 1\!\!1_{[i=j]} + 2\left[F(i) + F(i-1)\right] 1\!\!1_{[i<J]} + 2\left[F(j) + F(j-1)\right] 1\!\!1_{[i>j]},$$

and writing $F(i) - F(i-1) = p_i$, we get from (2.6) that

$$\begin{aligned}
\rho_{\max} \quad &= \quad 3\sum_{i=-\infty}^{\infty}\left[F(i) + 3F(i-1)\right]\left[F(i) - F(i-1)\right] p_i \\
&+ \quad 6\sum_{i=-\infty}^{\infty}\left[F(i) + F(i-1)\right] p_i \sum_{j=i+1}^{\infty}\left[F(j) - F(j-1)\right] \\
&+ \quad 6\sum_{i=-\infty}^{\infty} p_i \sum_{j=-\infty}^{i-1}\left[F(j) - F(j-1)\right]\left[F(j) + F(j-1)\right] - 3,
\end{aligned}$$

which may be simplified as

$$\begin{aligned}
\rho_{\max} \quad &= \quad 3\mathrm{E}\left\{\left[F(X) + 3F(X-1)\right]\left[F(X) - F(X-1)\right]\right\} \\
&+ \quad 6\mathrm{E}\left\{\left[F(X) + F(X-1)\right]\left[1 - F(X)\right]\right\} \\
&+ \quad 6\mathrm{E}\left[F^2(X-1)\right] - 3.
\end{aligned}$$

The result then follows from the fact that $\mathrm{E}\left[F(X)+F(X-1)\right]=1$. Now, choose $H(i,j)=\sup\left[F(i)+F(j)-1,0\right]$ and put $H^{+}(i,j)=F(i)+F(j)-1$. From (2.6), we see that

$$\begin{aligned}\rho_{\min} &= 3\sum_{i=-\infty}^{\infty}\sum_{j=\phi(i)}^{\infty}H^{+}(i,j)p_ip_j+3\sum_{i=-\infty}^{\infty}\sum_{j=\phi(i)}^{\infty}H^{+}(i,j)p_{i+1}p_{j+1}\\ &+ 3\sum_{i=-\infty}^{\infty}\sum_{j=\phi(i)}^{\infty}H^{+}(i,j)p_ip_{j+1}+3\sum_{i=-\infty}^{\infty}\sum_{j=\phi(i)}^{\infty}H^{+}(i,j)p_{i+1}p_j-3.\end{aligned}$$

It follows that

$$\rho_{\min}=3\sum_{i=-\infty}^{\infty}\left(p_i+p_{i+1}\right)\sum_{j=\phi(i)}^{\infty}\left[F(i)+F(j)-1\right]\left(p_j+p_{j+1}\right)-3 \tag{3.4}$$

which may be rewritten as

$$\rho_{\min}=3\sum_{i=-\infty}^{\infty}\left[\psi(i)+\psi(i-1)-1\right]p_i \tag{3.5}$$

where,

$$\psi(i)=\sum_{j=\phi(i)}^{\infty}\left[F(i)+F(j)-1\right]\left[F(j+1)-F(j-1)\right]. \tag{3.6}$$

The result is therefore obtained from (3.6) and (3.5). □

Using (2.5) with $H(i,j)=\min[F(i),F(j)]$, we notice that the upper bound of Kendall's τ in the space $\Gamma(F,F)$ can be expressed as

$$\tau_{\max}=2\mathrm{E}\left[F(X-1)\right]. \tag{3.7}$$

Note that (3.7) is equivalent to (15) in Denuit and Lambert (2005) coincides with (3.7) in $\Gamma(F,F)$ The following proposition gives an explicit form of Kendall's tau lower bound in $\Gamma(F,F)$.

Proposition 3. *The best lower bounds of τ in $\Gamma(F,F)$ is*

$$\tau_{\min} = 2E\Big[F[\phi(X-1)]\Big]-2\sum_{k=-\infty}^{\infty}\xi(k)-2, \tag{3.8}$$

where

$$\phi(i)=min\left\{j\in\mathbb{Z}:F(i)+F(j)>1\right\},\quad i\in\mathbb{Z},$$

and

$$\xi(k)=\left[F(k-1)+F[\phi(k-1)]-1\right]\left[F(k)+F[\phi(k-1)-1]-1\right]\mathbb{1}_{[\phi(k)<\phi(k-1)]}.$$

■

Proof. From (2.7) and (2.8) , we observe that

$$\begin{aligned} S_{ij}T_{ij} &= H^2(i,j) + H^2(i-1,j-1) - H^2(i-1,j) - H^2(i,j-1) \\ &+ 2H(i,j)H(i-1,j-1) - 2H(i-1,j)H(i,j-1) \end{aligned}$$

and

$$\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\left[H^2(i,j) + H^2(i-1,j-1) - H^2(i-1,j) - H^2(i,j-1)\right] = 1.$$

Consider now $H(i,j) = \sup\left[F(i) + F(j) - 1, 0\right]$ and write $H^+(i,j) = F(i) + F(j) - 1$. From (2.5), we get

$$\begin{aligned} \tau_{\min} &= 2\sum_{i=-\infty}^{\infty}\sum_{j=\phi(i-1)+1}^{\infty} H^+(i,j)H^+(i-1,j-1) \\ &- 2\sum_{i=-\infty}^{\infty}\sum_{j=\max[\phi(i-1),\phi(i)+1]}^{\infty} H^+(i-1,j)H^+(i,j-1). \end{aligned}$$

Using the fact that

$$H^+(i,j)H^+(i-1,j-1) - H^+(i-1,j)H^+(i,j-1) = -p_ip_j,$$

we have,

$$\begin{aligned} \tau_{\min} &= -2\sum_{i=-\infty}^{\infty} p_i\left[\sum_{j=\phi(i-1)+1}^{\infty} p_j\right]\mathbb{1}_{[\phi(i)=\phi(i-1)]} - 2\sum_{i=-\infty}^{\infty} p_i\left[\sum_{j=\phi(i-1)+1}^{\infty} p_j\right]\mathbb{1}_{[\phi(i)<\phi(i-1)]} \\ &- 2\sum_{i=-\infty}^{\infty}\left[F(i-1) + F(\phi(i-1)) - 1\right]\left[F(i) + F(\phi(i-1) - 1) - 1\right]\mathbb{1}_{[\phi(i)<\phi(i-1)]}, \end{aligned}$$

which is equivalent to

$$\tau_{\min} = -2\sum_{i=-\infty}^{\infty} p_i\left[1 - F[\phi(i-1)]\right] - 2\sum_{i=-\infty}^{\infty} r(i),$$

with,

$$r(i) = \left[F(i-1) + F(\phi(i-1)) - 1\right]\left[F(i) + F(\phi(i-1) - 1) - 1\right]\mathbb{1}_{[\phi(i)<\phi(i-1)]},$$

which completes the proof. □

In the above results are illustrated in the following example using the uniform discrete variables.

Example 2. *Let X_n be a uniform discrete random variable with n atoms and cdf F_n. In the Fréchet space $\Gamma(F_n, F_n)$, we obtain that $F_n(i) = i/n$, $\phi(i) = n - i + 1$, $\xi(i) = 1/n^2$, $\psi(i) = i^2/n^2$,*

$$E(X_n) = \frac{(n+1)}{2}, \qquad E(X_n^2) = \frac{(n+1)(2n+1)}{6}, \qquad E\left[(X_n - 1)^2\right] = \frac{(n-1)(2n-1)}{6},$$

$$E[\psi(X_n)] = \frac{(n+1)(2n+1)}{6n^2} \quad \textit{and} \quad E[\psi(X_n - 1)] = \frac{(n-1)(2n-1)}{6n^2}.$$

Therefore, proposition 2 and proposition 3 together with (3.7) imply

$$[\tau_{\min}, \tau_{\max}] = \left[-\frac{n-1}{n}, \frac{n-1}{n}\right] \quad \textit{and} \quad [\rho_{\min}, \rho_{\max}] = \left[-\frac{n^2-1}{n^2}, \frac{n^2-1}{n^2}\right].$$

Remark 1. *Let $F_{n,p}$ be a binomial distribution with parameters n and p, and denote the extreme values of τ and ρ in $\Gamma(F_{n,p}, F_{n,p})$ by $\tau_{\max}(n,p)$ and $\rho_{\max}(n,p)$. One can show the following symmetry properties, namely:*

$$\tau_{\max}(n,p) = \tau_{\max}(n, 1-p) \quad \textit{and} \quad \rho_{\max}(n,p) = \rho_{\max}(n, 1-p).$$

Indeed, since $F_{n,p}(k) = F_{n,1-p}(n-k)$, then from (3.7), we have

$$\begin{aligned}
\tau_{\max}(n,p) &= 2\sum_{k=0}^{n} F_{n,p}(k-1)\left[F_{n,p}(k) - F_{n,p}(k-1)\right] \\
&= 2\sum_{k=0}^{n} F_{n,1-p}(k-1)\left[F_{n,1-p}(k) - F_{n,1-p}(k-1)\right] \\
&= \tau_{\max}(n, 1-p).
\end{aligned}$$

Similar arguments provide $\rho_{\max}(n,p) = \rho_{\max}(n, 1-p)$.

4 Understanding dependence structure of the bivariate Poisson distribution

Our purpose in this section is to study dependence properties of the bivariate Poisson distribution H of a random couple (X, Y) and the relationship between τ and ρ and the parameters of H. Several bivariate Poisson distributions have been proposed in the statistical literature, for example, Kocherlakota and Kocherlakota (1992). In applied statistics, however, the focus is on the trivariate reduction method described by Johnson et al. (1997) who construct the Bivariate Poisson distribution using three independent random variables X_1, X_2 and Z all distributed as Poisson with parameters λ_1, λ_2 and α, respectively

$$X = X_1 + Z \quad \text{and} \quad Y = X_2 + Z.$$

The cumulative distribution of (X, Y) is given by

$$H_{\alpha,\lambda_1,\lambda_2}(i,j) = \sum_{k=0}^{i \wedge j} F_{\lambda_1}(i-k) F_{\lambda_2}(j-k) \frac{\alpha^k e^{-\alpha}}{k!} \tag{4.1}$$

where F_{λ_i} denotes the *cdf* of X_i, $i = 1, 2$. We notice that X and Y are Poisson model with means $\lambda_1 + \alpha$ and $\lambda_2 + \alpha$, respectively. Note that the covariance and the correlation between X and Y are expressed by

$$\mathrm{cov}(X, Y) = \alpha \quad \text{and} \quad \mathrm{corr}(X, Y) = \frac{\alpha}{\sqrt{(\lambda_1 + \alpha)(\lambda_2 + \alpha)}}$$

which are positive and non-decreasing functions of α.

The next result shows the effect of the parameters λ_1, λ_2 and α on the *cdf* and the survival function of the bivariate Poisson model.

Proposition 4. *Let (X, Y) be a bivariate Poisson random vector with cdf and survival function $H_{\alpha,\lambda_1,\lambda_2}(i, j)$ and $\bar{H}_{\alpha,\lambda_1,\lambda_2}(i, j)$, respectively,*

1. *For fixed λ_1 and λ_2, we have*

$$\alpha_1 \leq \alpha_2 \quad \Rightarrow \quad H_{\alpha_1,\lambda_1,\lambda_2}(i, j) \geq H_{\alpha_2,\lambda_1,\lambda_2}(i, j) \tag{4.2}$$

and

$$\alpha_1 \leq \alpha_2 \quad \Rightarrow \quad \bar{H}_{\alpha_1,\lambda_1,\lambda_2}(i, j) \leq \bar{H}_{\alpha_2,\lambda_1,\lambda_2}(i, j). \tag{4.3}$$

2. *For fixed α, we have*

$$\lambda_1 \leq \hat{\lambda}_1 \quad \textit{or} \quad \lambda_2 \leq \hat{\lambda}_2 \quad \Rightarrow \quad H_{\alpha,\lambda_1,\lambda_2}(i, j) \geq H_{\alpha,\hat{\lambda}_1,\hat{\lambda}_2}(i, j) \tag{4.4}$$

and

$$\lambda_1 \leq \hat{\lambda}_1 \quad \textit{or} \quad \lambda_2 \leq \hat{\lambda}_2 \quad \Rightarrow \quad \bar{H}_{\alpha,\lambda_1,\lambda_2}(i, j) \leq \bar{H}_{\alpha,\hat{\lambda}_1,\hat{\lambda}_2}(i, j). \tag{4.5}$$

■

Proof. Let $H(i, j) = F_{\lambda_1}(i)F_{\lambda_2}(j)$ be a *cdf* of the vector (X, Y). We have

$$\frac{\partial H_{\alpha,\lambda_1,\lambda_2}(i, j)}{\partial \alpha} = \sum_{k=1}^{i \wedge j} H(i - k, j - k) \left[\frac{\alpha^{k-1} e^{-\alpha}}{(k - 1)!} - \frac{\alpha^k e^{-\alpha}}{k!} \right]$$

which may be rewritten as

$$\begin{aligned} \frac{\partial H_{\alpha,\lambda_1,\lambda_2}(i, j)}{\partial \alpha} &= \sum_{k=0}^{i \wedge j} \frac{\alpha^k e^{-\alpha}}{k!} \left[H(i - k - 1, j - k - 1) - H(i - k, j - k) \right] \\ &= H_{\alpha,\lambda_1,\lambda_2}(i - 1, j - 1) - H_{\alpha,\lambda_1,\lambda_2}(i, j) \leq 0 \end{aligned}$$

so that (4.2) is true. Similar arguments provide

$$\frac{\partial \bar{H}_{\alpha}(i, j)}{\partial \alpha} = \bar{H}_{\alpha_1,\lambda_1,\lambda_2}(i - 1, j - 1) - \bar{H}_{\alpha_1,\lambda_1,\lambda_2}(i, j) \geq 0$$

so that, (4.3) is verified as well. To show (4.4) and (4.5), it is sufficient to note that

$$\begin{aligned}\frac{\partial H_{\alpha,\lambda_1,\lambda_2}(i,j)}{\partial \lambda_1} &= -\sum_{k=0}^{i\wedge j} \frac{\lambda_1^{i-k}e^{-\lambda_1}}{(i-k)!} F_{\lambda_2}(j-k)\frac{\alpha^k e^{-\alpha}}{k!} \\ &= H_{\alpha,\lambda_1,\lambda_2}(i-1,j) - H_{\alpha,\lambda_1,\lambda_2}(i,j) \leq 0\end{aligned}$$

and

$$\frac{\partial \bar{H}_{\alpha,\lambda_1,\lambda_2}(i,j)}{\partial \lambda_1} = \bar{H}_{\alpha,\lambda_1,\lambda_2}(i-1,j) - \bar{H}_{\alpha,\lambda_1,\lambda_2}(i,j) \geq 0.$$

□

Remark 2. *Let $\tau_{\alpha,\lambda_1,\lambda_2}$ and $\rho_{\alpha,\lambda_1,\lambda_2}$ be Kendall's tau and Spearman's rho associated with the cdf $H_{\alpha,\lambda_1,\lambda_2}$. Note that we can not conclude from* (4.2) *and proposition 1 that $\tau_{\alpha,\lambda_1,\lambda_2}$ and $\rho_{\alpha,\lambda_1,\lambda_2}$ are decreasing functions of α, because $H_{\alpha_1,\lambda_1,\lambda_2}(i,j)$ and $H_{\alpha_2,\lambda_1,\lambda_2}(i,j)$ don't have the same marginals when $\alpha_1 \neq \alpha_2$. However, numerical calculations show that these parameters are rather non-decreasing functions of α .*

To study further the relationships between α and each of τ and ρ for the bivariate Poisson model, we propose an alternative parametrization which consist to fix the marginal parameters $\alpha + \lambda_1 = m_1$ and $\alpha + \lambda_2 = m_2$. In this context, the *cdf* (4.1) becomes

$$H_\alpha(i,j) = \sum_{k=0}^{i\wedge j} F_{m_1-\alpha}(i-k)F_{m_2-\alpha}(j-k)\frac{\alpha^k e^{-\alpha}}{k!}. \tag{4.6}$$

As a consequence of the above representation, we can see $\{H_\alpha\}$ as a family of bivariate Poisson models with fixed marginals which are univariate Poisson models with parameters m_1 and m_2, respectively. This means that the set $\{H_\alpha\}$, $0 \leq \alpha \leq m_1 \wedge m_2$ is included in the particular Fréchet space $\Gamma(F_{m_1}, F_{m_2})$, where F_{m_i} denotes the *cdf* of a Poisson model with mean m_i, $i = 1, 2$. The advantage of the parametrization (4.6) rather than (4.1), is that the coefficient α may be interpreted as a dependence parameter in the family $\{H_\alpha\}$.

Now, let τ_α and ρ_α be Kendall's τ and Spearman's ρ associated with the distribution H_α. The result below provides the monotonicity of τ_α and ρ_α as functions of α.

Proposition 5. *Let H_{α_1} and H_{α_2} be two cdf of the set $\{H_\alpha\}$. Then,*

$$\alpha_1 \leq \alpha_2 \quad \Rightarrow \quad H_{\alpha_1} \leq H_{\alpha_2} \tag{4.7}$$

and consequently,

$$\alpha_1 \leq \alpha_2 \quad \Rightarrow \quad \tau_{\alpha_1} \leq \tau_{\alpha_2} \quad \text{and} \quad \rho_{\alpha_1} \leq \rho_{\alpha_2}. \tag{4.8}$$

■

Proof. From (4.6),

$$\begin{aligned}\frac{\partial H_\alpha(i,j)}{\partial\alpha} &= \sum_{k=0}^{i\wedge j}\frac{\partial F_{m_1-\alpha}(i-k)}{\partial\alpha}F_{m_2-\alpha}(j-k)\frac{\alpha^k e^{-\alpha}}{k!}\\ &+ \sum_{k=0}^{i\wedge j}F_{m_1-\alpha}(i-k)\frac{\partial F_{m_2-\alpha}(j-k)}{\partial\alpha}\frac{\alpha^k e^{-\alpha}}{k!}\\ &+ \sum_{k=0}^{i\wedge j}F_{m_1-\alpha}(i-k)F_{m_2-\alpha}(j-k)\left[\frac{\alpha^{k-1}e^{-\alpha}}{(k-1)!}-\frac{\alpha^k e^{-\alpha}}{k!}\right], \qquad (4.9)\end{aligned}$$

and using the fact that

$$\frac{\partial F_{m_1-\alpha}(i-k)}{\partial\alpha} = F_{m_1-\alpha}(i-k)-F_{m_1-\alpha}(i-k-1)$$

and

$$\frac{\partial F_{m_2-\alpha}(j-k)}{\partial\alpha} = F_{m_2-\alpha}(j-k)-F_{m_2-\alpha}(j-k-1),$$

(4.9) becomes, upon simplifications,

$$\frac{\partial H_\alpha(i,j)}{\partial\alpha} = \sum_{k=0}^{i\wedge j}\frac{(m_1-\alpha)^{i-k}e^{-(m_1-\alpha)}}{(i-k)!}\frac{(m_2-\alpha)^{j-k}e^{-(m_2-\alpha)}}{(j-k)!}\frac{\alpha^k e^{-\alpha}}{k!}\geq 0. \qquad (4.10)$$

Therefore (4.10) together with proposition 1 provide (4.7) and (4.8). □

Many statistical research have focused on studying concepts of positive dependence for bivariate distributions, example right tail increasing and positive quadrant dependence which are widely used in actuarial literature (Dhaene and Govaerts (1996). There are natural relationships between dependence properties and measures of concordance. An interesting property of positive dependence is the concept of positive quadrant dependence (PQD) defined as follows: let (X,Y) be a random couple valued in $\mathbb{R}\times\mathbb{R}$ with joint *cdf* H, and marginals F and G. These random variables are said to be positively quadrant dependent if, and only if, for all $(x,y)\in\mathbb{R}^2$

$$H(x,y)\geq F(x)G(y).$$

The following corollary is a direct consequence of the previous result.

Corollary 3. *The family $\{H_\alpha\}$ is positively quadrant dependent.*

■

Proof. Since H_α is a non-decreasing function of α, then $H_0\leq H_\alpha$ for all $0\leq\alpha\leq m_1\wedge m_2$. Now, from (4.6), $H_0(i,j)=F_{m_1}(i)F_{m_2}(j)$ for all i,j. Therefore the family $\{H_\alpha\}$ is PQD. Consequently, $\tau_\alpha\geq 0$, $\rho_\alpha\geq 0$ and $3\tau_\alpha\geq\rho_\alpha$ for all $0\leq\alpha\leq m_1\wedge m_2$. □

Remark 3. *When $m_1 = m_2 = m$, the upper bound of the family $\{H_\alpha\}$ is given by the cdf H_m, and using (4.6), we then obtain that $H_m(i,j) = F_m(i \wedge j) = \min[F_m(i), F_m(j)]$, for all i, j, which is the upper Fréchet bound.*

In order to appreciate the corrections of τ and ρ given by (2.9) and (2.10), we consider the family of Poisson model $\{H_\alpha\}$ with marginal parameters $m_1 = m_2 = 2$. Using (3.1) and (3.7) with F_m instead of F, we obtain that $\rho_{\max} = 0.951$ and $\tau_{\max} = 0.792$. The table below provides τ_α and ρ_α with their corrections $\tau_{\alpha,c}$ and $\rho_{\alpha,c}$ for chosen values of α.

α	τ_α	$\tau_{\alpha,c}$	ρ_α	$\rho_{\alpha,c}$
0.2	0.059	0.075	0.089	0.094
0.4	0.120	0.152	0.180	0.189
0.6	0.183	0.231	0.272	0.286
0.8	0.248	0.313	0.365	0.383
1.0	0.316	0.398	0.459	0.482
1.2	0.388	0.490	0.554	0.582
1.4	0.467	0.589	0.651	0.684
1.6	0.556	0.701	0.749	0.787
1.8	0.660	0.832	0.849	0.892

Table 4.1: τ_α , ρ_α , $\tau_{\alpha,c}$ and $\rho_{\alpha,c}$ for the Poisson model.

From table (4.1), we note that the differences $D_{\tau,\alpha} = \tau_{\alpha,c} - \tau_\alpha$ and $D_{\rho,\alpha} = \rho_{\alpha,c} - \rho_\alpha$ are increasing as function of the dependence parameter α. This constatation is true in general because $D_{\tau,\alpha}$ and $D_{\rho,\alpha}$ can be expressed as

$$D_{\tau,\alpha} = \frac{(1-\tau_{\max})\tau_\alpha}{\tau_{\max}} \quad \text{and} \quad D_{\rho,\alpha} = \frac{(1-\rho_{\max})\rho_\alpha}{\rho_{\max}},$$

which shows that these parameters are in fact increasing with α.

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