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BOOTSTRAPPING THE POISSON LOG-BILINEAR MODEL FOR MORTALITY FORECASTING

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Abstract

This paper proposes bootstrap procedures for expected ramining lifetimes and life annuity single premiums in a dynamic mortality environment. Assuming a further continuation of the stable pace of mortality decline, a Poisson log-bilinear projection model is applied to the forecasting of the gender- and age-specific mortality rates for Belgium on the basis of mortality statistics relating to the period 1950-2000. Bootstrap procedures are then used to obtain confidence intervals on various actuarial quantities.

Key words and phrases: Poisson regression, age-sex-specific mortality, projected lifetables, mortality forecasting, confidence intervals, bootstrap

1 Introduction and Motivation

Lee & Carter (1992) proposed a simple model for describing the secular change in mortality as a function of a single time index. This model is fit to historical data. The resulting estimate of the time-varying parameter is then modeled and forecast as a stochastic time series using standard Box-Jenkins methods. From this forecast of the general level of mortality, the actual age-specific rates are derived using the estimated age effects. For a review of recent applications of the Lee-Carter methodology, we refer the interested readers to Lee (2000).

The main statistical tool of Lee & Carter (1992) is least-squares estimation via singular value decomposition of the matrix of the log age-specific observed forces of mortality. The mortality data (death counts and exposures-to-risk) have to fill a rectangular matrix which may pose a problem. Singular value decomposition also implicitly means that the errors are assumed to be homoskedastic, which is quite unrealistic: the logarithm of the observed force of mortality is much more variable at older ages than at younger ages because of the much smaller absolute number of deaths at older ages. Sithole, Haberman & Verrall (2000) and Renshaw & Haberman (2003a,b) have recently implemented an alternative approach to mortality forecasting based on heteroskedastic Poisson error structures. A closely related model has been proposed by Brouhns, Denuit & Vermunt (2002a,b), keeping the Lee-Carter log-bilinear form for the forces of mortality but replacing ordinary least-squares regression with Poisson regression for the death counts. There is thus a key difference between Renshaw & Haberman (2003a) and the method proposed by Brouhns et al. (2002a,b) that is developed in the present paper: the difference centres on the interpretation of time which in the Lee-Carter and Brouhns et al. (2002a,b) approach is modeled as a factor and under the approach proposed by Renshaw & Haberman (2003a) is modelled as a known covariate.

Of course, the projection of the mortality itself is affected by uncertainty. The effects of uncertainty coming from projections are investigated. Such an analysis is particularly important in demographic or actuarial applications. In Brouhns et al. (2002a), confidence intervals (for annuities and life expectancies) were obtained by ignoring all the errors except those in forecasting the mortality index. According to Appendix B of Lee & Carter (1992), these errors dominate the others for annuities and expected remaining lifetimes. Because of the importance of appropriate measures of uncertainty in an actuarial context, Brouhns et al. (2002b) derived confidence intervals taking into account all the sources of variability. The nonlinear nature of the quantities of interest makes an analytical approach not tractable and therefore Monte-Carlo simulation (or parametric bootstrap) was used. In this paper, we aim to continue the study initiated in Brouhns et al. (2002a,b) and to explore alternative bootstrap procedures to derive error margins on life expectancies or annuity pure premiums.

The paper is organized as follows. Section 2 introduces the notation used in this paper. In Section 3, the Poisson log-bilinear model for mortality projection is described. Section 4 is devoted to the derivation of confidence intervals for expected remaining lifetimes with the help of the bootstrap. Belgian mortality statistics are investigated using these techniques. The final Section 5 concludes.

2 Notation, assumption and data

2.1 Notation

We analyze the changes in mortality as a function of both age x and calendar time t. Henceforth,

- $T_x(t)$ is the remaining lifetime of an individual aged x on January the first of year t; this individual will die at age $x + T_x(t)$ in year $t + T_x(t)$.
- $q_x(t)$ is the probability that an x-aged individual in calendar year t dies before reaching age $x + 1$, i.e. $q_x(t) = \Pr[T_x(t) \leq 1]$.
- $p_x(t) = 1 - q_x(t)$ is the probability that an x-aged individual in calendar year t reaches age $x + 1$, i.e. $p_x(t) = \Pr[T_x(t) > 1]$.
- $\mu_x(t)$ is the mortality force at age x during calendar year t.
- $e_x(t) = \mathbb{E}[T_x(t)]$ is the expected remaining lifetime of an individual aged x in year t.
- ETR_{xt} is the exposure-to-risk at age x during year t, i.e. the total time lived by people aged x in year t.
- D_{xt} is the number of deaths recorded at age x during year t, from an exposure-to-risk ETR_{xt}.
- L_{xt} is the number of individuals aged x on January 1 of year t.

2.2 Assumption

In this paper, we assume that the age-specific mortality rates are constant within bands of age and time, but allowed to vary from one band to the next. Specifically, given any integer age x and calendar year t, it is supposed that

$$\mu_{x+\xi}(t + \tau) = \mu_x(t) \text{ for } 0 \leq \xi, \tau < 1. \tag{2.1}$$

Under (2.1), we have for integer age x and calendar year t that

$$p_x(t) = \exp(-\mu_x(t)) \text{ and } \mathrm{ETR}_{xt} = \frac{-L_{xt} q_x(t)}{\ln(1 - q_x(t))}. \tag{2.2}$$

The formula giving $e_x(t)$ under (2.1) is

$$\begin{aligned} e_x(t) &= \frac{1 - \exp\big(-\mu_x(t)\big)}{\mu_x(t)} \\ &\quad + \sum_{k \geq 1} \left\{ \prod_{j=0}^{k-1} \exp\big(-\mu_{x+j}(t+j)\big) \right\} \frac{1 - \exp\big(-\mu_{x+k}(t+k)\big)}{\mu_{x+k}(t+k)}. \end{aligned} \tag{2.3}$$

The net single premium $a_x(t)$ of a life annuity sold to an x-year-old individual in year t is then given by

$$a_x(t) \quad = \quad \sum_{k\geq 1}\left\{\prod_{j=0}^{k-1}\exp\big(-\mu_{x+j}(t+j)\big)\right\}v^k, \tag{2.4}$$

where v is the (deterministic) discount rate.

2.3 Data

The data used to illustrate this paper relate to the Belgian population, males and females separately. They cover the period 1950-2000 and have been provided by the National Institute of Statistics (they are available from the authors upon request). These data comprise the series of L_{xt} for $x = 0$ to 94 and $t = 1950$ to 2000 as well as the corresponding death counts D_{xt}. Formula (2.2) is used to derive the exposure-to-risk.

We assume that the remaining lifetimes of the L_{xt} individuals aged x on January 1 of year t are independent and identically distributed. The unconstrained MLE of $\mu_x(t)$, denoted as $\widehat{\mu}_x(t)$, is thus given by the ratio of the observed number of deaths D_{xt} for age x and year t to ETR_{xt}, that is

$$\widehat{\mu_x}(t) = \frac{D_{xt}}{\text{ETR}_{xt}}. \tag{2.5}$$

3 Poisson log-bilinear methodology

3.1 Lee-Carter classical methodology

Before describing the Poisson model, we first recall the basic features of the classical Lee-Carter approach. The latter is in essence a relational model

$$\ln\widehat{\mu_x}(t) = \alpha_x + \beta_x\kappa_t + \epsilon_x(t) \tag{3.1}$$

where $\widehat{\mu_x}(t)$ is given by (2.5), the $\epsilon_x(t)$'s are homoskedastic centered error terms and where the parameters are subject to the constraints

$$\sum_t \kappa_t = 0 \text{ and } \sum_x \beta_x = 1 \tag{3.2}$$

ensuring model identification.

The model (3.1) is fitted to a matrix of age-specific observed forces of mortality using singular value decomposition (SVD). Specifically, the $\widehat{\alpha}_x$'s, $\widehat{\beta}_x$'s and $\widehat{\kappa}_t$'s are such that they minimize

$$\sum_{x,t}\Big(\ln\widehat{\mu_x}(t) - \alpha_x - \beta_x\kappa_t\Big)^2. \tag{3.3}$$

The minimization of (3.3) consists in taking for $\widehat{\alpha_x}$ the row average of the $\ln\widehat{\mu_x}(t)$'s, and to get the $\widehat{\beta}_x$'s and $\widehat{\kappa}_t$'s from the first term of a SVD of the matrix $\ln\widehat{\mu_x}(t) - \widehat{\alpha_x}$. This yields a single time-varying index of mortality κ_t.

When the model (3.1) is fit by minimizing (3.3), interpretation of the parameters is quite simple:

- the fitted value of α_x exactly equals the average of $\ln \widehat{\mu_x}(t)$ over time t so that $\exp \alpha_x$ is the general shape of the mortality schedule;

- the actual forces of mortality change according to an overall mortality index κ_t modulated by an age response β_x. The shape of the β_x profile tells which rates decline rapidly and which slowly over time in response of change in κ_t.

Before modeling the parameter $\widehat{\kappa}_t$ as a time series process, the $\widehat{\kappa}_t$'s are adjusted (taking $\widehat{\alpha}_x$ and $\widehat{\beta}_x$ estimates as given) to reproduce the observed number of deaths $\sum_x D_{xt}$, that is the $\widehat{\widehat{\kappa}}_t$'s solve

$$\sum_x D_{xt} = \sum_x \mathrm{ETR}_{xt} \exp(\widehat{\alpha}_x + \widehat{\beta}_x \widehat{\widehat{\kappa}}_t). \tag{3.4}$$

The latter equation means that the κ_t's are reestimated so that the resulting death rates (with the previously estimated $\widehat{\alpha}_x$ and $\widehat{\beta}_x$), applied to the actual risk exposure, produce the total number of deaths actually observed in the data for the year t in question. There are several advantages to making this second stage estimate of the parameters κ_t. In particular, it avoids sizable discrepancies between predicted and actual deaths (occurring because the first step is based on logarithms of death rates). Other advantages are discussed by Lee (2000).

The time factor $\widehat{\widehat{\kappa}}_t$ is intrinsically viewed as a stochastic process and Box-Jenkins techniques are then used to estimate and forecast κ_t within an ARIMA times series model.

3.2 Poisson log-bilinear model

According to Alho (2000), the model described in equation (3.1) is not well suited to the situation of interest. As already mentioned, the main drawback of the OLS estimation via SVD is that the errors are assumed to be homoskedastic. This is related to the fact that for inference we are actually assuming that the errors are normally distributed, which is quite unrealistic. The logarithm of the observed force of mortality is much more variable at older ages than at younger ages because of the much smaller absolute number of deaths at older ages.

The approach of Brouhns et al. (2002a) consists in substituting Poisson random variation for the number of deaths for an additive error term on the logarithm of mortality rates keeping the log-bilinear form for the $\mu_x(t)$'s unchanged. It is worth to mention that the Poisson distribution is well-suited to mortality analyses; see e.g. Brillinger (1986) for more details. Log-linear Poisson regression has been successfully applied by Renshaw & Haberman (1996, 2003a) and Sithole, Haberman & Verrall (2000) to the forecasting of mortality trends. Log-bilinear Poisson specifications are also considered in Renshaw & Haberman (2003b).

We now consider that

$$D_{xt} \sim \mathrm{Poisson}\Big(\mathrm{ETR}_{xt}\mu_x(t)\Big) \text{ with } \mu_x(t) = \exp\left(\alpha_x + \beta_x \kappa_t\right) \tag{3.5}$$

where the parameters are still subject to the constraints (3.2). The force of mortality is thus assumed to have the same log-bilinear form $\ln \mu_x(t) = \alpha_x + \beta_x \kappa_t$ as in the Lee-Carter model. The meaning of the α_x, β_x, and κ_t parameters is essentially the same as in the classical Lee-Carter model. Only the random part of the model is modified.

3.3 Maximum likelihood estimation

Instead of resorting to SVD for estimating α_x, β_x and κ_t, we now determine these parameters by maximizing the log-likelihood

$$L(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\kappa}) = \sum_t \sum_x \Big\{ D_{xt}(\alpha_x + \beta_x \kappa_t) - \mathrm{ETR}_{xt} \exp(\alpha_x + \beta_x \kappa_t) \Big\} + \text{constant}.$$

based on model (3.5). Details of the fitting procedure can be found in Brouhns et al. (2002a); see also Renshaw & Haberman (2003b) for related results.

Differentiating the loglikelihood with respect to α_x gives the equation

$$\sum_t D_{xt} = \sum_t \mathrm{ETR}_{xt} \exp(\widehat{\alpha}_x + \widehat{\beta}_x \widehat{\kappa}_t). \tag{3.6}$$

So, the estimated κ_t's are such that the resulting death rates applied to the actual risk exposure produce the total number of deaths actually observed in the data for each age x. Sizable discrepancies between predicted and actual deaths are thus avoided and there is thus no need of a second-stage estimation like (3.4).

We apply the Poisson modelling to the Belgian population data. The Poisson parameters α_x, β_x and κ_t involved in (3.5) are estimated via maximum likelihood, separately for men and women. Figure 3.1 plots the estimated α_x, β_x and κ_t. We can see that the $\widehat{\alpha_x}$'s summarize the average mortality accross time: the $\widehat{\alpha_x}$'s clearly increase in x, reflecting higher mortality at older ages, as expected. The $\widehat{\beta}_x$'s decrease with age but remain positive. The $\widehat{\kappa}_t$'s exhibit regular behavior revealing the improvements of mortality during the observation period.

Since we work in a regression framework, it is essential to inspect the residuals. With Poisson random component, deviance residuals are appropriate to monitor the quality of the fit. These residuals are defined as

$$\mathrm{sign}(D_{xt} - \widehat{D}_{xt}) \sqrt{D_{xt} \ln \frac{D_{xt}}{\widehat{D}_{xt}} - (D_{xt} - \widehat{D}_{xt})},$$

where

$$\widehat{D}_{xt} = \mathrm{ETR}_{xt} \exp(\widehat{\alpha}_x + \widehat{\beta}_x \widehat{\kappa}_t).$$

Figure 3.2 displays the evolution of residuals through time at different ages. The absence of structure at most ages (except the very youngest ones that are usually not used in actuarial computations) supports the model.

3.4 Modelling the index of mortality

As in the Lee-Carter methodology the time factor κ_t is intrinsically viewed as a stochastic process. Box-Jenkins techniques are therefore used to estimate and forecast κ_t within an

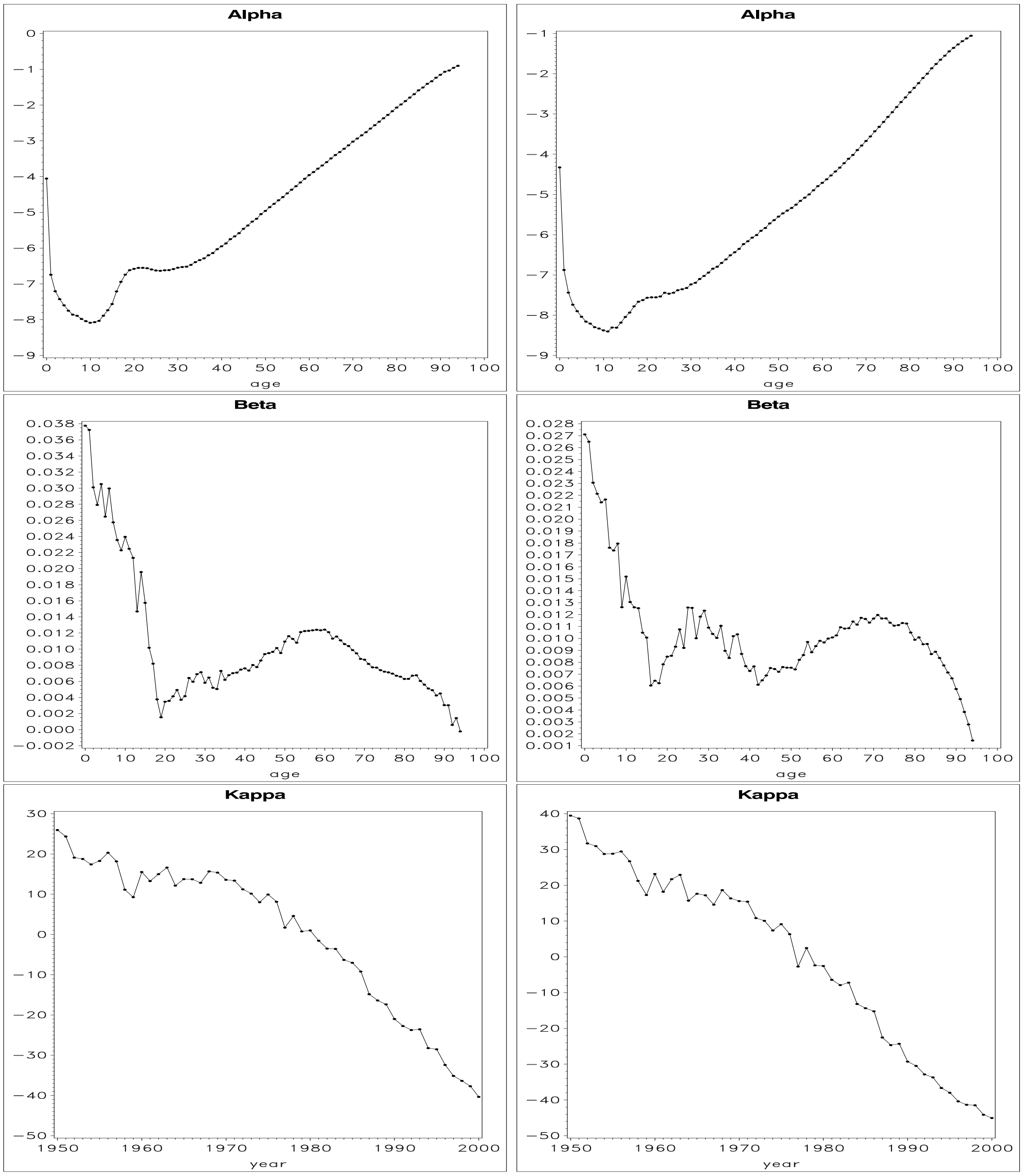

Figure 3.1: Estimations of the Poisson log-bilinear parameters involved in (3.5) (men are on the left, women on the right).

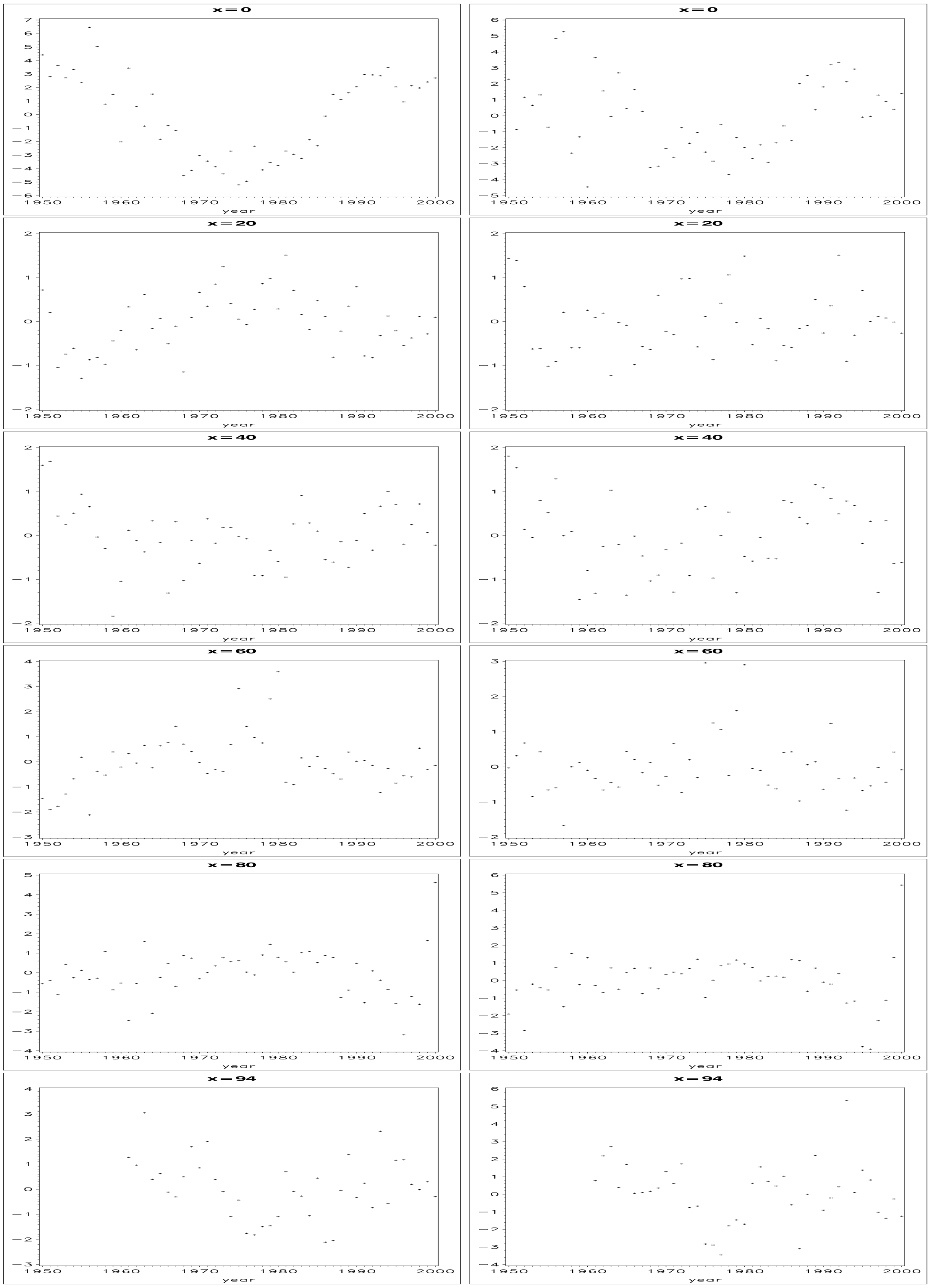

Figure 3.2: Deviance residuals for the Poisson log-bilinear model (3.5) applied to ages 0-94 (men are on the left, women on the right).

ARIMA times series model. Henceforth, a superscript "m" (resp. "w") indicates that the corresponding quantity relates to men (resp. to women). The models selected on the basis of the Box-Jenkins methodology are ARIMA(0,1,0)

$$\kappa_t^m - \kappa_{t-1}^m = \rho^m + \varepsilon_t^m$$

for men and ARIMA(0,1,1) for women

$$\kappa_t^w - \kappa_{t-1}^w = \rho^w + \theta^w \varepsilon_{t-1}^w + \varepsilon_t^w$$

where the ε_t^m's and ε_t^w's are white noises with variances σ_m^2 and σ_w^2, respectively. Estimations of the parameters are

Parameter	Estimation
ρ^m	-1.2735
ρ^w	-1.6604
θ^w	-0.4410
σ_m	2.3820
σ_w	2.7326

which all significantly differ from 0 (at 5%).

3.5 Projected lifetables

We are now ready to forecast the κ_t's. Figure 3.3 displays the projection of the κ_t's up to 2050, together with 95% confidence intervals.

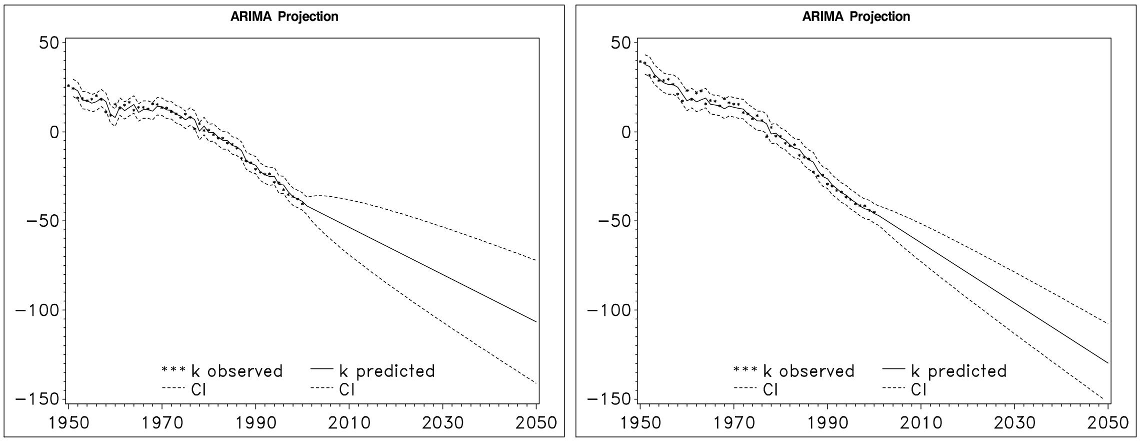

Figure 3.3: ARIMA forecasts (with 95% confidence intervals) up to 2050 (men are on the left, women on the right).

The projected κ_{2000+s}, $s = 1, 2, \ldots$, are inserted into the formulas giving the force of mortality and provide

$$\widehat{\mu_x}(2000 + s) = \exp(\widehat{\alpha}_x + \widehat{\beta}_x \widehat{\kappa}_{2000+s}).$$

From these projected forces of mortality, we can build projected lifetables and compute life expectations. Forecast mortality rates can also be computed as

$$\widehat{\mu_x}(2000+s) = \widehat{\mu_x}(2000)\exp(\widehat{\beta}_x\widehat{\kappa}_{2000+s} - \widehat{\kappa}_{2000})$$

thereby ensuring the forecasts are aligned to the latest available mortality rates $\widehat{\mu_x}(2000)$. As pointed out by Bell (1997), if the latest data are judged to generate atypically shaped crude mortality shapes (by age), it is possible to average accross a few years at the end of the observation period.

The interest of the approach developed in this paper is that we are now able to follow a generation. Figure 3.4 displays the forces of mortality applicable to different generations (i.e. those people born in 1950, 1960, ..., 2000).

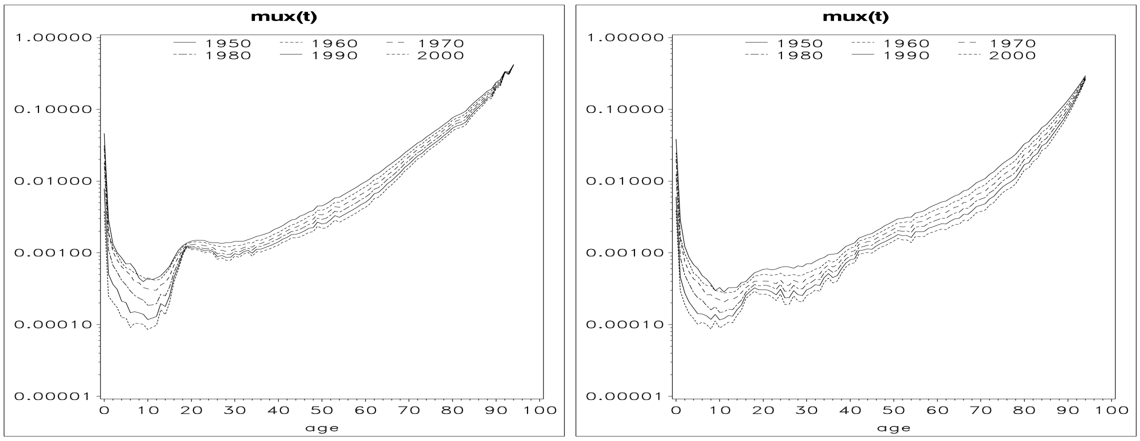

Figure 3.4: Forces of mortality for the generations born in years 1950 until 2000 obtained from model (3.5) (men are on the left, women on the right).

4 Confidence intervals for actuarial indicators

4.1 Why bootstrapping?

In forecasting, it is important to provide information on the uncertainty affecting the forecasted quantities. In that respect, confidence intervals are particularly useful. However, in the current application it is impossible to derive the relevant confidence intervals analytically. The reason for this is that two very different sources of uncertainty have to be combined: sampling errors in the parameters of the Poisson model and forecast errors in the projected ARIMA parameters. An additional complication is that the measures of interest – mortality rates and life expectancies – are complicated non-linear functions of the Poisson parameters α_x, β_x, and κ_t and the ARIMA parameters. The key idea behind the bootstrap is to resample from the original data (either directly or via a fitted model) to create replicate data sets, from which the variability of the quantities of interest can be assessed. Because this approach involves repeating the original data analysis procedure with many replicate sets of data, it is sometimes called a computer-intensive method. Bootstrap techniques are particularly useful when, as it will be the case in our problem, theoretical calculation with the fitted model is too complex.

The two sources of uncertainty that have to be combined are the sampling fluctuation in the α_x, β_x, and κ_t parameters and the forecast error in the κ_t parameters. Brouhns et al. (2002b) sampled directly from the approximate multivariate normal distribution of the maximum likelihood estimators $\widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}, \widehat{\boldsymbol{\kappa}}$. We propose here two alternative approaches.

4.2 Poisson bootstrap

Starting from the observations $(\mathrm{ETR}_{xt}, D_{xt})$, we create N bootstrap samples $(\mathrm{ETR}_{xt}, D^n_{xt})$, $n = 1, \ldots, N$, where the D^n_{xt}'s are realizations from the Poisson distribution with mean

$$\mathrm{ETR}_{xt}\widehat{\mu}_x(t) = D_{xt}.$$

The bootstrapped death counts D^n_{xt} are thus obtained by applying a Poisson noise to the observed numbers of deaths.

For each bootstrap sample, the α_x's, β_x's and κ_t's are estimated and the κ_t's are then projected on the basis of the reestimated ARIMA model. Note that we do not select a new ARIMA model but keep the ARIMA(0,1,0) for men and ARIMA(0,1,1) for women selected on the basis of original data. Nevertheless, the parameters of these models are reestimated with bootstrapped data. This yields N realizations α^n_x, β^n_x, κ^n_t and projected κ^n_t on the basis of which we compute the measure of interest.

We have applied this methodology to the Belgian data analyzed in Section 3. Specifically, we purpose to derive a confidence interval for $e_{65}(2000)$, the expected remaining lifetime for an individual aged 65 in year 2000 (separately for men and women). This represents the expected retirement period for people getting retired in year 2000 (and is therefore a key actuarial indicator for the management of public pension regimes).

Ten thousand bootstrapped samples have been generated, yielding $e_{65}(2000)^n$, $n = 1, \ldots, 10000$. An histogram of these values is given in Figure 4.1. The average of the 10 000 $e_{65}(2000)^n$'s is 16.04 for men and 20.06 for women (to be compared with the point forecasts 16.01 and 20.04). As 90% of the $e_{65}(2000)^n$'s fall in the interval [14.34;17.75] for men and [20.07;21.26] for women, the latter intervals can be considered as approximate 90% intervals for the unknown values of $e_{65}(2000)$. The larger width for men can be attributed to less regular κ_t's and corresponding higher order of the ARIMA model.

Let us now consider the net single premiums $a_{65}(2000)$. Based on the same 10 000 bootstrapped samples, we have computed $a_{65}(2000)^n$, $n = 1, \ldots, 10000$. An histogram of these values is given in Figure 4.2. The average of the 10 000 $a_{65}(2000)^n$'s is 10.71 for men and 13.01 for women (to be compared with the point forecasts 10.69 and 13.02). As 90% of the $a_{65}(2000)^n$'s fall in the interval [9.86;11.55] for men and [12.76;13.28] for women, the latter intervals can be considered as approximate 90% intervals for the unknown values of $a_{65}(2000)$. The accuracy of the projections can be assessed through the width of these intervals: a relative error of 15.7% for men and 3.9% for women can be considered as reasonably accurate (compared to the projection 50 years in the future required to perform these computations).

Remark 4.1. The bootstrapping procedure can also be achieved in a number of alternative ways. For instance, we could follow here a "generation" $(\mathrm{ETR}_{xt}, D_{xt})$, $(\mathrm{ETR}_{x+1,t+1}, D_{x+1,t+1})$,

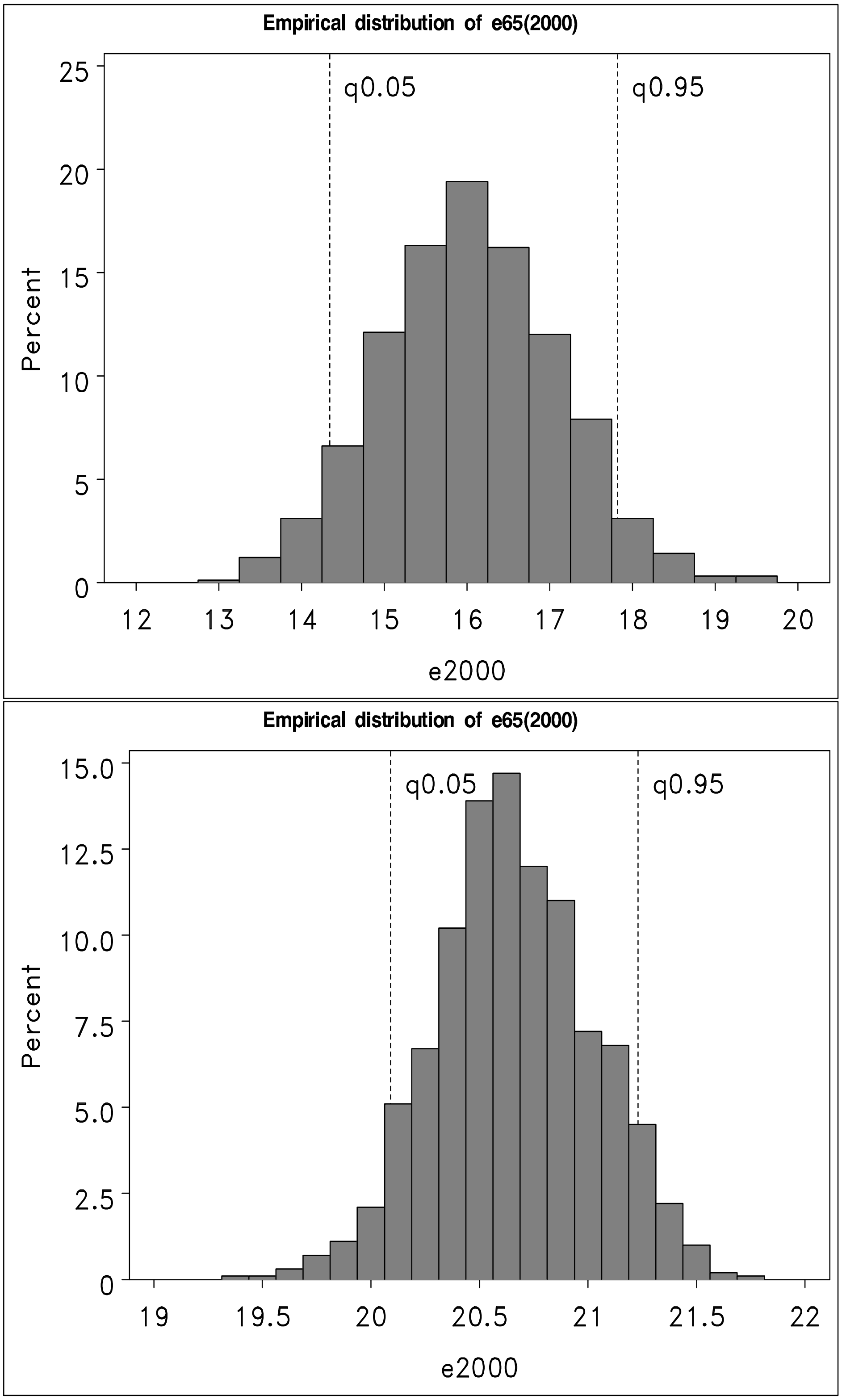

Figure 4.1: Histograms for $e_{35}(2000)$ coming from Poisson bootstrap (men are above, women below).

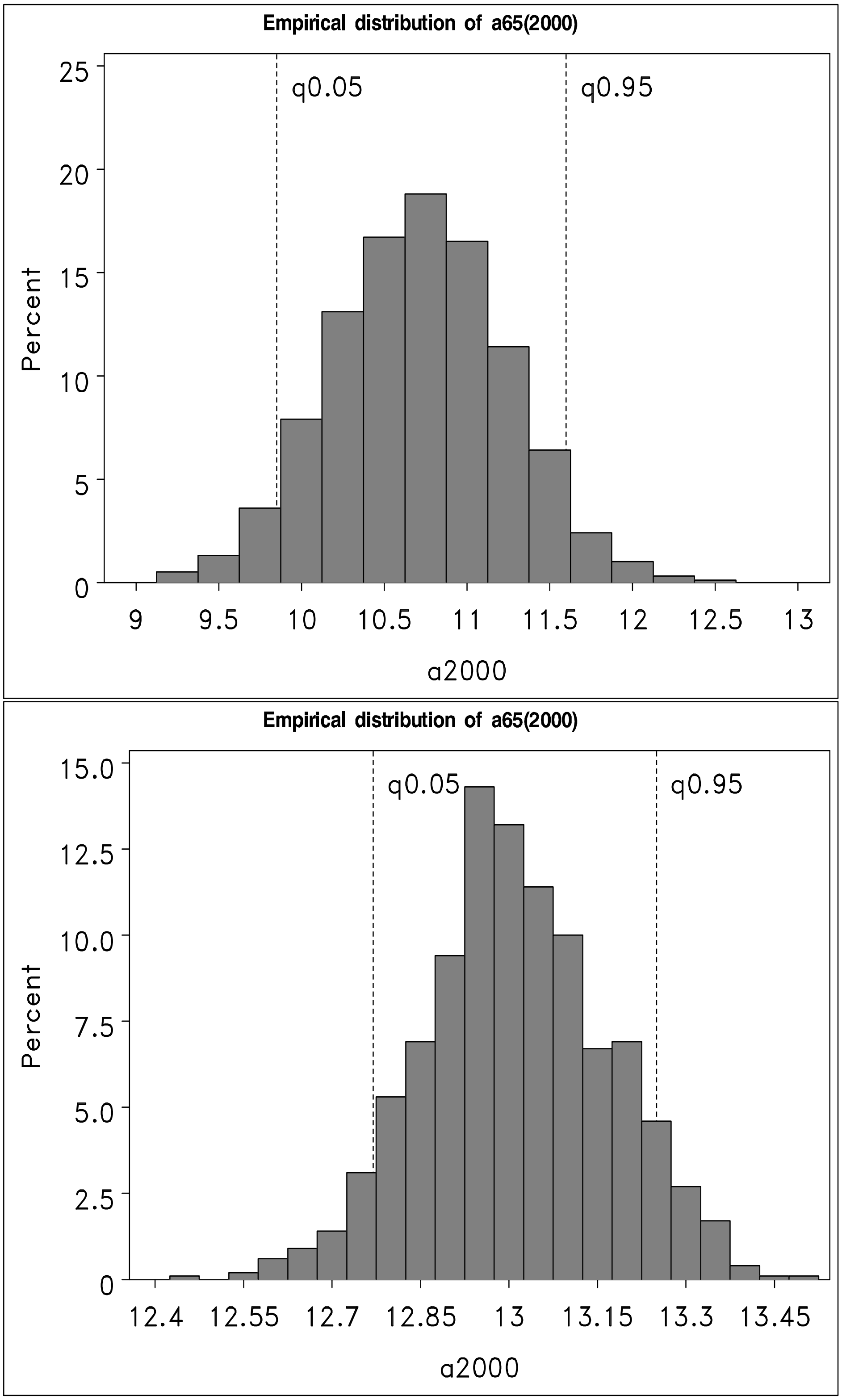

Figure 4.2: Histograms for $a_{65}(2000)$ coming from Poisson bootstrap (men are above, women below).

$(\mathrm{ETR}_{x+2,t+2}, D_{x+2,t+2})$. Pseudo death counts $D^n_{xt}, D^n_{x+1,t+1}, \ldots$, $n = 1, \ldots, N$, are generated from a multinomial distribution with exponent

$$D_\bullet = \sum_{k \geq 0} D_{x+k,t+k}$$

and parameters

$$\frac{D_{xt}}{D_\bullet}, \quad \frac{D_{x+1,t+1}}{D_\bullet}, \ldots$$

We then proceed as described above. This approach is very close to the Poisson bootstrap for the D^n_{xt} since the conditional distribution of the yearly death counts given their sum conforms to the multinomial law when the yearly death counts are modelled by independent Poisson random variables. Of course, the relation $D_\bullet = \sum_{k\geq 0} D^n_{x+k,t+k}$ is not necessarily satisfied in the Poisson bootstrap.

Another possibility is to bootstrap from the residuals of the fitted Poisson log-bilinear model. The deviance residuals should be independent and identically distributed (provided the model is well specified). Therefore, it is possible to reconstitute bootstrapped residuals, and therefrom bootstrapped mortality data.

5 Conclusion

This paper presents the Poisson log-bilinear mortality projection model proposed by Brouhns, Denuit & Vermunt (2002a) and goes further into the use of bootstrap procedures for the calculation of confidence intervals. The width of the confidence intervals derived for expected remaining lifetimes $e_{65}(2000)$ and life annuity net single premiums $a_{65}(2000)$ is moderate enough to allow for practical purposes.

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