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STOP-LOSS BOUNDS ON FUNCTIONS OF POSSIBLY DEPENDENT RISKS IN THE PRESENCE OF PARTIAL INFORMATION ON THEIR MARGINALS

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A general strategy is proposed for deriving upper and lower stop-loss bounds on directionally convex, increasing functions of possibly dependent, non-negative random variables when partial information is available about their joint copula or their marginal distributions. Illustrations are presented which involve looking for a stop-loss bound on a sum of insurance risks whose means, variances or range are finite and known. The key to these developments, which generalize recent findings of Genest, Marceau and Mesfioui (2002), is a result due to Müller and Scarsini (2001) on the stochastic comparison of random vectors having a common copula.

Key words. Archimedean copulas, conditionally increasing property, convex ordering, dependent risks, directional convexity, Fréchet bounds., stop-loss ordering, supermodular ordering.

1. Introduction. Consider a portfolio consisting of $n \geq 2$ insurance policies and let $X_1, \ldots, X_n$ be associated non-negative claim amounts over a given reference period. There is an abundant literature on the behavior of the aggregate claim amount $X = X_1 + \cdots + X_n$ in the case where the risks X_i are mutually independent, but relatively little is known to date about the more realistic scenario in which these variables are associated in some way.

Under the assumption that the marginal cumulative distribution function F_i of X_i is known for every $1 \leq i \leq n$, Dhaene and Goovaerts (1996, 1997) and Müller (1997) identified the riskiest dependence structure, that is, that which yields the largest possible value for the stop-loss premium

$$\pi(X, d) = \mathrm{E}\{\max(0, X - d)\}$$

representing the expected claim amount in excess of a deductible $d \geq 0$. They showed that $\pi(X, d)$ is maximized when the X_i are comonotonic, that is, when there exists a uniform random variable on $(0, 1)$ that simultaneously governs the values of all the claims via the relation

$$X_i = F_i^{-1}(U), \quad 1 \leq i \leq n. \tag{1}$$

Under additional conditions, Dhaene and Denuit (1999) derived the dependence structure minimizing $\pi(X, d)$, which they called countermonotonicity.

In practice, however, the F_i are rarely completely determined. This led Hürlimann (1998) and, more recently, Genest, Marceau and Mesfioui (2002) to look for an upper bound on $\pi(X, d)$ in the presence of partial information about the F_i. In the latter paper, two natural strategies were briefly investigated for arriving at such a bound in the case where

$$\mu_i = \mathrm{E}(X_i) > 0 \quad \text{and} \quad \sigma_i^2 = \mathrm{var}(X_i) > 0$$

was known for each i. One option consisted in computing $\pi(X, d)$ with

$$\mathrm{E}(X) = \sum_{i=1}^{n} \mu_i, \quad \mathrm{var}(X) = \sum_{i=1}^{n} \sigma_i^2 + 2 \sum_{i<j} \rho_{ij} \sigma_i \sigma_j$$

for all possible values of the ρ_{ij}, and using the greatest value as an upper bound. A second, more fruitful strategy consisted in computing

$$\pi\left(\sum_{i=1}^{n} F_{i,\max}^{-1}(U), d \right), \tag{2}$$

where for each i, $F_{i,\max}$ is the cumulative distribution function that provides the best upper stop-loss bound on F_i given the available information on X_i. The quantity (2) is necessary larger than $\pi(X, d)$, in the light of Theorem 5 of Dhaene et al. (2000). It was made explicit only in the case $n = 2$, however.

The purpose of this paper is to describe, more generally, how results due to Müller and Scarsini (2001) can be used to extend the second approach of Genest et al. (2002) to the determination of an upper stop-loss bound on any directionally convex, increasing function $\Psi(X_1, \ldots, X_n)$ of $n \geq 2$ possibly dependent risks in situations where the marginals are either totally or partially known, or even when information may be available about the copula regulating the dependence between the X_i.

The general strategy is described in Section 2 and detailed calculations are then given in three illustrative cases involving $\Psi(X_1, \ldots, X_n) = X_1 + \cdots + X_n$ for arbitrary $n \geq 2$. Section 3 treats the case where the X_i are bounded and $\mu_i = \mathrm{E}(X_i)$ is known for all $1 \leq i \leq n$. Section 4 revisits the case considered by Genest et al. (2002) in which the X_i are unbounded and both $\mu_i = \mathrm{E}(X_i)$ and $\sigma_i^2 = \mathrm{var}(X_i)$ are available for each i. Section 5 shows what can be done when the copula is fixed and Archimedean with a completely monotone generator. The issue of finding a lower stop-loss bound is also considered briefly in Section 6. Concluding comments are given in Section 7.

2. The strategy. Let $X_1, \ldots, X_n$ be continuous non-negative random variables with marginal cumulative distribution functions $F_1, \ldots, F_n$, respectively, and unique underling copula C, defined implicitly by

$$\Pr\left(X_1 \leq x_1, \ldots, X_n \leq x_n\right) = C\left\{F_1(x_1), \ldots, F_n(x_n)\right\} \tag{3}$$

for all $\mathbf{x} = (x_1, \ldots, x_n) \in I\!\!R^n$.

Assume that upper and lower stop-loss bounds are sought for $\Psi(X_1, \ldots, X_n)$, where $\Psi : I\!\!R^n \to I\!\!R$ represents an increasing, directionally convex function. In other words, one has

$$\Delta_i^\epsilon \Psi(\mathbf{x}) \geq 0 \quad \text{and} \quad \Delta_i^\epsilon \Delta_j^\delta \Psi(\mathbf{x}) \geq 0 \tag{4}$$

for all $1 \leq i, j \leq n$ and $\epsilon, \delta > 0$, where the ith difference operator Δ_i^ϵ is defined as

$$\Delta_i^\epsilon \Psi(\mathbf{x}) = \Psi(\mathbf{x} + \epsilon \mathbf{1}_i) - \Psi(\mathbf{x})$$

in terms of the ith unit vector $\mathbf{1}_i$.

Following Theorem 2.6 of Müller and Scarsini (2001), directionally convex functions are precisely those that are supermodular and convex in each of their coordinates. Supermodular functions are those for which $\Delta_i^\epsilon \Delta_j^\delta \Psi(\mathbf{x}) \geq 0$ holds for all $1 \leq i < j \leq n$ and $\epsilon, \delta > 0$. Possibly the simplest examples of directionally convex, increasing functions are linear combinations

$$\Psi(x_1, \ldots, x_n) = \sum_{i=1}^{n} \alpha_i x_i$$

with $\alpha_1, \ldots, \alpha_n > 0$.

The suggested procedure to construct upper and lower stop-loss bounds on such functions of risks $X_1, \ldots, X_n$ varies, according as their marginal distributions are known or not.

2.1 Known marginals. Suppose that $F_1, \ldots, F_n$ are known and that, in addition, copulas C^- and C^+ may be found for which

$$C^- \prec_{\rm sm} C \prec_{\rm sm} C^+,$$

where $\prec_{\rm sm}$ denotes the supermodular ordering (Shaked and Shanthikumar 1997, Müller 1997). Recall that in general, $C \prec_{\rm sm} D$ means that if $(U_1, \ldots, U_n)$ and $(V_1, \ldots, V_n)$ are distributed as C and D, respectively, then

$$\mathrm{E}\{\psi(U_1, \ldots, U_n)\} \leq \mathrm{E}\{\psi(V_1, \ldots, V_n)\}$$

for any supermodular function ψ for which these expectations exist.

Under condition (4) on Ψ, a slight adaptation of Theorem 3.1 in Müller (1997) implies that

$$\Psi(X_1^-, \ldots, X_n^-) \prec_{\rm icx} \Psi(X_1, \ldots, X_n) \prec_{\rm icx} \Psi(X_1^+, \ldots, X_n^+) \quad (5)$$

whenever $(X_1^-, \ldots, X_n^-)$ and $(X_1^+, \ldots, X_n^+)$ are distributed as the right-hand side of (3), but with C^- and C^+ instead of C.

Here, $\prec_{\rm icx}$ denotes the standard increasing convex ordering, between two variables S and T with cumulative distribution functions F_S and F_T, respectively. Specifically, one writes either $S \prec_{\rm icx} T$ or $F_S \prec_{\rm icx} F_T$, if, and only if,

$$\mathrm{E}\{\psi(S)\} \leq \mathrm{E}\{\psi(T)\} \quad (6)$$

for every increasing convex function $\psi : \mathbb{R} \to \mathbb{R}$ making these expectations finite. Similarly, $S \prec_{\rm cx} T$ or $F_S \prec_{\rm cx} F_T$ means that (6) holds true for every convex ψ for which the expectations exist.

Since the mapping $\psi(x) = \max(0, x - d)$ is convex and increasing, it follows from inequality (5) that

$$\pi\left\{\Psi\left(X_1^-, \ldots, X_n^-\right), d\right\} \le \pi\left\{\Psi\left(X_1, \ldots, X_n\right), d\right\} \le \pi\left\{\Psi(X_1^+, \ldots, X_n^+), d\right\}$$

for all $d \ge 0$. The extreme terms in this chain of inequalities thus provide explicit upper and lower bounds on the stop-loss premium for $\Psi(X_1, \ldots, X_n)$, insofar as C^-, C^+ and the marginal cumulative distribution functions $F_1, \ldots, F_n$ are available.

Since the inequality

$$C(u_1, \ldots, u_n) \preceq_{\text{sm}} M(u_1, \ldots, u_n) = \min(u_1, \ldots, u_n)$$

holds uniformly in $0 < u_1, \ldots, u_n < 1$ (see, for example, Theorem 2.10.12 in Nelsen 1999), the upper Fréchet bound M constitutes a rather natural choice for C^+. It corresponds to comonotonic risks X_i as defined in (1). In the special case where $\Psi(X_1, \ldots, X_n) = X_1 + \cdots + X_n$, the inequality

$$\pi\left(\sum_{i=1}^n X_i, d\right) \le \pi\left\{\sum_{i=1}^n F_i^{-1}(U), d\right\}$$

due to Dhaene et al. (2000) obtains.

When the risks X_i are known to be associated in the sense of Esary, Proschan and Walkup (1967), Christofides and Vaggelatou (2003) show that an appropriate choice for C^- is the independence copula $\Pi(u_1, \ldots, u_n) = u_1 \cdots u_n$.

2.2. Unknown marginals. Assume more realistically that the marginal distributions of $X_1, \ldots, X_n$ are unknown but that partial information about them allows one to write

$$F_{i,\min} \prec_{\text{cx}} F_i \prec_{\text{cx}} F_{i,\max}, \quad 1 \le i \le n.$$

Suppose further that C^- and C^+ are symmetric in their arguments and conditionally increasing in sequence (Lehmann 1966). What this means specifically is that if $(U_1, \ldots, U_n)$ is distributed as C^- or C^+, then for all $2 \le i \le n$,

$$\Pr\left(U_i \le u | U_1 = u_1, \ldots, U_{i-1} = u_{i-1}\right)$$

is non-decreasing in $u_1, \ldots, u_{i-1}$. The independence copula and the Fréchet upper bound clearly meet this requirement, but additional examples will be mentioned in Section 5.

Under these conditions on C^- and C^+, a straightforward application of Theorem 4.5 in Müller and Scarsini (2001) yields

$$\Psi\left(X_{1,\min}^-, \ldots, X_{n,\min}^-\right) \prec_{\text{icx}} \Psi\left(X_1^-, \ldots, X_n^-\right) \tag{7}$$

and

$$\Psi\left(X_1^+, \ldots, X_n^+\right) \prec_{\text{icx}} \Psi\left(X_{1,\max}^+, \ldots, X_{n,\max}^+\right), \tag{8}$$

where $(X_{1,\max}^-, \ldots, X_{n,\max}^-)$ has copula C^- and marginals $F_{i,\min}$, while $(X_{1,\max}^+, \ldots, X_{n,\max}^+)$ has copula C^+ and marginals $F_{i,\max}$, $1 \le i \le n$.

The derivation of (7) and (8) hinges on the observation that if $\varphi : I\!\!R \to I\!\!R$ is increasing and convex, then $\varphi \circ \Psi$ is directionally convex so long as Ψ itself is. The combination of (5), (7) and (8) then yields the desired stop-loss bounds on $\Psi(X_1, \ldots, X_n)$ and hence on $\pi\{\Psi(X_1, \ldots, X_n), d\}$. A definite advantage of this strategy is that the choice of stop-loss bounds $F_{i,\min}$ and $F_{i,\max}$ on F_i can then rely on the univariate literature, as will be illustrated in the following sections.

3. Bounded risks with given means. Assume that $\mathrm{E}(X_i) = \mu_i$ and $X_i \in [0, b_i]$ with probability one for known constants μ_i and b_i, $1 \le i \le n$. For simplicity, suppose without further loss of generality that

$$\frac{\mu_1}{b_1} \ge \cdots \ge \frac{\mu_n}{b_n} \ge \frac{\mu_{n+1}}{b_{n+1}} = 0.$$

Then it is well known (see, for example, de Vylder 1996) that $F_i \prec_{\text{cx}} F_{i,\max}$, where

$$F_{i,\max}(x) = \begin{cases} 0 & \text{if} \quad x < 0, \\ 1 - \frac{\mu_i}{b_i} & \text{if} \quad 0 \le x < b_i, \\ 1 & \text{if} \quad x \ge b_i. \end{cases} \tag{9}$$

Thus if for instance $\Psi(X_1, \ldots, X_n) = X_1 + \cdots + X_n$, one may conclude from (5) and (8) that

$$X = \sum_{i=1}^n X_i \prec_{\text{cx}} \sum_{i=1}^n F_{i,\max}^{-1}(U) = X_{\max}, \tag{10}$$

where U is a uniform random variable on $(0,1)$. The exact distribution of $X_{\max}$ is given explicitly by

$$F_{\max}(x) = \left(1 - \frac{\mu_i}{b_i}\right) 1_{\left(b_{i-1}^* \le x < b_i^*\right)}, \quad 1 \le i \le n+1 \tag{11}$$

and its associated stop-loss premium $\pi\left(X_{\max}, d\right)$, which constitutes an upper bound on $\pi\left(X, d\right)$, is of the form

$$\pi\left(X_{\max}, d\right) = \left\{\mu_n^* - \mu_{i+1}^* + \frac{\mu_{i+1}}{b_{i+1}}\left(b_{i+1}^* - d\right)\right\} 1_{\left(b_i^* \leq d < b_{i+1}^*\right)} \tag{12}$$

for $0 \leq i \leq n-1$, where

$$\mu_i^* = \sum_{i=1}^{i} \mu_j, \quad b_i^* = \sum_{j=1}^{i} b_j, \quad 1 \leq i \leq n$$

with $b_0^* = 0$ and $b_{n+1}^* = \infty$.

To prove (11), use the fact that for all $x \geq 0$, one has

$$\begin{aligned} F_{\max}(x) &= \Pr\left\{\sum_{i=1}^{n} F_{i,\max}^{-1}(U) \leq x\right\} \\ &= 1 - \frac{\mu_1}{b_1} + \sum_{i=1}^{n} \Pr\left(b_i^* \leq x \,\middle|\, \frac{\mu_{i+1}}{b_{i+1}} \leq 1 - U \leq \frac{\mu_i}{b_i}\right) \\ &\qquad\qquad \times \Pr\left(\frac{\mu_{i+1}}{b_{i+1}} \leq 1 - U \leq \frac{\mu_i}{b_i}\right). \end{aligned}$$

Since U is uniform on $(0, 1)$, this expression amounts to

$$F_{\max}(x) = 1 - \frac{\mu_1}{b_1} + \sum_{i=1}^{n} \left(\frac{\mu_i}{b_i} - \frac{\mu_{i+1}}{b_{i+1}}\right) 1_{\left(b_i^* \leq x\right)},$$

which reduces at once to (11).

To establish (12), observe that for $b_i^* \leq d < b_{i+1}^*$, one has

$$\pi\left(X_{\max}, d\right) = \mathrm{E}\left\{\max\left(0, X_{\max} - d\right)\right\} = \sum_{j=i+1}^{n} \left(b_j^* - d\right)\left(\frac{\mu_j}{b_j} - \frac{\mu_{j+1}}{b_{j+1}}\right),$$

which may be recast into

$$\begin{aligned} \pi\left(X_{\max}, d\right) &= \frac{\mu_{i+1}}{b_{i+1}} b_{i+1}^* + \sum_{j=i+2}^{n} \frac{\mu_j}{b_j}\left(b_j^* - b_{j-1}^*\right) - \frac{\mu_{i+1}}{b_{i+1}} d \\ &= \frac{\mu_{i+1}}{b_{i+1}}\left(b_{i+1}^* - d\right) + \mu_n^* - \mu_{i+1}^*. \end{aligned}$$

4. Unbounded risks with given means and variances. Let $X_1, \ldots, X_n$ be unbounded risks such that $\mathrm{E}(X_i) = \mu_i$ and $\mathrm{var}(X_i) = \sigma_i^2$ are known for each $i \in \{1, \ldots, n\}$. Also assume without loss of generality that

$$a_0 = 0 \le a_1 = \frac{\sigma_1^2}{\sigma_1^2 + \mu_1^2} \le \cdots \le a_n = \frac{\sigma_1^2}{\sigma_n^2 + \mu_n^2} \le a_{n+1} = 1.$$

Jansen, Haezendonck and Goovaerts (1986) proved that $F_i \prec_{\mathrm{cx}} F_{i,\max}$, where

$$F_{i,\max}(x) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{\sigma_i^2}{\sigma_i^2+\mu_i^2} & \text{if } 0 \le x < \frac{\mu_i^2+\sigma_i^2}{2\mu_i}, \\ \frac{1}{2} + \frac{1}{2}\frac{x-\mu_i}{\sqrt{(x-\mu_i)^2+\sigma_i^2}} & \text{if } x \ge \frac{\mu_i^2+\sigma_i^2}{2\mu_i}. \end{cases} \tag{13}$$

When $\Psi(X_1, \ldots, X_n) = X_1 + \cdots + X_n = X$, an upper bound on $\pi(X, d)$ is thus given by $\pi(X_{\max}, d)$, where $X_{\max} = F_{1,\max}^{-1}(U) + \cdots + F_{n,\max}^{-1}(U)$ and U is a uniform random variable on $(0, 1)$. The explicit distribution of $X_{\max}$ is given by

$$F_{\max}(x) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{\sigma_i^2}{\sigma_i^2+\mu_i^2} & \text{if } b_{i-1,i} \le x < b_{i,i}, \quad 1 \le i \le n \\ \frac{1}{2} + \frac{1}{2}\frac{x-\mu_i^*}{\sqrt{(x-\mu_i^*)^2+\sigma_i^{*2}}} & \text{if } b_{i,i} \le x < b_{i,i+1}, \quad 1 \le i \le n \end{cases} \tag{14}$$

with

$$\mu_i^* = \sum_{j=1}^{i} \mu_j, \quad \sigma_i^* = \sum_{j=1}^{i} \sigma_j, \quad b_{i,j} = \mu_i^* + \frac{\sigma_i^*}{2\sigma_j\mu_j}\left(\sigma_j^2 - \mu_j^2\right)$$

and $b_{0,1} = 0$, $b_{n,n+1} = \infty$.

The corresponding stop-loss premium is then

$$\pi(X_{\max}, d) = \begin{cases} \frac{\mu_i^2}{\sigma_i^2+\mu_i^2}(b_{i,i} - d) + \mu_n^* - \mu_i^* + \frac{\sigma_i^*\mu_i}{2\sigma_i} & \text{if } b_{i-1,i} \le d \le b_{i,i}, \\ & \quad 1 \le i \le n \\ \pi_i^*(d) + \mu_n^* - \mu_i^* & \text{if } b_{i,i} \le d \le b_{i,i+1}, \\ & \quad 1 \le i \le n \end{cases} \tag{15}$$

with

$$\pi_i^*(d) = \frac{\mu_i^* - d}{2} + \frac{1}{2}\sqrt{(d-\mu_i^*)^2 + \sigma_i^{*2}}.$$

The above formulas for $F_{\max}$ and $\pi(X_{\max}, d)$ obviously reduce to those given by Genest et al. (2002) when $n = 2$.

In order to establish (14), one may first combine (10) and (13) to get

$$\begin{aligned} F_{\max}(x) &= \sum_{i=0}^{n} \Pr\left\{ \sum_{j=1}^{n} F_{j,\max}^{-1}(U) \le x \cap a_i \le U \le a_{i+1} \right\} \\ &= a_1 + \sum_{i=1}^{n} \Pr\left\{ \sum_{j=1}^{i} \left(\mu_j + \sigma_j \frac{2U-1}{\sqrt{1-(2U-1)^2}} \right) \le x \right. \\ &\qquad \left. \cap\, a_i \le U \le a_{i+1} \right\}, \end{aligned}$$

which may be rewritten as

$$\begin{aligned} F_{\max}(x) &= a_1 + \sum_{i=1}^{n} \Pr\left\{ \mu_i^* + \sigma_i^* \frac{2U-1}{\sqrt{1-(2U-1)^2}} \le x \cap a_i \le U \le a_{i+1} \right\} \\ &= a_1 + \sum_{i=1}^{n} \Pr\left\{ U \le F_{\max,i}^*(x) \cap a_i \le U \le a_{i+1} \right\}, \end{aligned}$$

where

$$F_{i,\max}^*(x) = \frac{1}{2} + \frac{1}{2}\frac{x-\mu_i^*}{\sqrt{(x-\mu_i^*)^2 + \sigma_i^{*2}}}.$$

Thus if $b_{i,j}$ denotes $F_{i,\max}^{*-1}(a_j)$, then

$$\begin{aligned} \Pr\left\{ U \le F_{i,\max}^*(x) \cap a_i \le U \le a_{i+1} \right\} &= \left\{ F_{i,\max}^*(x) - a_i \right\} I_{(b_{i,i} \le x \le b_{i,i+1})} \\ &\quad + \ (a_{i+1} - a_i)\, I_{(x > b_{i,i+1})}, \end{aligned}$$

whence the result.

Now to prove (15), observe that for $b_{i-1,i} \le d \le b_{i,i}$, one has

$$\begin{aligned}
\pi(X_{\max}, d) &= \int_d^\infty \{1 - F_{\max}(x)\}\, dx \\
&= \int_d^{b_{i,i}} \frac{\mu_i^2}{\sigma_i^2 + \mu_i^2}\, dx + \sum_{j=i}^{n} \int_{b_{j,j}}^{b_{j,j+1}} \{1 - F_{\max}(x)\}\, dx \\
&\qquad + \sum_{j=i}^{n-1} \int_{b_{j,j+1}}^{b_{j+1,j+1}} \{1 - F_{\max}(x)\}\, dx.
\end{aligned}$$

This, in turn, may be rewritten as

$$\begin{aligned}
= \; & \frac{\mu_i^2}{\sigma_i^2 + \mu_i^2}\, (b_{i,i} - d) + \sum_{j=i}^{n-1} \frac{\mu_{j+1}^2}{\sigma_{j+1}^2 + \mu_{j+1}^2}\, (b_{j+1,j+1} - b_{j,j+1}) \\
& + \sum_{j=i}^{n} \left\{ \pi_j^*(b_{j,j}) - \pi_j^*\,(b_{j,j+1}) \right\}.
\end{aligned}$$

Elementary calculations further yield

$$\pi_j^*(b_{j,j}) = \frac{\sigma_j^* \mu_j}{2\sigma_j} \quad \text{and} \quad \pi_j^*(b_{j,j+1}) = \frac{\sigma_j^* \mu_{j+1}}{2\sigma_{j+1}}.$$

It follows that

$$\begin{aligned}
\sum_{j=i}^{n} \left\{ \pi_j^*(b_{j,j}) - \pi_j^*(b_{j,j+1}) \right\} &= \sum_{j=i+1}^{n} \frac{\mu_j}{2\sigma_j}\, \left(\sigma_j^* - \sigma_{j-1}^* \right) + \frac{\sigma_i^* \mu_i}{2\sigma_i} \\
&= \frac{\mu_n^* - \mu_i^*}{2} + \frac{\sigma_i^* \mu_i}{2\sigma_i}
\end{aligned}$$

and

$$\sum_{j=i}^{n-1} \frac{\mu_{j+1}^2}{\sigma_{j+1}^2 + \mu_{j+1}^2}\, (b_{j+1,j+1} - b_{j,j+1}) = \frac{\mu_n^* - \mu_i^*}{2},$$

which implies that

$$\pi(X_{\max}, d) = \frac{\mu_i^2}{\sigma_i^2 + \mu_i^2}\, (b_{i,i} - d) + \mu_n^* - \mu_i^* + \frac{\sigma_i^* \mu_i}{2\sigma_i}.$$

As previously, one can get for $b_{i,i} \le d \le b_{i,i+1}$

$$\begin{aligned}\pi(X_{\max}, d) &= \pi_i^*(d) - \pi_i^*(b_{i,i}) + \pi(X_{\max}, b_{i,i}) \\ &= \pi_i^*(d) - \frac{\sigma_i^* \mu_i}{2\sigma_i} + \mu_n^* - \mu_i^* + \frac{\sigma_i^* \mu_i}{2\sigma_i} \\ &= \pi_i^*(d) + \mu_n^* - \mu_i^*.\end{aligned}$$

5. Bounded risks with given means and Archimedean copula. Suppose that $\mathrm{E}(X_i) = \mu_i$ is given for every $i \in \{1, \ldots, n\}$ and that it is further known that in formula (3), the underlying copula associated with the vector $(X_1, \ldots, X_n)$ is of the form

$$C_\phi(u_1, \ldots, u_n) = \phi^{-1}\{\phi(u_1) + \cdots + \phi(u_n)\} \tag{16}$$

for some function $\phi : (0, 1] \to [0, \infty)$ whose inverse ϕ^{-1} is such that

$$\frac{d^k}{dt^k}\,\phi^{-1}(t) > 0, \tag{17}$$

for every $1 \le k \le n$.

Such copulas, which are called Archimedean (Genest and MacKay 1986), have recently become popular as a model for dependence. They have attractive theoretical properties (summarized by Nelsen 1999, Chapter 4), are easy to simulate, and are readily interpretable in terms of a mixture model when condition (17) is verified for all integer $n \ge 2$. In the latter case, ϕ^{-1} is a Laplace transform and the induced dependence is necessarily positive (Marshall and Olkin 1988). For recent actuarial and financial applications involving Archimedean copula modelling, see Frees and Valdez (1998), Klugman and Parsa (1999), Carrière (2000), Belguise and Lévi (2001–2002), Cherubini and Luciano (2002), or Hennessy and Lapan (2002), among others.

In the present context, the key property of Archimedean copulas is the fact that C_ϕ is symmetric and conditionally increasing in sequence whenever $(-1)^n d^n \phi^{-1}(t)/dt^n$ is log-convex (see Müller and Scarsini 2003). Thus, as an immediate consequence of Theorem 4.5 of Müller and Scarsini (2001), one gets

$$X = \sum_{i=1}^n X_i \prec_{\mathrm{cx}} \sum_{i=1}^n X_{i,\max} = X_{\max},$$

for any vector $(X_{1,\max}, \ldots, X_{n,\max})$ with copula C_ϕ and marginals $F_{1,\max}$, $\ldots, F_{n,\max}$ such that $F_i \prec_{\mathrm{cx}} F_{i,\max}$ for $1 \le i \le n$.

The distribution of $(X_{1,\max}, \ldots, X_{n,\max})$ and hence a bound on $\pi(X, d)$ can sometimes be found explicitly. For illustration purposes, suppose as in Section 3 that for each i, $X_i \in [0, b_i]$ and $\mathrm{E}(X_i) = \mu_i$. In this situation, the appropriate $F_{i,\max}$ is that given by (9). Now suppose that the $X_{i,\max}$ are linked by an Archimedean copula C_ϕ for which ϕ^{-1} is a Laplace transform, which implies that $(-1)^n d^n \phi^{-1}(t)/dt^n$ is log-convex for all $n \geq 2$. According to Marshall and Olkin (1988), there then exists a latent random variable Θ conditional upon which the random variables $X_{1,\max}, \ldots, X_{n,\max}$ are mutually independent and marginally distributed as

$$F_{\theta,i,\max}(x) = \Pr\left(X_{i,\max} \leq x | \Theta = \theta\right) = \exp\left[-\theta\phi\{F_{i,\max}(x)\}\right]. \tag{18}$$

The stop-loss premium of $X_{\max}$ is then of the form

$$\pi(X_{\max}, d) = \sum_{\ell=0}^{n} \sum_{A \in \mathcal{I}_\ell} s(A) \max\left(0, -d + \sum_{i \notin A} b_i\right)$$

with $\mathcal{I}_\ell = \mathcal{I}_\ell(\emptyset)$, where in general $\mathcal{I}_\ell(B)$ denotes the collection of subsets of size ℓ of $\{1, \ldots, n\} \setminus B$. Here, the coefficient $s(A)$ is defined for each set A of size ℓ as

$$\sum_{j=0}^{n-\ell} (-1)^j \sum_{B \in \mathcal{I}_j(A)} \phi^{-1}\left\{\sum_{i \in A} \phi\left(1 - \frac{\mu_i}{b_i}\right) + \sum_{k \in B} \phi\left(1 - \frac{\mu_k}{b_k}\right)\right\}, \tag{19}$$

with the understanding that a sum over an empty set is null.

To prove this, introduce for each possible θ a set $X_{\theta,1,\max}, \ldots, X_{\theta,n,\max}$ of mutually independent random variables with $X_{\theta,i,\max}$ distributed as $F_{\theta,i,\max}$ for $1 \leq i \leq n$. Then

$$\pi(X_{\max}, d) = \mathrm{E}_\Theta\left\{\pi\left(X_{\theta,\max}, d\right)\right\}$$

with $X_{\theta,\max} = X_{\theta,1,\max} + \cdots + X_{\theta,n,\max}$. Using the fact that $X_{\theta,i,\max}$ is either equal to 0 or b_i, one may then write

$$\pi\left(X_{\theta,\max}, d\right) = \sum_{\ell=0}^{n} \sum_{A \in \mathcal{I}_\ell} s_\theta(A) \max\left(0, -d + \sum_{i \notin A} b_i\right),$$

where

$$s_\theta(A) = \prod_{i \in A} p_{\theta,i} \prod_{i \notin A} (1 - p_{\theta,i})$$

with

$$p_{\theta,i} = \exp\{-\theta\phi(1-\mu_i/b_i)\}, \quad 1 \le i \le n.$$

Next observe that

$$\prod_{i\notin A}(1-p_{\theta,i}) = \sum_{j=0}^{n-\ell}(-1)^j \sum_{B\in\mathcal{I}_j(A)} \prod_{k\in B} p_{\theta,k},$$

so that

$$s_\theta(A) \quad = \quad \sum_{j=0}^{n-\ell}(-1)^j \sum_{B\in\mathcal{I}_j(A)} \prod_{i\in A} p_{\theta,i} \prod_{k\in B} p_{\theta,k}$$

which upon substitution, may be written explicitly as

$$\sum_{j=0}^{n-\ell}(-1)^j \sum_{B\in\mathcal{I}_j(A)} \exp\left\{-\theta\sum_{i\in A}\phi\left(1-\frac{\mu_i}{b_i}\right) - \theta\sum_{k\in B}\phi\left(1-\frac{\mu_k}{b_k}\right)\right\}.$$

The proof is concluded by noting that $\mathrm{E}_\Theta\left\{s_\theta(A)\right\}$ is precisely the expression given by (19).

6. Lower stop-loss bounds. While all the applications covered so far produced upper stop-loss bounds, the strategy outlined in Section 2 also applies to the construction of lower bounds. To emphasize this point, consider a set $X_1,\ldots,X_n$ of random variables whose underlying copula $C(u_1,\ldots,u_n)$ dominates $u_1\cdots u_n$ in the supermodular ordering. By Theorem 1 of Christofides and Vaggelatou (2003), this happens whenever the X_i are associated in the sense of Esary, Proschan and Walkup (1967), for example.

Further assume, for illustration purposes, that

$$X_i \in [0,b_i], \quad \mathrm{E}(X_i) = \mu_i, \quad \mathrm{var}(X_i) = \sigma_i^2$$

are know for each $1 \le i \le n$. Then

$$X = \sum_{i=1}^n X_i \succ_{\mathrm{cx}} \sum_{i=1}^n X_{i,\min} = X_{\min},$$

where $X_{1,\min},\ldots,X_{n,\min}$ are mutually independent and $X_{i,\min} \prec_{\mathrm{icx}} X_i$ for $1 \le i \le n$. According to De Vylder and Goovaerts (1982), the optimal choice for the marginal distribution $X_{i,\min}$ is

$$F_{i,\min}(x) = \begin{cases} 0 & \text{if } x < \mu_i - \frac{\sigma_i^2}{b_i - \mu_i}\,, \\ 1 - \frac{\mu_i}{b_i} & \text{if } \mu_i - \frac{\sigma_i^2}{b_i-\mu_i} \le x < \frac{\mu_i^2+\sigma_i^2}{\mu_i}\,, \\ 1 & \text{if } x \ge \frac{\mu_i^2+\sigma_i^2}{\mu_i}\,. \end{cases}$$

Therefore, $\pi(X, d) \ge \pi(X_{\min}, d)$, and the latter may be expressed in the form

$$\sum_{\ell=0}^{n} \sum_{A \in \mathcal{I}_\ell} s(A) \max\left\{0, \sum_{i \in A}\left(\mu_i - \frac{\sigma_i^2}{b_i - \mu_i}\right) + \sum_{i \notin A}\left(\frac{\mu_i^2 + \sigma_i^2}{\mu_i}\right) - d\right\}, \tag{20}$$

with $\mathcal{I}_\ell$ defined as in Section 5 and

$$s(A) = \prod_{i \in A}\left(1 - \frac{\mu_i}{b_i}\right) \prod_{i \notin A} \frac{\mu_i}{b_i}\,,$$

The proof simply proceeds by conditioning on the two possible values of each of the $X_{i,\min}$, namely $\mu_i - \sigma_i^2/(b_i - \mu_i)$ and $(\mu_i^2 + \sigma_i^2)/\mu_i$, which occur with probability $1 - \mu_i/b_i$ and μ_i/b_i, respectively.

As a second illustration, consider once again the context of Archimedean copulas. Suppose that the vectors $(X_1, \ldots, X_n)$ and $(X_{1,\min}, \ldots, X_{n,\min})$ have the same Archimedean copula C_ϕ for which ϕ^{-1} is a Laplace transform. By virtue of Theorem 4.5 of Müller and Scarsini (2001), one then has

$$X_{\min} = \sum_{i=1}^{n} X_{i,\min} \prec_{\text{cx}} \sum_{i=1}^{n} X = X_{\max}.$$

Proceeding as in Section 5, one can then check easily that the lower stop-loss bound $\pi(X_{\min}, d)$ is expressed by (20) where the coefficient $s(A)$ is given by (19).

7. Concluding remarks. A general strategy was presented above for deriving upper and lower stop-loss bounds on directionally convex, increasing functions $\Psi(X_1, \ldots, X_n)$ of possibly dependent non-negative risks $X_1, \ldots, X_n$. Illustrations involving the important special case where $\Psi(X_1, \ldots, X_n) = X_1 + \cdots + X_n$ were presented in Sections 3–6.

Clearly, additional variations on the theme are possible. When a lower stop-loss bounds is sought on $\Psi(x_1, \ldots, x_n) = x_1 + \cdots + x_n$, for instance, weaker conditions than association between the risks X_i can be invoked to conclude that

$$\sum_{i=1}^{n} X_i^{\perp} \preceq_{\text{cx}} \sum_{i=1}^{n} X_i,$$

where the $X_i^{\perp}$ have the same marginals as the X_i but are mutually independence. One such condition, called positive cumulative dependence, was recently identified by Denuit, Dhaene and Ribas (2002). It is met whenever X_j and $S_{\mathcal{I}} = \sum_{i \in \mathcal{I}} X_i$ are in positive quadrant dependence for all choices of $\mathcal{I} \subset \{1, \ldots, n\}$ and $j \notin \mathcal{I}$.

As shown by Denuit et al. (2002), positive cumulative dependence is a weak form of dependence extends the classical bivariate notion of positive quadrant dependence to arbitrary dimension while keeping its intuitive meaning: if a vector $(X_1, \ldots, X_n)$ is positive cumulative dependent, the probability that $S_{\mathcal{I}}$ and X_j both assume "large" values is greater than if the X_i were independent. In particular, the inequality

$$\Pr\left(\sum_{i \neq j} X_i > s \middle| X_j > t\right) \geq \Pr\left(\sum_{i \neq j} X_i > s\right)$$

then holds true for any $1 \leq j \leq n$ with $\Pr(X_j > t) > 0$. Thus it follows that

$$\mathrm{E}\left\{\max\left(0, \sum_{i \neq j} X_i - s\right) \middle| X_j > t\right\} \geq \mathrm{E}\left\{\max\left(0, \sum_{i \neq j} X_i - s\right)\right\}.$$

In other words, the knowledge that a given component X_j is larger than some level t increases the probability that the sum of the $n-1$ remaining components is also large, as well as their expected excess over some threshold s.

In closing, it should be stressed that while the present approach to determining bounds on a function $\Psi(X_1, \ldots, X_n)$ of non-negative random variables was illustrated here in an actuarial context, it also has obvious applications in finance. For instance, path-dependent options depend on the history of an asset price, not just on its value on exercise. An example is an option to purchase an asset for the arithmetic average value of the asset over the month before expiry. Formally, denoting as X_i the value of some asset at dates t_i, $1 \leq i \leq n$, the path-dependent option has payoff

$$\max\left(0, -K + \sum_{i=1}^{n} \alpha_i X_i\right), \tag{21}$$

where K is the exercise price and α_i is the weight allotted to the value recorded at time t_i. As an additional example, consider an option on an index relating to some stock exchange market. A payoff of the form (21) results once again if the index is defined in terms of a basket of leading-firm shares $X_1, \ldots, X_n$, each of which is weighted according to the amount α_i of capital invested.

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