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COHERENCE ANALYSIS OF NONSTATIONARY TIME SERIES: A LINEAR FILTERING POINT OF VIEW

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Coherence Analysis of Nonstationary Time Series: A Linear Filtering Point of View

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Abstract

Coherence is a widely used measure for characterizing linear dependence between two time series. Classical books on time series analysis present coherence as "the frequency domain analogue of the autocorrelation function" which lacks intuitive appeal. The first goal of this paper is to present a more illuminating and yet still precise interpretation of coherence. Consider a filter whose power transfer function is concentrated on a particular frequency band Ω. We show that coherence at Ω is equivalent to the correlation between the two filtered time series. The second goal of this paper is to develop a novel adaptive statistical procedure for estimating coherence when the time series are non-stationary, that is, the nature of linear dependence between time series may evolve with time. The proposed method for estimating local coherence automatically selects, via repeated tests of homogeneity (in time) of coherence, the optimal width of the time window on which one computes the estimated local coherence. This approach is point-wise adaptive in the sense that the width of the optimal interval is allowed to change across time. Under the locally stationary process framework, we develop a central limit theorem on the Fisher-z transform of our time-localized band coherence. We apply our method to a pair of highly dynamic brain waves signals whose coherence is shown evolve during an epileptic seizure.

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1 Introduction

Neuroscientists use electroencephalograms (EEGs) to study the highly complex spatio-temporal dynamics of brain processes. One important feature of underlying neuronal activity is that brain regions do not act in isolation. In fact, epileptiform activity and cognitive processes require synchronous activation of multiple large aggregates of neurons [see, e.g., Gotman (1987), Duckrow and Spencer (1992) and Ahlfors et al (1999)]. Thus, it is important to both establish an easily interpretable measure of synchrony between brain regions and develop an estimation procedure that is consistent. There are other many interesting and useful ways of analyzing inter-relationships between time series. For example, Toda and Philipps (1993, 1994) develop Wald tests for Granger causality. In addition, Dahlhaus and Eichler (2003) and Eichler (2005, 2006) develop graphical models and the significant contribution of these work is that they allow one to test for directionality and causality between time series. This approach uses parametric representations like the multivariate autoregressive models. In this paper we focus on coherence which is a widely used measure of linear dependence between two time series. We give an intuitive interpretation to coherence in a locally stationary setting and develop a novel data-adaptive non-parametric approach to estimating it.

In standard textbooks, coherence is interpreted as "the frequency domain analogue of the autocorrelation function" (Brillinger, 1981) or as the "correlation between the stochastic increments in the spectral representation" (Brockwell and Davis, 1991). This notion of coherence does not offer an intuitive interpretation. We now consider the formal definition of coherence. Let $\{(X_{1,t}, X_{2,t}), t = 0, \pm 1, \pm 2, \ldots\}$ be a bivariate stationary time series with spectral density matrix $\boldsymbol{f}(\omega)$ which is hermitian and whose diagonal elements $f_{11}(\omega)$ and $f_{22}(\omega)$ are the auto-spectra and the off-diagonal element $f_{12}(\omega)$ is the cross-spectra. Coherence between $X_{1,t}$ and $X_{2,t}$ is defined to be $\mathcal{K}(\omega) = f_{12}(\omega)/\sqrt{f_{11}(\omega) f_{22}(\omega)}$, whose form is similar to that of correlation. That is, coherence is the covariance (cross-spectrum) normalized by the variances (auto-spectra). In this paper, we will present coherence from

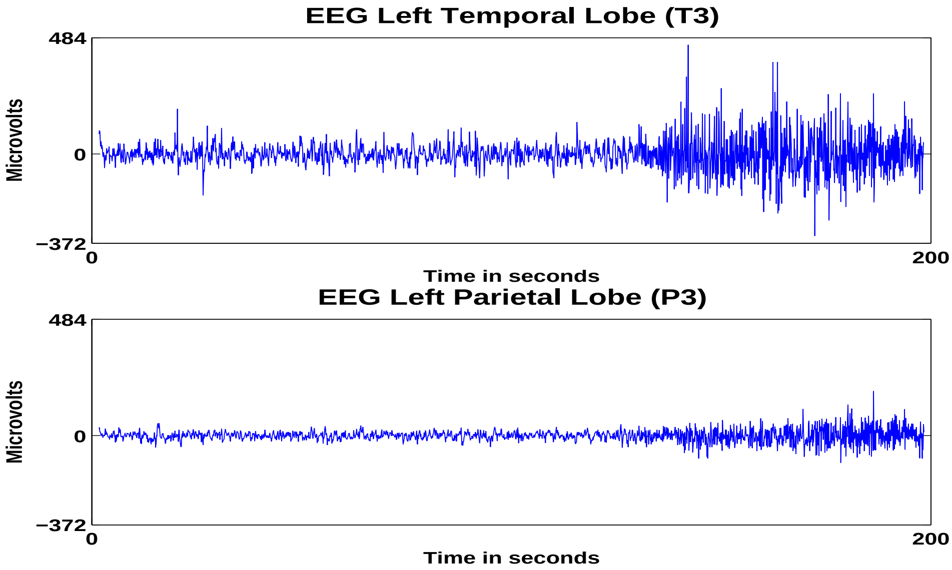

Figure 1: EEG surface recordings at the left temporal lobe channel ($T3$) and the left parietal lobe channel ($P3$). Length of the time series is $T = 16000$; sampling rate is 100 hertz; total time of recording is 160 seconds.

a time domain point of view which we hope can give a more illuminating interpretation to this measure of linear dependence. Moreover, we develop an estimator that is consistent and performs at least as well as the classical approach based on smoothing auto and cross-periodograms.

For stationary time series, coherence varies only according frequency ω but remains constant over time. In other words, under stationarity, the linear dependence between two time series does not change with time. It is our practical experience that most data are non-stationary. Consider for example the bivariate EEGs in Figure 1; a visual inspection of which reveals that the variance of the EEGs evolve over time. Equivalently, the auto-spectra may vary over time. Moreover, the nature of linear dependence between $X_{1,t}$ and $X_{2,t}$ may change with time as well. There are time-dependent analogues of coherence for multivariate non-stationary time series developed in the literature. Li and Klemm (2000) use the non-decimated wavelets; Maraun and Kurths (2004) define the wavelet cross-coherence which depends on time and scale (or frequency); Ombao, Raz, von Sachs and Malow (2001) and Ombao, von Sachs and Guo (2005), both of which use the SLEX transform (an orthogonal time-localized generalization of the Fourier transform). In this paper, we develop a procedure for estimating the time-varying coherence between time series via a linear filtering approach. We will apply our procedure for estimating the evolutionary coherence between the EEG signals at the left temporal lobe ($T3$ channel) and the left parietal lobe ($P3$ channel).

The goals of this research are to present a more illuminating interpretation of coherence via filtering and to develop a novel point-wise adaptive statistical procedure for estimating the time-evolutionary coherence of non-stationary time series. To fix ideas, we first look into the stationary setting. Consider a filter whose power transfer function is concentrated at a frequency band around ω_0 which we denote to be $\Omega = [\omega_0 - \delta, \omega_0 + \delta]$. Let us denote the filtered signals to be $Y_{1,t}$ and $Y_{2,t}$. We will show that coherence between $X_{1,t}$ and $X_{2,t}$ at frequency band Ω (which we call **band coherence**) is simply the cross-correlation between the filtered time series $Y_{1,t}$ and $Y_{2,t}$. Thus, based on this more intuitive interpretation, a natural estimator for the band coherence is the sample correlation between the filtered observed time series.

We generalize our estimation method to the non-stationary setting. Again, to fix ideas, suppose that the scientist is interested in how coherence, at a particular band Ω, evolves over time. First, fix a filter whose transfer function is concentrated in Ω. To estimate the band coherence around time point t_0, the procedure is simple. First, form a time window centered around t_0; apply the filter on the windowed series and finally compute the sample correlation between the filtered signals. Note that this correlation is "localized in time" because it is derived from observations in the time window around t_0. A natural interest is in the choice of the width of the window. One of the contributions in this paper is a consistent procedure that automatically selects the optimal width of the time window on which the local band coherence is to be computed. The principle is that one should use the largest possible window within which the band coherence is almost constant. This approach is point-wise adaptive in the sense that it finds an"optimal" window for each time point t_0 under consideration.

This work on coherence via filtering is also closely related to band spectrum regression for bivariate stationary time series $[X_1(t), X_2(t)]$ in Hannan (1963) and Engle (1974) who modelled the linear relationship between the two time series via $X_2(t) = \beta X_1(t) + \epsilon(t)$ where the errors $\epsilon(t)$ are also stationary. The goal in the above-mentioned papers was to estimate β which naturally characterizes the linear relationship between the two time series. Their approach was to convert the time domain model into the frequency domain thus resulting in a model that is essentially a regression between band spectra (or averaged periodograms) of the two time series X_2 and X_1. This approach does not impose any parametric structure on the autocovariance function of $\epsilon(t)$. The estimator for β depends on the bandwidth used for estimating the spectrum of the error $\epsilon(t)$. The efficiency of frequency domain estimator in the regression setting is studied rigorously in Xiao and Phillips (1998) who considered higher order expansion of the coefficient estimates. In contrast to these approaches, this present paper studies coherence in investigating the relationship between X_1 and X_2. Moreover, coherence is studied under both the stationary and non-stationary setting.

The rest of this paper is organized as follows. In Section 2, we develop the procedure for stationary time series and then extend this to the non-stationary setting in Section 3. Results from the simulation studies are presented in Section 4; theoretical investigations are developed in Section 5 and finally an application of the method to epileptic seizure EEG signals are presented in Section 6.

2 Band Coherence Estimation by Filtering for Stationary Time Series

Suppose we observe a vector-valued time series $\boldsymbol{X}_t = (X_{1,t}, X_{2,t})$, $t = 1, \ldots, T$, from a bivariate, discrete-time, weakly stationary, zero mean process having a spectral representation $\boldsymbol{X}_t = \int_{-\pi}^{\pi} \exp(i\omega t) d\boldsymbol{Z}(\omega)$ where $\boldsymbol{Z}(\omega)$ is a bivariate zero mean orthogonal increment random process whose covariance structure $\text{cov}(d\boldsymbol{Z}(\omega_1), d\boldsymbol{Z}(\omega_2)) = \delta(\omega_1 - \omega_2)\boldsymbol{f}(\omega_1)$ where δ is the Dirac-delta function and $\boldsymbol{f}(\omega)$ is the spectral density matrix. The auto-spectra of $X_{1,t}$ and $X_{2,t}$ are the diagonal elements denoted as $f_{11}(\omega)$ and $f_{22}(\omega)$ while the cross-spectrum between $X_{1,t}$ and $X_{2,t}$ is denoted $f_{12}(\omega)$.

The band coherence between $X_{1,t}$ and $X_{2,t}$ over frequency $\Omega = [\omega_0 - \delta, \omega_0 + \delta] \subset [-\pi, \pi]$ is defined as

$$\mathcal{K}(\Omega) = \frac{\int_\Omega f_{12}(\omega)d\omega}{\sqrt{\int_\Omega f_{11}(\omega)d\omega \int_\Omega f_{22}(\omega)d\omega}}. \tag{1}$$

Band coherence is a complex-valued number. Its magnitude lies in the unit interval $[0, 1]$ so that a value that is close to 1 indicates a strong linear association between $X_{1,t}$ and $X_{2,t}$. Its phase measures the time lag between the two time series. Another widely-used index is the squared band coherence $|\mathcal{K}(\Omega)|^2$. We note that band coherence is zero at all frequency bands if and only if the time series $X_{1,t}$ and $X_{2,t}$ are uncorrelated.

2.1 Estimation via the periodogram matrix.

The classical approach to estimating the spectral matrix for a fixed frequency ω is via the periodogram matrix

$$\boldsymbol{I}_T(\omega) = (2\pi T)^{-1} \left(\sum_{t=0}^{T-1} \boldsymbol{X}_t \exp(-i\lambda t) \right) \left(\sum_{t=0}^{T-1} \boldsymbol{X}_t \exp(-i\lambda t) \right)^*. \tag{2}$$

The periodogram matrix is unbiased for the spectral density matrix. However, it is not consistent. A consistent estimator for band coherence $\mathcal{K}(\Omega)$ is given by $\widetilde{I}_{12}(\Omega)/\sqrt{\widetilde{I}_{11}(\Omega)\widetilde{I}_{22}(\Omega)}$ where $\widetilde{I}_{\ell\ell}(\Omega)$ is the averaged auto periodogram and $\widetilde{I}_{\ell m}(\Omega)$ is the averaged cross periodogram across the frequency band Ω. The properties of these estimators, including their asymptotic distribution are developed in Brillinger (1981) and Brockwell and Davis (1991).

2.2 Coherence: interpretation in terms of linear filtering

An alternate estimator of band coherence can be derived from an appropriate linear filtering of the time series $X_{1,t}$ and $X_{2,t}$. Suppose that the goal is to estimate band coherence at $\Omega = [\omega_0 - \delta, \omega_0 + \delta]$. The first task is to construct an absolutely summable sequence $\{b_k\}$ with transfer function $B(\omega) = \sum_{k=-\infty}^{\infty} b_k \exp(-i\omega k)$ that is concentrated on Ω, i.e.,

$$B(\omega) = \begin{cases} \sqrt{1/(2\delta)} & \text{for } \omega \in \Omega \\ 0 & \text{elsewhere} \end{cases} \tag{3}$$

where 2δ is the length of the interval Ω. The corresponding time sequence is given by the inverse Fourier transform of B:

$$b_k = \begin{cases} \frac{1}{2\pi}\sqrt{2\delta} & \text{for } k = 0 \\ \frac{2}{2\pi k} e^{i\omega_0 k}\sin(\delta k)\sqrt{1/(2\delta)} & \text{for } k \neq 0 \end{cases} \tag{4}$$

We first note that the filter sequence is allowed to be complex so that, if desired, it would be possible to analyze phase-lag relationships between the resulting filtered time series. However, if one must use a real-valued filter, then it is straightforward to generalize this by imposing the transfer function to be concentrated at both ω_0 and $-\omega_0$.

Applying the same filter $\{b_k\}$ on both $X_{1,t}$ and $X_{2,t}$ gives the output filtered time series $\boldsymbol{Y}_t = (Y_{1,t}, Y_{2,t})$ where $Y_{\ell,t} = \sum_{s=-\infty}^{\infty} b_{t-s} X_{\ell,s}$ for $\ell = 1, 2$. The spectral density matrix $\boldsymbol{f}^Y$ of the bivariate filtered process $\boldsymbol{Y}_t$ is related to the spectral density matrix $\boldsymbol{f}^X$ of the original process $\boldsymbol{X}_t$ by $f^Y_{\ell n}(\omega) = |B(\omega)|^2 f^X_{\ell n}(\omega) \qquad 1 \leqslant \ell, n \leqslant 2$. We derive the variances and the cross-covariance between $Y_{1,t}$ and $Y_{2,t}$ to be

$$\mathbb{V}\mathrm{ar}(Y_{1,t}) = \int_{-\pi}^{\pi} |B(\omega)|^2 f^X_{11}(\omega) d\omega = \int_{\Omega} \frac{1}{2\delta} f^X_{11}(\omega) d\omega = f^X_{11}(\Omega) \tag{5}$$

$$\mathbb{V}\mathrm{ar}(Y_{2,t}) = \int_{-\pi}^{\pi} |B(\omega)|^2 f^X_{22}(\omega) d\omega = \int_{\Omega} \frac{1}{2\delta} f^X_{22}(\omega) d\omega = f^X_{22}(\Omega) \tag{6}$$

$$\mathbb{C}\mathrm{ov}(Y_{1,t}, Y_{2,t}) = \int_{-\pi}^{\pi} |B(\omega)|^2 f^X_{12}(\omega) d\omega = \int_{\Omega} \frac{1}{2\delta} f^X_{12}(\omega) d\omega = f^X_{12}(\Omega). \tag{7}$$

This implies that $\mathbb{C}\mathrm{orr}(Y_{1,t}, Y_{2,t}) = f^X_{12}(\Omega)/\sqrt{f^X_{11}(\Omega) f^X_{22}(\Omega)}$ which is identical to the the band coherence in Equation (1). We believe that the above sheds light on the interpretation of the band coherence between two time series and gives a precise meaning to “frequency analogue of correlation”. That is, coherence $\mathcal{K}(\Omega)$ may be seen as the correlation between two filtered signals, where the (band pass) filter is concentrated at the band Ω.

Consequently, a reasonable estimator of $\mathcal{K}(\Omega)$ is given by the sample cross correlation

$$\widehat{\mathcal{K}}(\Omega) = \frac{T^{-1}\sum_{t=1}^{T} Y_{1,t} Y^*_{2,t}}{\sqrt{T^{-1}\sum_{t=1}^{T} Y_{1,t} Y^*_{1,t}}\sqrt{T^{-1}\sum_{t=1}^{T} Y_{2,t} Y^*_{2,t}}}. \tag{8}$$

We state a central limit theorem for the band coherence estimator. The proof is given in the appendix.

Proposition 2.1. *Let $\boldsymbol{X}(t)$ be a stationary linear process $\boldsymbol{X}(t) = \sum_{s=-\infty}^{\infty} \boldsymbol{C}(t+s)\boldsymbol{Z}(s)$ where $\{\boldsymbol{Z}(s)\}$ is an iid innovation process with non-singular covariance matrix $\boldsymbol{\Sigma}$ such that for $\ell = 1, 2$, $\mathbb{E}\ Z_\ell(t)^4 < \infty$. Moreover, for $\ell = 1, 2$, the sequence of matrices C fulfills $\sum_{u=-\infty}^{\infty} \sqrt{|u|}|C_{j,\ell}(u)| < \infty$. Then $T^{\frac{1}{2}} \tanh^{-1}(|\widehat{\mathcal{K}}(\Omega)|)$ is $AN\left(\tanh^{-1}(|\mathcal{K}(\Omega)|), \delta^{-1}\right)$.*

Consequently, we can form an approximate $100(1-\alpha)\%$ confidence interval for $|\mathcal{K}(\Omega)|$. Let $\Phi(.)$ be the cdf of a standard normal. Then an approximate $100(1-\alpha)\%$ confidence interval for $\tanh^{-1}|\mathcal{K}(\Omega)|$ is of the form $[W_0, W_1] = \tanh^{-1}|\widehat{\mathcal{K}}(\Omega)| \pm \Phi(\alpha/2)(\delta T)^{-\frac{1}{2}}$. Finally, an approximate $100(1-\alpha)\%$ confidence interval for $|\mathcal{K}(\Omega)|$ is $\left[\frac{\exp(2W_0)-1}{\exp(W_0)+1}, \frac{\exp(2W_1)-1}{\exp(W_1)+1}\right]$ which is contained in $[0, 1]$.

2.3 Remarks.

For a band pass filter whose power transfer function is concentrated at band Ω, we note that $\mathbb{V}\mathrm{ar}(Y_{\ell,t}) = \int_\Omega \frac{1}{2\delta} f_{\ell\ell}^X(\omega)d\omega$. Thus, the sample variance of $Y_{\ell,t}$, $\widehat{\mathbb{V}\mathrm{ar}}(Y_{\ell,t}) = \frac{1}{T}\sum_{t=0}^{T} Y_{\ell,t}Y_{\ell,t}^*$, would lead to a consistent estimator of the auto spectra $f_{\ell\ell}^X(\Omega)$ ($\ell = 1, 2$). It is quite remarkable that this approach was already proposed in Pupin (1894) more than a century ago and, according to Brillinger (1981), could be the first spectral estimator used in practice.

Moreover, similar to the above alternate estimator of the auto spectra, a consistent estimator the cross spectrum $f_{12}^Y(\Omega)$ can be obtained via the sample cross-covariance between the two filtered time series $\widehat{\mathbb{C}\mathrm{ov}}(Y_{1,t}, Y_{2,t}) = \frac{1}{T}\sum_{t=1}^{T} Y_{1,t}Y_{2,t}^*$.

3 Evolutionary Band Coherence of Non-Stationary Time Series

Consider a sequence of zero mean Gaussian bivariate locally stationary stochastic process [as in Dahlhaus (2000))] $\boldsymbol{X}_{t,T} = (X_{1,T}(t), X_{2,T}(t))$, $t = 1, \ldots, T$, with transfer function matrix $\mathbf{A}^{(o)}$ having a spectral representation $\mathbf{X}_{t,T} = \int_{-\pi}^{\pi} \mathbf{A}_{t,T}^{(o)}(\omega)\exp(i\omega t)d\mathbf{Z}(\omega)$ with the following properties (i.) $\mathbf{Z}(\omega)$ is a zero mean complex-valued Gaussian bivariate process on $[-\pi, \pi]$ with $d\mathbf{Z}^*(\omega) = d\mathbf{Z}(-\omega)$ and $\mathbb{E}\ \{d\mathbf{Z}_j(\omega), d\mathbf{Z}_\ell(\omega)\} = \delta_{j\ell}\ \eta(\omega + \lambda)\ d\omega\ d\lambda$, where $\eta(\lambda) = \sum_{k=-\infty}^{\infty} \delta(\lambda + 2\pi k)$ is the 2π periodic extension of the Dirac delta function; and

(ii.) There exists a constant C and a 2π-periodic matrix-valued function $\mathbf{A} : [0, 1] \times \mathbb{R} \rightarrow \mathbb{C}^{2\times 2}$ with $\mathbf{A}^*(u, \omega) = \mathbf{A}(u, -\omega)$ and $\sup_{t,\omega} |\mathbf{A}_{t,T}^{(o)}(\omega)_{j\ell} - \mathbf{A}(t/T, \omega)_{j,\ell}| \leq CT^{-1}$ for $j, \ell = 1, 2$ and $T \in \mathbb{N}$.

The function $\mathbf{A}(u, \omega)$ is continuous in rescaled time u and the time-varying spectral matrix is $\mathbf{f}(\omega) = \mathbf{A}(u, \omega)\mathbf{A}^*(u, \omega)$. In other words, the auto-spectra and cross-spectrum at rescaled time u frequency ω are, respectively, $f_{11}(u, \omega)$, $f_{22}(u, \omega)$ and $f_{12}(u, \omega)$. Our goal is to estimate coherence at rescaled time $u \in [0, 1]$ and frequency band Ω, which we define to be

$$\mathcal{K}(u, \Omega) = \frac{\int_{-\pi}^{\pi} |B(\omega)|^2 f_{12}(u, \omega) d\omega}{\sqrt{\int_{-\pi}^{\pi} |B(\omega)|^2 f_{11}(u, \omega) d\omega \int_{-\pi}^{\pi} |B(\omega)|^2 f_{22}(u, \omega) d\omega}}. \tag{9}$$

The essential idea in our approach is to find the largest possible U that contains the target time point u where for any $v \in U$, $|\mathcal{K}(u, \Omega) - \mathcal{K}(v, \Omega)|$ is small. In this case, we denote the local coherence averaged over the interval $U \subset [0, 1]$ and frequency band $\Omega \subset [-\pi, \pi]$ is defined to be

$$\mathcal{K}(U, \Omega) = \frac{\int_{-\pi}^{\pi} \int_U |B(\omega)|^2 f_{12}(u, \omega) d\omega du}{\sqrt{\int_{-\pi}^{\pi} \int_U |B(\omega)|^2 f_{11}(u, \omega) d\omega du \int_{-\pi}^{\pi} \int_U |B(\omega)|^2 f_{22}(u, \omega) d\omega du}}. \tag{10}$$

3.1 Local coherence: interpretation in terms of linear filtering

We now extend the ideas in the stationary case to the situation where coherence is allowed to evolve over time. First, we define the interval U on rescaled time to be $[u_0, u_1] \subset [0, 1]$. The corresponding interval of U on real time is $\{[u_0 T], \ldots, [u_1 T]\}$. Consider the time series at the local interval $\{[u_0 T], \ldots, [u_1 T]\}$ to be:

$$X_{1,T}(t) \cdot \mathbb{1}_U(t) \qquad \text{and} \qquad X_{2,T}(t) \cdot \mathbb{1}_U(t)$$

where $\mathbb{1}_U(t) = 1$ if $t/T \in U$ and $\mathbb{1}_U(t) = 0$ if $t/T \notin U$. We apply the filter b_k to the windowed time series to produce

$$Y_{1,T}(t) = \sum_{k=-\infty}^{\infty} b_{t-k} X_{1,T}(k) \mathbb{1}_U(k) \qquad \text{and} \qquad Y_{2,T}(t) = \sum_{k=-\infty}^{\infty} b_{t-k} X_{2,T}(k) \mathbb{1}_U(k)$$

The variances of and cross-variance between $Y_{1,T}(t)$ and $Y_{2,T}(t)$ are, respectively,

$$\begin{aligned}
\mathbb{V}\mathrm{ar}(Y_{1,T}(t)) &= \int_{-\pi}^{\pi} \int_U |B(\omega)|^2 f_{11}(u,\omega) d\omega du + o_T(1) \\
&= \int_\Omega \int_U \frac{1}{2\delta |U|} f_{11}(u,\omega) d\omega du + o_T(1); \\
\mathbb{V}\mathrm{ar}(Y_{2,T}(t)) &= \int_{-\pi}^{\pi} \int_U |B(\omega)|^2 f_{22}(u,\omega) d\omega du + o_T(1) \\
&= \int_\Omega \int_U \frac{1}{2\delta |U|} f_{22}(u,\omega) d\omega du + o_T(1); \\
\mathbb{C}\mathrm{ov}(Y_{1,T}(t), Y_{2,T}(t)) &= \int_{-\pi}^{\pi} \int_U |B(\omega)|^2 f_{12}(u,\omega) d\omega du + o_T(1) \\
&= \int_\Omega \int_U \frac{1}{2\delta |U|} f_{12}(u,\omega) d\omega du + o_T(1)
\end{aligned}$$

Denote for $\ell, m = 1, 2$, $f_{\ell m}(U,\Omega) = \int_\Omega \int_U \frac{1}{2\delta} \frac{1}{|U|} f_{\ell m}(u,\omega) d\omega du$. It follows that the correlation between $Y_{1,T}(t)$ and $Y_{2,T}(t)$ is $\mathbb{C}\mathrm{orr}(Y_{1,T}(t), Y_{2,T}(t)) = f_{12}(U,\Omega)/\sqrt{f_{11}(U,\Omega) f_{22}(U,\Omega)}$ which is clearly identical to the quantity $\mathcal{K}(U,\Omega)$ in Equation (10). This suggests that a good estimator for $\mathcal{K}(U,\Omega)$ is

$$\widehat{\mathcal{K}}(U,\Omega) := \frac{\sum_{t:t/T\in U} Y_{1,T}(t) Y_{2,T}^*(t)}{\sqrt{\sum_{t:t/T\in U} Y_{1,T}(t) Y_{2,T}^*(t) \sum_{t:t/T\in U} Y_{2,T}(t) Y_{2,T}^*(t)}}\,. \tag{11}$$

We state a central limit theorem for the band coherence estimator under the Dahlhaus locally stationary process. The proof is given in the appendix.

Proposition 3.1. *Let $\mathbf{X}_T(t) = [X_{1,T}(t), X_{2,T}(t)]'$, $t = 1, \ldots, T$, be a Gaussian locally stationary process (as in Dahlhaus, 2000). Define $2m_T + 1$ to be the number of discrete frequencies in the band Ω. Then, as $T \to \infty$, $m_T \to \infty$, $m_T/T \to \frac{|U||\Omega|}{2\pi}$ and*

$$\frac{\sqrt{2(2m_T+1)}}{1 - \mathbb{E}|\widehat{\mathcal{K}}(U,\Omega)|^2} \left(|\widehat{\mathcal{K}}(U,\Omega)| - \mathbb{E}|\widehat{\mathcal{K}}(U,\Omega)| \right) \to N(0,1).$$

3.2 Point-wise adaptive selection method for the optimal time window U

Suppose we want to estimate the absolute band coherence of a locally stationary bivariate time series at frequency band Ω, around a given time point u. The basic idea of our procedure is to find the largest possible interval $U = [u_0, u_1]$, where $u_0 < u < u_1$ where the local band coherence is approximately constant. This locally adaptive approach is related to the work of Lepski (1990) in the non-parametric regression context for iid data; see also Mercurio and Spokoiny (2004) and Van Bellegem and von Sachs (2005).

First, we set some notation. Let n_U be the number of time points included in the real time interval $\{[u_0T], \ldots, [u_1T]\}$; let $(2m+1)$ be the number of discrete frequencies in the frequency interval Ω; and $|\Omega|$ be the length of the frequency band Ω. Given that the corresponding real time interval has n_U observations, then the resolution in frequency is $2\pi/n_U$. Consequently, the number of discrete frequencies in Ω corresponding to this resolution is $1+n_U|\Omega|/(2\pi)$. This leads to the relationship $2m+1 = \frac{n_U|\Omega|}{2\pi}+1$ By replacing $2m+1$ in the above central limit theorem and applying the delta-method we derive

$$a_U \left(\tanh^{-1}(|\widehat{\mathcal{K}}(U,\Omega)|) - \tanh^{-1}\mathbb{E}(|\widehat{\mathcal{K}}(U,\Omega)|)\right) \xrightarrow{d} N(0,1) \tag{12}$$

where a_U is such that $a_U^2 = 2 + n_U|\Omega|/\pi$. We now give a complete working algorithm to select U.

Initialization. Select the smallest interval U_0 containing u.

Iteration. Set $U = U_0$ and calculate the corresponding estimate $\widehat{\mathcal{K}}(\Omega, U)$. Expand the interval U to $U^* = U_1$, i.e., $U \subset U^*$ and then calculate the corresponding estimate $\widehat{\mathcal{K}}(\Omega, U^*)$.

Testing heterogeneity. Define the following quantities:

$$\begin{aligned}
\Delta(U,U^*) &= |\tanh^{-1}(|\mathcal{K}(U,\Omega)|) - \tanh^{-1}(|\mathcal{K}(U^*,\Omega)|)| \\
\widehat{\Delta}(U,U^*) &= |\tanh^{-1}(|\widehat{\mathcal{K}}(U,\Omega)|) - \tanh^{-1}(|\widehat{\mathcal{K}}(U^*,\Omega)|)| \\
D(U,U^*) &= \frac{\widehat{\Delta}(U,U^*)}{a_U^{-1} + a_{U^*}^{-1}}.
\end{aligned} \tag{13}$$

If $D(U, U^*) > \zeta$, for some preselected constant ζ, then heterogeneity in coherence within the larger interval U^* is detected. The decision is to keep $U = U_1$ as the optimal interval of homogeneity.

Loop. If $D(U, U^*) \leq \zeta$, then there is no sufficient evidence for heterogeneity within U^*. Set $U = U_1$ and expand to $U^* = U_2$. Calculate the corresponding estimate $\widehat{\mathcal{K}}(\Omega, U^*)$. Test for heterogeneity in Equation (13) and continue expanding until heterogeneity is detected.

This procedure determines the largest possible interval around the time point u on which we can estimate the local band absolute coherence using the filtering method explained in the previous section. This approach is *pointwise adaptive* in the sense that it finds a different "optimal" interval for each time point u under consideration. It is also important to note that we do not make any global assumption on the behavior of the coherency structure between the two time series. In particular, we do *not* assume the process to be piecewise stationary. The only implicit assumption is that around each time point t_0 there is some interval where coherence is approximately constant (homogenous) so that we can compute the band coherence as the sample correlation between to filtered time series.

The *theoretical* justification of (13) comes from the Central Limit Theorem (Proposition 3.1), which implies:

$$
\begin{aligned}
&\mathbb{P}\left(\frac{\left|\tanh^{-1}(|\widehat{\mathcal{K}}(U,\Omega)|) - \tanh^{-1}(|\widehat{\mathcal{K}}(U^*,\Omega)|)\right|}{a_U^{-1} + a_{U^*}^{-1}} > \zeta\right) \\
&\leqslant \mathbb{P}\left(\left|\tanh^{-1}(|\widehat{\mathcal{K}}(U,\Omega)|) - \tanh^{-1}(|\mathcal{K}(U,\Omega)|)\right| > a_U^{-1}\zeta\right) \\
&\qquad + \mathbb{P}\left(\left|\tanh^{-1}(|\widehat{\mathcal{K}}(U^*,\Omega)|) - \tanh^{-1}(|\mathcal{K}(U^*,\Omega)|)\right| + \Delta(U,U^*) > a_{U^*}^{-1}\zeta\right) \quad (14)
\end{aligned}
$$

Under the assumption of near-constant coherence (or "homogeneity"), $\Delta(U, U^*) \approx 0$. Thus, by the central limit theorem, the above probability is asymptotically bounded by $4\Phi(\zeta)$.

In practice, the choice of the threshold ζ in (13) based on the central limit theorem leads to a very conservative test rule. That is, the method tends to choose a larger interval even in the presence of heterogeneity of the band coherence. This can cause severe bias problems. Thus, one has to seek an alternative data-driven approach to determining the critical value ζ. Note that the choice of this parameter is global in the sense that it does not depend on u.

We propose a data-driven procedure to select the critical value based on the minimization of some prediction error. First we define a grid of potential critical values ζ and a set of test time points $\mathcal{T} = \{t_1, \ldots, t_N\}$. For each candidate critical value ζ and each test point $t \in \mathcal{T}$, we consider the corresponding segment U_t^{ζ} given by the above procedure around the time point t. The selected segment U_t^{ζ} can be used to compute the localized coherency around u, where $[uT] = t$ using the filtered estimator. We denote this estimator by $\widehat{\mathcal{K}}^{\zeta}(u, \Omega)$. By doing so for different time points $t \in \mathcal{T}$, we select the critical value $\widehat{\zeta}$ such that minimizes the prediction error is reached

$$\widehat{\zeta} = \arg\min_{\zeta} \sum_{t \in \{t_1, \ldots, t_N\}} \left\{ Y_{1,T}(t) - \widehat{\mathcal{K}}_{21}^{\zeta}(t/T, \Omega) \frac{\widehat{\sigma}_2}{\widehat{\sigma}_1} Y_{2,T}(t) \right\}^2$$

where $\widehat{\sigma}_\ell^2 = n_{U_t^{\zeta}}^{-1} \sum_{s \in U_t^{\zeta}} Y_{\ell,T}^2(s)$.

The above criterion is motivated by the mean-squared error of linear predictors. Assume Y_1 and Y_2 to be zero mean random variables. Define βY_1 to be the linear predictor for Y_2. The "best" linear predictor minimizes the mean squared prediction error $\mathbb{E}(Y_2 - \beta Y_1)^2$. The value of β that minimizes the mean squared prediction error is $\beta = \mathbb{C}\mathrm{ov}(Y_2, Y_1)/\, \mathbb{V}\mathrm{ar}(Y_1) = \mathcal{K}_{21}\sqrt{\mathbb{V}\mathrm{ar}\, Y_2/\, \mathbb{V}\mathrm{ar}\, Y_1}$. The next section shows that this procedure gives reasonable results in simulations and analysis of EEG data.

4 Simulations and Coherence Analysis of EEGs

We investigated the performance of our estimator on a small simulation study and then analyzed a pair of EEGs recorded during an epileptic seizure.

4.1 Simulation Study

We describe the model from which we generated the non-stationary time series. First, we considered a latent independent stationary processes $Z_1(t)$ and $Z_2(t)$ with spectra $h_1(\omega)$

and $h_2(\omega)$ respectively. Next, we generated the observed bivariate non-stationary time series from the model

$$X_{1,T}(t) = A_{11}(t)Z_1(t) + A_{12}(t)Z_2(t) + W_{1,t} \tag{15}$$

$$X_{2,T}(t) = A_{21}(t)Z_1(t) + A_{22}(t)Z_2(t) + W_{2,t} \tag{16}$$

where W_1 are W_2 are mutually independent stationary white noise processes. Moreover, the W's and the Z's are independent. We note that if $A_{\ell n}(t)$ is constant in time, then $\boldsymbol{X}_t$ is stationary. Otherwise, $\boldsymbol{X}_{t,T}$ is non-stationary. We now derive the band coherence between $X_{1,T}(t)$ and $X_{2,T}(t)$ based on the above model. Denote the spectra of the white noise process to be identically $\frac{\sigma^2}{2\pi}$. Consequently, the auto-spectra of $X_{1,T}(t)$ and $X_{2,T}(t)$ at rescaled time u such that $[uT] = t$, are

$$\begin{aligned} f_{11}(u,\omega) &= |A_{11}(t)|^2 h_{11}(\omega) + |A_{12}(t)|^2 h_{22}(\omega) + \frac{\sigma^2}{2\pi} \\ f_{22}(u,\omega) &= |A_{21}(t)|^2 h_{11}(\omega) + |A_{22}(t)|^2 h_{22}(\omega) + \frac{\sigma^2}{2\pi}, \end{aligned}$$

respectively. The cross-spectrum between $X_{1,t}$ and $X_{2,t}$ is derived to be

$$f_{12}(\omega, u) = A_{11}(t)A_{21}(t)h_{11}(\omega) + A_{12}(t)A_{22}(t)h_{22}(\omega).$$

For illustration purposes, the goal was to estimate evolutionary band coherence $\mathcal{K}(u,\Omega)$ at frequency band $\Omega = [\omega_0 - \delta, \omega_0 + \delta]$ where $\omega_0 = \frac{2\pi}{1024}64$ and $\delta = \frac{2\pi}{1024}2$. The plots of the true curve and the average of the estimated curves from 1000 data sets using the filtering and the periodogram methods are given in Figure (2). The 10-th and 90-th percentile curves for the two methods are given in Figure (3). The estimated bias and standard deviation of filtering method is $(0.0063; 0.0070)$ while that of the periodogram smoothing method is $(0.0066; 0.0070)$. This study indicates that the filtering method is not worse than the classical auto and cross-periodogram smoothing approach. In fact, the filtering approach has a slightly smaller bias.

4.2 Evolutionary Band Coherence Analysis of EEGs

We illustrate the method for estimating the evolutionary absolute band coherence between the EEGs recorded at the $T3$ (left temporal lobe) and $P3$ (left parietal lobe) channels

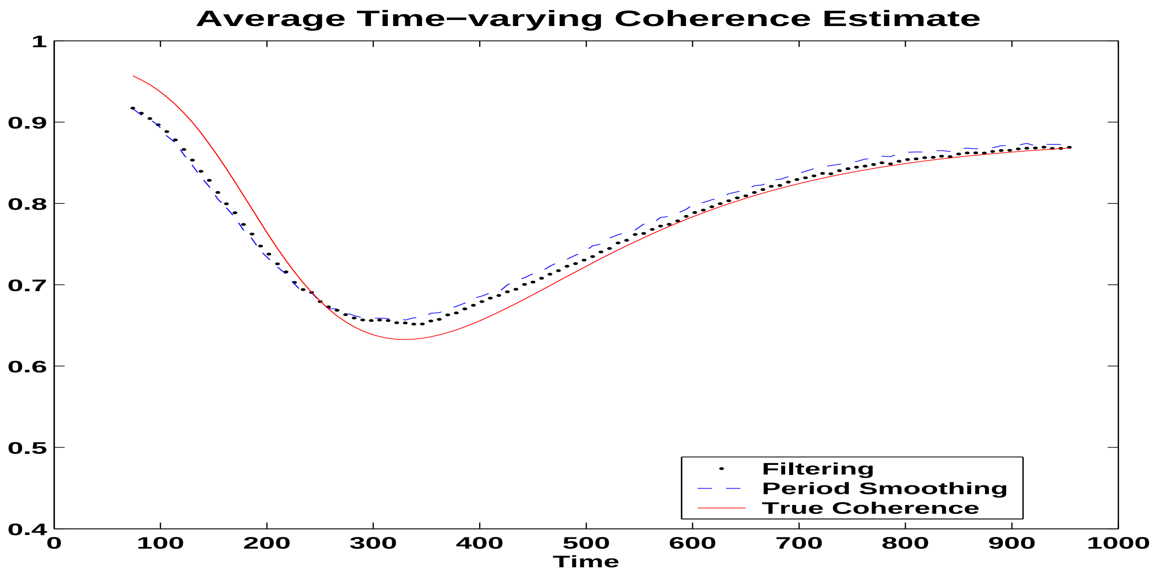

Figure 2: Band coherence for the non-stationary process. Plots of the true band coherence and average of the estimates from 1000 data sets obtained from the filtering method and the periodogram smoothing method.

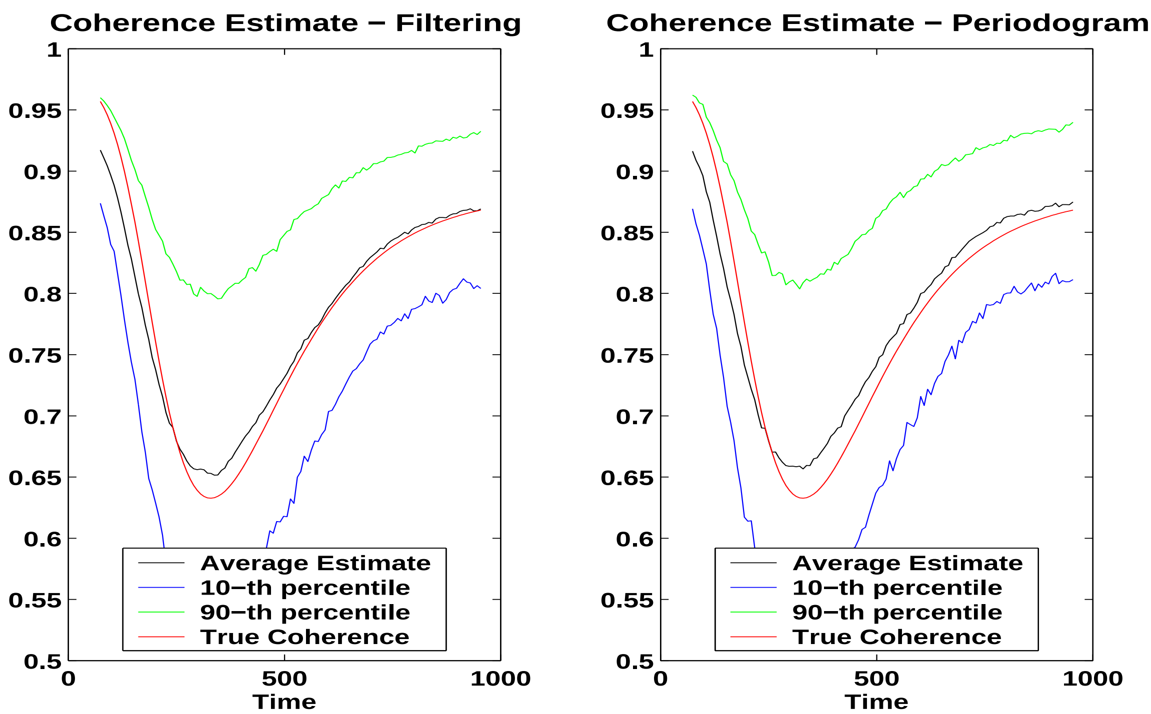

Figure 3: Band coherence for the non-stationary process. Plots of the 10-th percentile; 90-th percentile and average curves of the estimates from 1000 data sets obtained from the filtering method and the periodogram smoothing method.

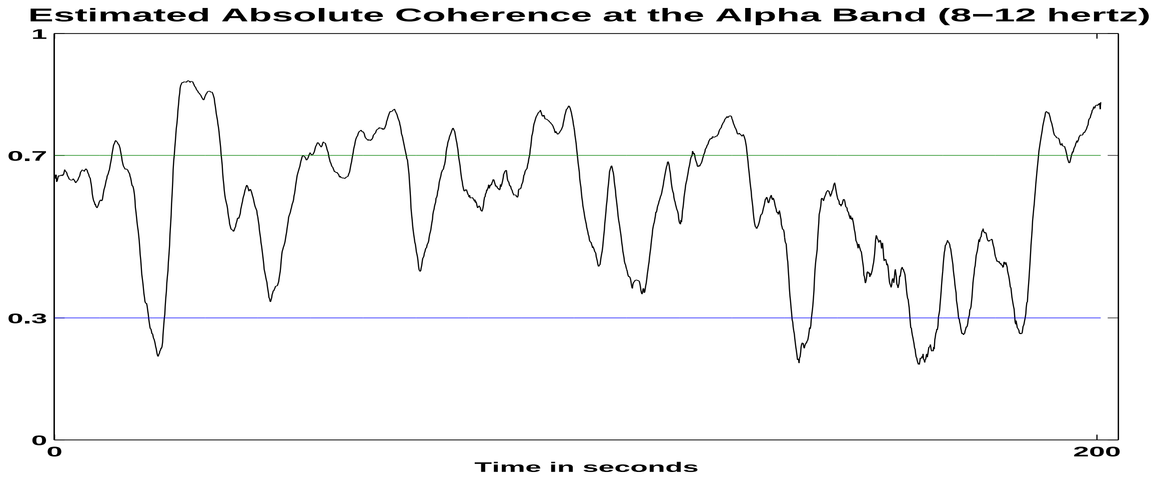

Figure 4: Estimate of the evolutionary absolute coherence between the EEGs at the $T3$ and $P3$ channels at the alpha band $8 - 12$ hertz.

during an epileptic seizure. Band coherence was analyzed at the alpha ($8 - 12$ hertz) and the beta ($12 - 30$ hertz) bands. It is typical in the EEG literature to categorize absolute coherence in the interval $[0, 0.3)$ to be “weak”; $[0.3, 0.7)$ to be “moderate” and $[0.7, 0.1]$ to be “strong”. The results of our analysis further confirm clinicians’ theories about epileptic seizure and bring about some useful information that would not have been obvious by a mere visual inspection of the EEGs. We first observe that the coherence profiles at the beta and alpha bands are quite different. In particular, coherence at the beta band fluctuates from weak to moderate while that at the alpha band fluctuates from moderate to strong. Compare Figures (4) and (5). It is possible to relate our findings with the model in Equation (15) which is a perfectly reasonable model that gives a spectral decomposition of EEGs. Suppose that the spectra of the latent processes $h_{11}(\omega)$ and $h_{22}(\omega)$ are concentrated, respectively, at the alpha and beta bands. Then mixing coefficients $A_{11}(t)$ and $A_{21}(t)$ are larger in magnitude than $A_{12}(t)$ and $A_{22}(t)$. In other words, the underlying neuronal aggregates firing at the alpha frequency band has a more pronounced excitatory effect on the electrical activity at the left temporal and parietal brain regions. Finally, our analysis clearly demonstrates that the strength of linear dependence between the two brain regions can change very quickly during an epileptic seizure.

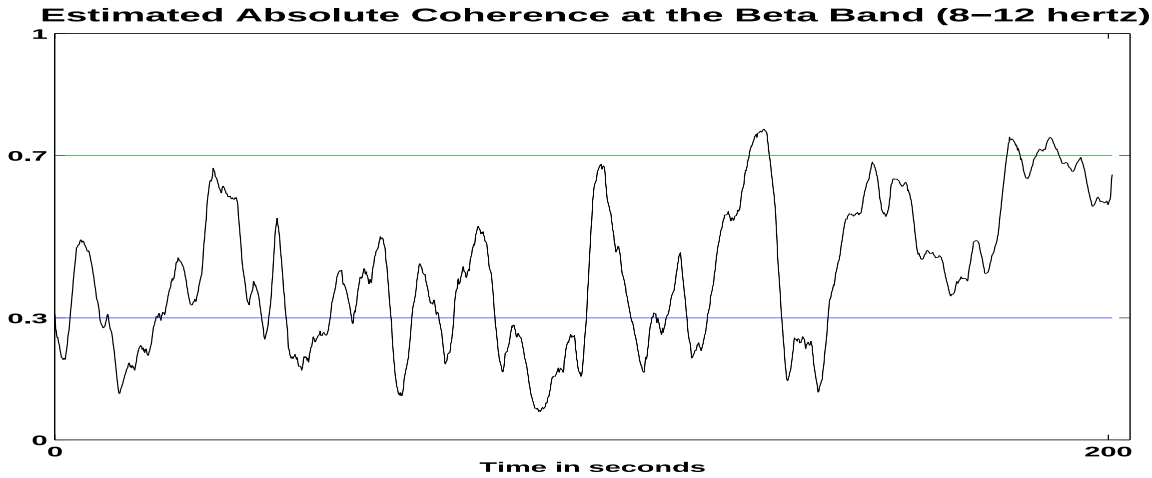

Figure 5: Estimate of the evolutionary absolute coherence between the EEGs at the $T3$ and $P3$ channels at the beta band $12 - 30$ hertz.

5 Conclusion and Discussion

In summary, this paper presented an intuitive interpretation of coherence at a frequency band. That is, coherence is equivalent to the correlation between the two filtered time series. Moreover, we derived asymptotic normality of the Fisher-z transform of the absolute coherence and developed a novel adaptive non-parametric procedure for estimating coherence when the time series are locally stationary, that is, the nature of linear dependence between time series may evolve with time. The procedure automatically selects, via repeated tests of homogeneity (in time) of coherence, the optimal width of the time window on which one computes the estimated local coherence. This approach is point-wise adaptive in the sense that the width of the optimal interval is allowed to change across time.

Furthermore, this interpretation of coherence allows one to generalize the concept to *partial* coherence. To illustrate the idea, let $\mathbf{X}_t = [X_{1,T}(t), X_{2,T}(t), X_{3,T}(t)]'$ be a trivariate time series. By applying the local filter on $\mathbf{X}_t$ at local time window U, we obtain the filtered series $[Y_{1,T}(t), Y_{2,T}(t), Y_{3,T}(t)]'$. The correlation between filtered series $Y_{1,T}(t)$ and $Y_{2,T}(t)$ with the linear effect of $Y_{3,T}(t)$ removed is the partial coherence at frequency Ω: $\mathcal{K}_{12.3}(U,\Omega) = \frac{\mathcal{K}_{12}(U,\Omega) - \mathcal{K}_{13}(U,\Omega)\mathcal{K}^*_{23}(U,\Omega)}{\sqrt{[1-|\mathcal{K}_{13}(U,\Omega)|^2][1-|\mathcal{K}_{23}(U,\Omega)|^2]}}$, where $\mathcal{K}_{\ell m}(U,\Omega)$ is defined as in Equation (10), which can be estimated consistently using the procedure in Section 3.

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This work was supported in part by the contract 'Projet d'Actions de Recherche Concertées' nr 98/03–217 from the Belgian government, the IAP research network nr P5/24 of the Belgian Government (Belgian Science Policy) and NSF-DMS 04-05243 (USA).

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APPENDIX

Proof of Proposition 2.1. Denote $\widehat{k}_{j\ell} = T^{-1}\sum_t Y_j(t)Y_\ell^*(t)$ for $j,\ell = 1$ or 2. The modulus of the coherence can be written

$$|\widehat{\mathcal{K}}(\Omega)| = \frac{\sqrt{\widehat{c}_{12}^2 + \widehat{d}_{12}^2}}{\sqrt{\widehat{k}_{11}\widehat{k}_{22}}} = h\left(\widehat{k}_{11}, \widehat{k}_{22}, \widehat{c}_{12}, \widehat{d}_{21}\right)$$

where $\widehat{c}_{12} := (\widehat{k}_{12} + \widehat{k}_{21})/2$ and $\widehat{d}_{12} := (\widehat{k}_{12} - \widehat{k}_{21})/(2i)$.

We first consider the asymptotic distribution of the vector $(\widehat{k}_{11}, \widehat{k}_{22}, \widehat{c}_{12}, \widehat{d}_{21})$. To compute the expectation we note that, by definition of $Y_j(t)$,

$$\mathbb{E}\widehat{k}_{j\ell} = T^{-1}\sum_{t,s,m} b_{t-s}b_{t-m}\mathbb{E}\{X_j(s)X_\ell^*(m)\}$$

which, using the spectral representation of the (cross-)covariance $\mathbb{E}\{X_j(s)X_\ell^*(m)\}$ and the definition of the sequence b_k, leads to

$$\mathbb{E}\ \widehat{k}_{j\ell} = (2\delta)^{-1}\int_\Omega f_{j\ell}(\omega)d\omega \equiv f_{j\ell}(\Omega).$$

Consequently, $\mathbb{E}\ (\widehat{k}_{11}, \widehat{k}_{22}, \widehat{c}_{12}, \widehat{d}_{21}) = (f_{11}(\Omega), f_{22}(\Omega), c_{12}(\Omega), d_{12}(\Omega))$ with $c_{12}(\Omega) = (f_{12}(\Omega)+ f_{21}(\Omega))/2$ and $d_{12}(\Omega) = (f_{12}(\Omega) - f_{21}(\Omega))/(2i)$.

To derive the covariance and asymptotic normality, we first expand the formula of $\widehat{k}_{j\ell}$ for using the definition of $\{b_k\}$

$$\widehat{k}_{j\ell} = \frac{(2\pi)^2}{T}\iint B(\omega)B(\lambda)d_j(\omega)\overline{d_\ell(\lambda)}\sum_t \exp(-i(\lambda-\omega)t)\ d\omega\ d\lambda$$

where $d_j(\omega) = T^{-1/2}\sum_s X_j(s)\exp(-i\omega s)$ is the Fourier transform of the time series $X_j(s)$.

It follows from the Poisson formula $\sum_{n=-\infty}^{\infty}\exp(-inT\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - \frac{2\pi k}{T})$ [see Mallat (1998), Section 2, for a proof] that the above quantity has the same limiting distribution as $\tilde{k}_{j\ell} := (2\pi)\int B(\omega)I_{j\ell}(\omega)\ d\omega$, where $I_{j\ell}$ denotes the cross-periodogram. The limiting distribution of $\tilde{k}_{j\ell}$ clearly follows using Hannan (1970, Section V.5). The vector $(\widehat{k}_{11}(\Omega), \widehat{k}_{22}(\Omega), \widehat{c}_{12}(\Omega), \widehat{d}_{21}(\Omega))'$ is asymptotically Normal with

$$\mathbb{E}(\widehat{k}_{11}(\Omega), \widehat{k}_{22}(\Omega), \widehat{c}_{12}(\Omega), \widehat{d}_{21}(\Omega))' \;=\; (f_{11}(\Omega), f_{22}(\Omega), c_{12}(\Omega), d_{21}(\Omega))'$$

and covariance matrix

$$\begin{pmatrix} f_{11}^2(\Omega) & |f_{12}(\Omega)|^2 & f_{11}(\Omega)c_{12}(\Omega) & f_{11}(\Omega)d_{12}(\Omega) \\ |f_{12}(\Omega)|^2 & f_{22}^2(\Omega) & f_{22}(\Omega)c_{12}(\Omega) & f_{22}(\Omega)d_{12}(\Omega) \\ f_{11}(\Omega)c_{12}(\Omega) & f_{22}(\Omega)d_{12}(\Omega) & \frac{1}{2}(f_{11}(\Omega)f_{22}(\Omega)+c_{12}^2(\Omega)-d_{12}^2(\Omega)) & c_{12}(\Omega)d_{12}(\Omega) \\ f_{11}(\Omega)d_{12}(\Omega) & f_{22}(\Omega)d_{12}(\Omega) & c_{12}(\Omega)d_{12}(\Omega) & \frac{1}{2}(f_{11}(\Omega)f_{22}(\Omega)+d_{12}^2(\Omega)-c_{12}^2(\Omega)) \end{pmatrix}$$

Finally, by an application of the Delta method, the claimed asymptotic distribution of $\tanh^{-1}(|\widehat{\mathcal{K}}(\Omega)|)$ follows.

Proof of Proposition 3.1 The proof is similar to the proof of Proposition 2.1. The main difference is the derivation of the asymptotic distribution of the vector $(\widehat{k}_{11}, \widehat{k}_{22}, \widehat{c}_{12}, \widehat{d}_{21})$, where $\widehat{k}_{j\ell}$ is now defined by $\widehat{k}_{j\ell} := |U|^{-1}\sum_{t\in U} Y_j(t)Y_\ell^*(t)$ for $j,\ell = 1$ or 2. A straightforward expansion leads to

$$E\widehat{k}_{j\ell} = \frac{1}{T|U|}\iint B(\omega)B(\lambda)\sum_k \exp(-i(\lambda-\omega)k)\sum_{s,t\in U}\exp(-i(s\omega-t\lambda))EX_{j;T}(s)X_{\ell;T}(t)$$

With the change of variables $h := s-t$, this expectation can be written

$$\frac{1}{T|U|}\iint B(\omega)B(\lambda)\sum_k \exp(-i(\lambda-\omega)k)\sum_\delta\sum_t c_{j\ell}\left(\frac{t}{T}+\frac{h}{2T},h\right)\exp(-it(\omega-\lambda))\exp(-ih\omega)+o_T(1).$$

where $c_{j\ell}(u,h)$ is the evolutionary cross-covariance function at lag h, i.e.,

$$c_{j\ell}(u,h) := 2\pi\int_{-\pi}^{\pi} f_{j\ell}(u,\omega)\exp(-i\omega h)d\omega.$$

Using that the total variation of the function $x\mapsto c_{j\ell}(x,h)$ is finite, and using the Poisson formula for the sum over k, the leading term of expectation is

$$|U|^{-1}\int |B(\omega)|^2\sum_{h=-\infty}^{\infty}\int_U c_{j\ell}\,(u,h)\exp(-ih\omega)du = |2\delta U|^{-1}\int_\Omega\int_U f_{j\ell}\,(u,\omega)\,d\omega du \equiv f_{j\ell}(U,\Omega).$$

The asymptotic distribution follows using the same technique, and similarly to the proof of Proposition 2.1.