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Abstract

This paper revisits some asymptotic properties of the robust nonparametric estimators of order-m and order-α quantile frontiers and proposes isotonized version of these estimators. Previous convergence properties of the order-m frontier are extended (from weak uniform convergence to complete uniform convergence). Complete uniform convergence of the order-m (and of the quantile order-α) nonparametric estimators to the boundary is also established, for an appropriate choice of m (and of α, respectively) as a function of the sample size. The new isotonized estimators share the asymptotic properties of the original ones and a simulated example shows, as expected, that these new versions are even more robust than the original estimators. The procedure is also illustrated through a real data set.

Key words : Estimation of a monotone boundary; Robust nonparametric estimators; Isotonization procedure; Complete uniform convergence.

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1 Introduction and Basic Notation

Let Ψ be the support of the joint probability measure of a random vector $(X, Y) \in \mathbb{R}_+^p \times \mathbb{R}_+$ and let $(\Omega, \mathcal{A}, P)$ be the probability space on which the vector X and the variable Y are defined. Consider the problem of estimating non parametrically the upper boundary of Ψ, where "upper" is in the direction of the univariate Y. This boundary is assumed to be a monotone nondecreasing[1] function of X and we have a sample $(X_1, Y_1), \cdots, (X_n, Y_n)$ of independent random vectors with the same distribution as (X, Y).

Let us denote by $F(y|x) = F(x, y)/F_X(x)$ the conditional distribution function of Y given $X \leq x$, where F is the joint distribution function of (X, Y) and $F_X(x) = F(x, \infty)$. From now on we assume that $x \in \mathbb{R}_+^p$ is such that $F_X(x) > 0$. The monotone boundary of Ψ can then be characterized through the frontier function

$$\varphi(x) = \inf\{y \in \mathbb{R}_+ | F(y|x) = 1\},$$

which is the upper boundary of the support of the nonstandard conditional probability measure of Y given $X \leq x$.

This kind of problem appears naturally to be useful when analyzing production performance of firms, where X represents the vector of inputs (resources of production) and Y is the output (a quantity of produced goods). In this context, $\varphi(x)$ is the production frontier, i.e., the maximal achievable level of output for a firm working at the level of inputs x. The production efficiency of a firm operating at the level (x, y) can then be measured by the relative comparison of its output y with the reference frontier $\varphi(x)$.

Nonparametric envelopment estimators have been mostly used, like the Free Disposal Hull estimator (FDH, initiated by Deprins, Simar and Tulkens [6] in the context of measuring the efficiency of enterprises),

$$\widehat{\varphi}_n(x) = \inf\{y \in \mathbb{R}_+ | \widehat{F}_n(y|x) = 1\} = \max_{i|X_i \leq x} Y_i,$$

where $\widehat{F}_n(y|x) = \widehat{F}_n(x, y)/\widehat{F}_{X,n}(x)$, with $\widehat{F}_n(x, y) = (1/n)\sum_{i=1}^n 1(X_i \leq x, Y_i \leq y)$ and $\widehat{F}_{X,n}(x) = \widehat{F}_n(x, \infty)$. The convex hull of the FDH frontier $\widehat{\varphi}_n$ provides the Data Envel-

[1] For two vectors x and x' in $\mathbb{R}^p$ the inequality $x \leq x'$ has to be understood componentwise. A real valued function r on $\mathbb{R}^p$ is then said to be monotone nondecreasing with respect to this partial order if $x \leq x'$ implies $r(x) \leq r(x')$.

opment Analysis estimator (DEA, initiated by Farrell [7] and popularized as linear programming estimator by Charnes,Cooper and Rhodes [5]). The statistical inference based on these estimators is now mostly available either by using asymptotic results or by using the bootstrap (see Simar and Wilson [14] for a recent survey and Seiford [11] for a survey and more than 700 references on applications using these estimators). But, by construction, these estimators envelop all the data points and so, are very sensitive to extreme values.

Original robust non parametric estimators have been suggested recently by Cazals, Florens and Simar [4]. In place of looking for the full frontier, they estimate a partial frontier of order $m \geq 1$, which can be defined as follows. For a given level x, it is defined as the expected value of the maximum of m independent random variables $Y^1, \cdots, Y^m$, drawn from the conditional distribution of Y given $X \leq x$, *i.e.*,

$$\varphi_m(x) = E[\max(Y^1, \cdots, Y^m)|X \leq x] = \int_0^\infty (1 - [F(y|x)]^m)dy.$$

For all finite integer $m \geq 1$, $\varphi_m(x) \leq \varphi(x)$ and $\lim_{m\to\infty} \varphi_m(x) = \varphi(x)$. This expected frontier function of order m can be estimated non parametrically by plugging the empirical version $\widehat{F}_n(y|x)$ of the conditional distribution function $F(y|x)$ to obtain

$$\widehat{\varphi}_{m,n}(x) = \widehat{E}[\max(Y^1, \cdots, Y^m)|X \leq x] = \int_0^\infty (1 - [\widehat{F}_n(y|x)]^m)dy.$$

An explicit formula is available in order to compute $\widehat{\varphi}_{m,n}(x)$, but in practice it is more easy to approximate the empirical expectation by a Monte-Carlo algorithm (see, *e.g.*,Florens and Simar [8]). To summarize the properties of these functions, we have

$$\widehat{\varphi}_{m,n}(x) \leq \widehat{\varphi}_n(x), \quad \lim_{m\to\infty} \widehat{\varphi}_{m,n}(x) = \widehat{\varphi}_n(x)$$

$$\sqrt{n}(\widehat{\varphi}_{m,n}(x) - \varphi_m(x)) \to N\left(0, \sigma^2(x,m)\right) \quad \text{as} \quad n \to \infty,$$

where an expression of $\sigma^2(x,m)$ is available. By choosing m appropriately as a function of the sample size n, $\widehat{\varphi}_{m(n),n}(x)$ estimates the true frontier function $\varphi(x)$ itself and is more robust to extreme values than the FDH since it does not envelop all the data points : it is computed as the expectation of a maximum and not as an observed maximum. An explicit formula of the order $m(n)$ is given in [4], to sumarize, we must have $m(n) = O(n\log(n))$. In this case, this estimator keeps the asymptotic properties of the FDH estimator as derived in Park, Simar and Weiner [10].

Similarly, Aragon, Daouia and Thomas-Agnan [1] introduce the concept of an order-α quantile frontier function, which increases w.r.t. the continuous order $\alpha \in [0,1]$ and converges to the full frontier $\varphi(x)$ as $\alpha \nearrow 1$. It is defined, for a given level x, by the conditional α-quantile of the distribution of Y given $X \le x$, i.e.,

$$q_\alpha(x) := F^{-1}(\alpha|x) = \inf\{y \in \mathbb{R}_+ | F(y|x) \ge \alpha\}.$$

A nonparametric estimator of $q_\alpha(x)$, which increases and converges to the FDH $\widehat{\varphi}_n(x)$ as $\alpha \nearrow 1$, is easily derived by inverting the empirical version of the conditional distribution function,

$$\widehat{q}_{\alpha,n}(x) := \widehat{F}_n^{-1}(\alpha|x) = \inf\{y \in \mathbb{R}_+ | \widehat{F}_n(y|x) \ge \alpha\}.$$

As pointed out in [1], this estimator is very fast to compute, very easy to interpret and satisfies very similar statistical properties to those of the nonparametric estimator $\widehat{\varphi}_{m,n}(x)$. In summary, it converges at the rate $\sqrt{n}$, is asymptotically unbiased and normally distributed. Moreover, when the order α is considered as a function of n such that $n^{(p+2)/(p+1)}\,(1-\alpha(n)) \to 0$ as $n \to \infty$, $\widehat{q}_{\alpha(n),n}(x)$ estimates the true frontier function $\varphi(x)$ and shares the same asymptotic distribution of both the FDH estimator $\widehat{\varphi}_n(x)$ and the order-$m(n)$ frontier $\widehat{\varphi}_{m(n),n}(x)$.

The reliability of the two sequences of estimators $\{\widehat{q}_{\alpha,n}(x)\}$ and $\{\widehat{\varphi}_{m,n}(x)\}$ is analyzed from a robustness theory point of view in Daouia and Ruiz-Gazen [3]. Both of these nonparametric frontier estimators are qualitatively robust and bias-robust. But the order-α quantile frontiers can be more robust to extreme values than the order-m frontiers when estimating the true full frontier since the influence function is no longer bounded for order-m frontiers when m tends to infinity, while it remains bounded for the conditional quantile frontiers when the quantile order tends to one. The advantage of the order-m frontiers lies in the fact that they can be easily extended to the full multivariate case ($X \in \mathbb{R}_+^p$ and $Y \in \mathbb{R}_+^q$), where they can be computed by using a Monte-Carlo algorithm (Simar [13]). This full multivariate extension has not been obtained for the order-α quantile frontiers.

The drawback of the concepts of these partial frontiers lies in the fact that they are not necessarily monotone with respect to x, whereas the full frontier is monotone. In this paper, we propose an isotonized version $\varphi_m^{\#}(x)$ of $\varphi_m(x)$ and $q_\alpha^{\#}(x)$ of $q_\alpha(x)$, respectively, which converges uniformly to the full frontier $\varphi(x)$ as $m \to \infty$ and as $\alpha \nearrow 1$, respectively. In the same way, we introduce monotone versions $\widehat{\varphi}_{m,n}^{\#}(x)$ and $\widehat{q}_{\alpha,n}^{\#}(x)$ of the initial estimators $\widehat{\varphi}_{m,n}(x)$ and $\widehat{q}_{\alpha,n}(x)$. We first extend the results obtained in [8] about weak uniform consistency of $\widehat{\varphi}_{m,n}$ and $\widehat{\varphi}_n$ to the complete uniform convergence. We also establish the complete uniform convergence of both $\widehat{\varphi}_{m(n),n}$ and $\widehat{q}_{\alpha(n),n}$ to φ as $n \to \infty$. We then show that the isotone estimator $\widehat{\varphi}_{m,n}^{\#}$ converges completely and uniformly to the monotone order-m frontier $\varphi_m^{\#}$, and that the monotone versions $\widehat{\varphi}_{m(n),n}^{\#}$ and $\widehat{q}_{\alpha(n),n}^{\#}$ of the initial estimators $\widehat{\varphi}_{m(n),n}$ and $\widehat{q}_{\alpha(n),n}$ share the same strong uniform convergence property of the FDH estimator $\widehat{\varphi}_n$ to the full frontier φ. We illustrate the method through some numerical examples with real and simulated data.

2 Monotone Estimators of the Upper Boundary

The partial functions $\varphi_m(x)$ and $q_\alpha(x)$ converge to the nondecreasing full function $\varphi(x)$ as $m \to \infty$ and as $\alpha \nearrow 1$, respectively, but they are not nondecreasing themselves unless we assume that the conditional distribution function $F(y|x)$ is nonincreasing as a function of x (see [4], Theorem A.3, and [1], Proposition 2.5, respectively). Our goal is to make these partial frontier functions monotone nondecreasing on some given subset D interior to the support of X in a more general setup, *i.e.* without relying on such an assumption.

This is achieved through the following isotonization method : we denote by $||\cdot||$ the sup-norm of a real valued function over the domain D and we assume from now on that this domain is compact. For a real valued function r defined on D, let us define the following three functions

$$\begin{aligned} r^u(x) &= \sup_{x' \in D; x' \le x} r(x'), \\ r^l(x) &= \inf_{x' \in D; x' \ge x} r(x'), \\ r^{\#}(x) &= (r^u(x) + r^l(x))/2. \end{aligned} \tag{1}$$

It is clear that $r^u(x)$, $r^l(x)$ and $r^{\#}(x)$ are nondecreasing and that $r^l(x) \le r(x) \le r^u(x)$, for all x in their domain D.

A natural concept of a monotone order-m frontier can then be defined simply as the isotonized version $\varphi_m^{\#}(x)$ of $\varphi_m(x)$. This nondecreasing partial function can be estimated non parametrically by the isotonized version $\widehat{\varphi}_{m,n}^{\#}(x)$ of $\widehat{\varphi}_{m,n}(x)$.

The basic idea of this monotonization procedure is not new. Mukerjee and Stern [9] use a similar principle to isotonize a Nadaraya-Watson kernel estimator of the regression function, and with a slight difference, which is in fact a computational artifact : in their approach, the sup and inf in (1) are taken over a discrete grid instead of the whole domain D. In the context of production efficiency measurement, Aragon et al. [2] use the same technique to isotonize a smoothed estimator of the nonstandard conditional distribution function $F(y|x)$ with respect to x, but in the nonincreasing sense. They prove that when the initial smoothed estimator is strongly uniformly consistent and the function $x \mapsto F(y|x)$ is nonincreasing for y fixed, then the isotonized estimator is also strongly uniformly consistent. Their argument is based on the fact that the # operator, which provides in their approach a nonincreasing version of r on D, is sup-norm contracting (see [2], Lemma 3.7). In our setup, we only need to adapt this result to our # operator which rather provides a nondecreasing version of r on D.

Lemma 2.1. *If r and s are two functions defined on D, then*

$$||r^{\#} - s^{\#}|| \leq ||r - s||.$$

We know from Florens and Simar ([8], see the appendix, Proof of Lemma A.1) that $\widehat{\varphi}_{m,n}$ converges uniformly in probability to φ_m as $n \to \infty$. By applying Lemma 2.1, we obtain

$$||\widehat{\varphi}_{m,n}^{\#} - \varphi_m^{\#}|| \leq ||\widehat{\varphi}_{m,n} - \varphi_m||,$$

which implies the weak uniform consistency of $\widehat{\varphi}_{m,n}^{\#}$ for $\varphi_m^{\#}$. This result can be improved to obtain the complete uniform convergence by using the following lemma.

Lemma 2.2. *Assume that F_X is continuous on the compact D and that the upper boundary of the support of Y is finite. Then,*

$$||\widehat{\varphi}_{m,n} - \varphi_m|| \xrightarrow{co.} 0 \quad as \quad n \to \infty.$$

As an immediate consequence of Lemmas 2.1 and 2.2, we have

Theorem 2.3.

1. *Assume that φ and φ_m are continuous on the compact D, for every $m \geq 1$. Then,*

$$||\varphi_m^{\#} - \varphi|| \longrightarrow 0 \quad as \quad m \to \infty.$$

2. *Under the condition of Lemma 2.2, we have*

$$||\widehat{\varphi}_{m,n}^{\#} - \varphi_m^{\#}|| \xrightarrow{co.} 0 \quad as \quad n \to \infty.$$

3. *Under both above conditions, we have*

$$||\widehat{\varphi}_{m(n),n}^{\#} - \varphi|| \xrightarrow{co.} 0 \quad as \quad n \to \infty,$$

where the integer $m(n) \geq 1$ is such that

$$\lim_{n\to\infty} m(n) = \infty, \quad \lim_{n\to\infty} m(n) \left(\log n/n\right)^{1/2} = 0.$$

Note that the proof of the last result of Theorem 2.3 requires to extend the weak consistency of $\widehat{\varphi}_{m(n),n}(x)$ for $\varphi(x)$ proved in [4] to the complete uniform convergence. The next result gives a more subtle convergence rate of $m(n)$ as n tends to infinity, but the stochastic convergence here is only in the almost sure sense.

Theorem 2.4. *Under the conditions of Theorem 2.3, we have*

$$||\widehat{\varphi}_{m(n),n}^{\#} - \varphi|| \xrightarrow{a.s.} 0 \quad as \quad n \to \infty,$$

where $\lim_{n\to\infty} m(n) = \infty$, *and* $\lim_{n\to\infty} m(n) \left(\log\log n/n\right)^{1/2} = 0$.

Making use of Lemma 2.2, we also can improve the weak uniform consistency of the FDH estimator $\widehat{\varphi}_n$ by adapting the proof of Florens and Simar ([8], Lemma A.1).

Lemma 2.5. *Under the same regularity conditions of Theorem 2.3, we have*

$$||\widehat{\varphi}_n - \varphi|| \xrightarrow{co.} 0 \quad as \quad n \to \infty.$$

Likewise, in place of looking to the α-quantile function $q_\alpha(x)$ and its estimator $\widehat{q}_{\alpha,n}$, we rather concentrate on their isotonic versions $q_\alpha^{\#}(x)$ and $\widehat{q}_{\alpha,n}^{\#}(x)$.

Theorem 2.6.

1. *Assume that $x \mapsto q_\alpha(x)$ is continuous on the compact D, for every $\alpha \in [0,1]$. Then,*

$$||q_\alpha^{\#} - \varphi|| \longrightarrow 0 \quad as \quad \alpha \nearrow 1.$$

2. *Under the conditions of Lemma 2.5, we have*

$$||\widehat{q}^{\#}_{\alpha(n),n} - \varphi|| \xrightarrow{co.} 0 \quad as \quad n \to \infty,$$

where the order $\alpha(n)$ is such that $n(1-\alpha(n)) \to 0$ as $n \to \infty$.

Here, the initial estimator $\widehat{q}_{\alpha(n),n}(x)$ and its isotone version $\widehat{q}^{\#}_{\alpha(n),n}(x)$ estimate the full frontier $\varphi(x)$ itself. As expected by Aragon et al. ([1], Theorem 4.3), when the order $\alpha(n)$ converges to 1 at the rate $n^{(p+2)/(p+1)}$ as $n \to \infty$, $n^{1/(p+1)}\big(\varphi(x) - \widehat{q}_{\alpha(n),n}(x)\big)$ converges to a Weibull distribution whose parameters depend on the joint density of (X,Y) near the frontier point $(x, \varphi(x))$.

In practice, to compute the monotone frontier $\widehat{\varphi}^{\#}_{m,n}$ (in the same way $\widehat{q}^{\#}_{\alpha,n}$), we use a discrete grid instead of the whole domain D. For instance, we could consider the minimal rectangular set with edges parallel to the coordinate axes that covers all the observations X_i, and then choose a discrete grid $D_n = \{x_{n,1}, \cdots, x_{n,k}\}$ in this rectangular set containing the unique minimal and maximal (with respect to the partial order "$\leq$") points of this set (we could choose D_n to be simply the set of the observation points $\{X_i\}$ besides the minimal and maximal points of the minimal envelopment rectangular set). Such a choice makes it easier to compute both $\widehat{\varphi}_{m,n}(x)$ and $\widehat{\varphi}^{\#}_{m,n}(x)$ over the rectangular set. For example, if $p=1$ and $x_{n,1} \leq \cdots \leq x_{n,k}$, then $\widehat{\varphi}^{l}_{m,n}$ and $\widehat{\varphi}^{u}_{m,n}$ are constant between successive points such that

$$\widehat{\varphi}^{l}_{m,n}(x_{n,i}) = \widehat{\varphi}^{l}_{m,n}(x_{n,i+1}) \wedge \widehat{\varphi}_{m,n}(x_{n,i}), \quad \widehat{\varphi}^{u}_{m,n}(x_{n,i+1}) = \widehat{\varphi}^{u}_{m,n}(x_{n,i}) \vee \widehat{\varphi}_{m,n}(x_{n,i+1}),$$

for all $i = 1, \cdots, k-1$. Note that in this case, the choice of $D_n = \{X_i\}$ happens to be more natural for the quantile framework since the initial frontier $\widehat{q}_{\alpha,n}$ is by construction constant between successive observations X_i. For the general case ($p \geq 1$), first compute $\widehat{\varphi}^{u}_{m,n}$ successively along D_n starting from its minimal point, using the fact that

$$\widehat{\varphi}^{u}_{m,n}(x_{n,i}) = \widehat{\varphi}_{m,n}(x_{n,i}) \vee \max\left\{\widehat{\varphi}^{u}_{m,n}(x_{n,j}) : \ x_{n,j} \text{ is an immediate predecessor of } x_{n,i}\right\},$$

for all $x_{n,i} \in D_n$. Compute also $\widehat{\varphi}^{l}_{m,n}$ successively along D_n starting this time from its maximal point, using the fact that

$$\widehat{\varphi}^{l}_{m,n}(x_{n,i}) = \widehat{\varphi}_{m,n}(x_{n,i}) \wedge \min\left\{\widehat{\varphi}^{l}_{m,n}(x_{n,j}) : \ x_{n,j} \text{ is an immediate successor of } x_{n,i}\right\}.$$

The isotonic order-m frontier $\widehat{\varphi}^{\#}_{m,n}(x)$ can be therfore easily computed, for any x in the rectangular set, as the mean of

$$\widehat{\varphi}^{u}_{m,n}(x) = \max_{x_{n,i}\in D_n|x_{n,i}\leq x} \widehat{\varphi}^{u}_{m,n}(x_{n,i}) \quad \text{and} \quad \widehat{\varphi}^{l}_{m,n}(x) = \min_{x_{n,i}\in D_n|x_{n,i}\geq x} \widehat{\varphi}^{l}_{m,n}(x_{n,i}).$$

It is clear that a large value of k is necessary to get a good result in practice. We will see a numerical illustration in Section 4.

Mukerjee and Stern [9] perform a very closely similar isotonization algorithm by using an appropriate choice of D_n that leads to the strong uniform consistency of their isotonic estimator. We can easily adapt their setup to our problem by taking φ, $\widehat{\varphi}_{m(n),n}$, $\widehat{\varphi}^{\#}_{m(n),n}$, $\widehat{\varphi}^{u}_{m(n),n}$, $\widehat{\varphi}^{l}_{m(n),n}$ and D in place of the quantities τ, $\widehat{\tau}_n$, G_n, G_{1n}, G_{2n} and H in [9] (see Section 2), respectively (the same construction can be done for the quantile framework) :

For $\delta > 0$, let $D_\delta \supset D$ be the closed δ-neighborhood od D which we assume to be interior to the support of X. Let the initial estimator $\widehat{\varphi}_{m(n),n}(x)$ of the monotone upper boundary $\varphi(x)$ be defined on D_δ with $\widehat{\varphi}_{m(n),n}(x) = 0$ if $\widehat{F}_{X,n}(x) = 0$. Consider a positive sequence $\{b_n\}$ tending to 0, and let D_n be the set of vectors in D_δ with components that are integral multiples of b_n. For $\widehat{\varphi}^{\#}_{m(n),n}(x)$ to be well defined for $x \in D$ (see [9], Equation (2)), we assume that n is large enough.

As stated by Mukerjee and Stern, if D is rectangular with edges parallel to the coordinate axes, as is often the case, then we could consider only the minimal subset of D_n that covers D by convex combinations. The minimal and maximal points of this subset being unique, we then can isotonize $\widehat{\varphi}_{m(n),n}(x)$ over D, for a given order $m(n)$, by applying the computation method described above.

From a theoritical point of view, since D_n is not contained in D, we cannot apply Lemma 2.1 to obtain the complete uniform convergence of $\widehat{\varphi}^{\#}_{m(n),n}$ to φ on D (see Theorem 2.3). However, we can easily adapt the proof of Mukerjee and Stern to keep this asymptotic property. But such technique of proof requires more stringent conditions compared with those of Theorem 2.3. Indeed, if φ is uniformly continuous on D_δ and φ_m is continuous on this compact for every $m \geq 1$, then the same arguments used by Mukerjee and Stern (see [9], the paragraph after Equation (4), p. 78) show that

$$||\widehat{\varphi}^{\#}_{m(n),n} - \varphi|| \leq \sup_{x \in D_\delta} |\widehat{\varphi}_{m(n),n}(x) - \varphi(x)| + R_n,$$

where the remainder $R_n = o(1)$ in view of the appropriate characterization of D_n and the uniform continuity of φ on D_δ (for more details see [9], Theorem 2, the proof of Equation (6)). Finally, using the fact that $\sup_{x \in D_\delta} |\widehat{\varphi}_{m(n),n}(x) - \varphi(x)| \stackrel{co.}{\to} 0$ (replace D by D_δ in the proof of Theorem 2.3 to obtain this result), we obtain the complete uniform convergence of $\widehat{\varphi}^{\#}_{m(n),n}$ to φ on D. Under the same regularity conditions, we also get the complete uniform convergence of $\widehat{q}^{\#}_{\alpha(n),n}$ to φ on D by using similar arguments.

3 Proofs

Proof of Lemma 2.1. Let $M = \sup_{x \in D} |r(x) - s(x)|$. The lemma will follow from the following sets of inequalities

$$r^u - M \leq s^u \leq r^u + M,$$
$$r^l - M \leq s^l \leq r^l + M.$$

The two right inequalities follow from taking the $\sup_{x' \leq x}$ (resp : the $\inf_{x' \geq x}$) in the inequality $s(x') \leq r(x') + M$, and the left ones follow from taking the $\sup_{x' \leq x}$ (resp : the $\inf_{x' \geq x}$) in the inequality $r(x') - M \leq s(x')$. □

Proof of Lemma 2.2. Let $\nu < \infty$ be the upper boundary of the support of Y and let $x \in D$. Since $\widehat{\varphi}_n(x) \leq \varphi(x) \leq \nu$ with probability 1 (for a proof, see [1], Section 3), we have with probability 1,

$$\widehat{\varphi}_{m,n}(x) = \int_0^{\widehat{\varphi}_n(x)} (1 - [\widehat{F}_n(y|x)]^m) dy = \int_0^{\nu} (1 - [\widehat{F}_n(y|x)]^m) dy$$

We therefore obtain, with probability 1,

$$\begin{aligned}\widehat{\varphi}_{m,n}(x) - \varphi_m(x) &= \int_0^{\nu} \Big([F(y|x)]^m - [\widehat{F}_n(y|x)]^m \Big)\, dy \\ &= \int_0^{\nu} \Big(F(y|x) - \widehat{F}_n(y|x) \Big) \sum_{j=0}^{m-1} [F(y|x)]^{m-1-j} [\widehat{F}_n(y|x)]^j dy.\end{aligned}$$

This implies, with probability 1,

$$\begin{aligned}|\widehat{\varphi}_{m,n}(x) - \varphi_m(x)| &\leq m \int_0^{\nu} |F(y|x) - \widehat{F}_n(y|x)| dy \\ &= m \int_0^{\nu} \frac{|\widehat{F}_{X,n}(x)F(x,y) - F_X(x)\widehat{F}_n(x,y)|}{F_X(x)\widehat{F}_{X,n}(x)} dy \\ &\leq m \int_0^{\nu} \frac{F(x,y)|\widehat{F}_{X,n}(x) - F_X(x)| + F_X(x)|\widehat{F}_n(x,y) - F(x,y)|}{F_X(x)\widehat{F}_{X,n}(x)} dy \\ &\leq \frac{m\nu}{\widehat{F}_{X,n}(x)} \Big(||\widehat{F}_{X,n} - F_X|| + ||\widehat{F}_n - F|| \Big).\end{aligned}$$

Thus, we have with probability 1,

$$||\widehat{\varphi}_{m,n} - \varphi_m|| \leq \frac{m\nu}{\inf_{x\in D} \widehat{F}_{X,n}(x)} \Big(||\widehat{F}_{X,n} - F_X|| + ||\widehat{F}_n - F|| \Big). \quad (2)$$

To complete the proof, it suffices to show that the term on the right-hand side of Inequality (2) converges completely to 0 as $n \to \infty$. We know from Glivenko-Cantelli Theorem ([12], see the proof of Theorem A, p. 61) that $||\widehat{F}_{X,n} - F_X||$ and $||\widehat{F}_n - F||$ converge completely to 0 as $n \to \infty$. Hence, it only remains to show that

$$\exists \delta > 0 \quad \text{such that} \quad \sum_{n=1}^{\infty} P(\inf_{x\in D} \widehat{F}_{X,n}(x) \leq \delta) < \infty. \quad (3)$$

Indeed, it can be easily seen that, if $\{V_n\}$ and $\{W_n\}$ are two sequences of random variables s.t. V_n converges completely to 0 and there exists $\delta > 0$ s.t. $\sum_{n=1}^{\infty} P(|W_n| \leq \delta) < \infty$, then V_n/W_n converges completely to 0.

Since $|\inf_{x\in D} \widehat{F}_{X,n}(x) - \inf_{x\in D} F_X(x)| \leq ||\widehat{F}_{X,n} - F_X||$ and $||\widehat{F}_{X,n} - F_X||$ converges completely to 0, we obtain $\sum_{n=1}^{\infty} P(|\inf_{x\in D} \widehat{F}_{X,n}(x) - \inf_{x\in D} F_X(x)| \geq \delta) < \infty$, for every $\delta > 0$. This yields $\sum_{n=1}^{\infty} P(\inf_{x\in D} \widehat{F}_{X,n}(x) \leq \inf_{x\in D} F_X(x) - \delta) < \infty$, $\forall \delta > 0$. Thus, we can end the proof by putting $\delta = \inf_{x\in D} F_X(x)/2 > 0$. □

Proof of Theorem 2.3. We know from Florens and Simar ([8], see the proof of Lemma A.1) that φ_m converges uniformly to φ as $m \to \infty$. Therefore, by applying Lemma 2.1, we obtain the first result,

$$||\varphi_m^{\#} - \varphi|| = ||\varphi_m^{\#} - \varphi^{\#}|| \leq ||\varphi_m - \varphi|| \longrightarrow 0 \quad \text{as} \quad m \to \infty.$$

The second result follows from Lemmas 2.1 and 2.2,

$$||\widehat{\varphi}_{m,n}^{\#} - \varphi_m^{\#}|| \leq ||\widehat{\varphi}_{m,n} - \varphi_m|| \xrightarrow{co.} 0 \quad \text{as} \quad n \to \infty.$$

To prove the last result, first let us show that $||\widehat{\varphi}_{m(n),n} - \varphi_{m(n)}||$ converges completely to 0 as $n \to \infty$. Let $\varepsilon > 0$. We know from Kiefer's Inequality ([12], Theorem B, p. 61) that there exists finite positive constants C_1 and C_2 such that

$$P(||\widehat{F}_n - F|| > d) \leq C_1 e^{-nd^2}, \quad P(||\widehat{F}_{X,n} - F_X|| > d) \leq C_2 e^{-nd^2}$$

for every $d > 0$ and all $n \geq 1$. By taking $\varepsilon/m(n) > 0$ in place of d in the above inequalities, we obtain

$$P\left(m(n)||\widehat{F}_n - F|| > \varepsilon\right) \leq C_1 e^{-n\varepsilon^2/m^2(n)}, \quad P\left(m(n)||\widehat{F}_{X,n} - F_X|| > \varepsilon\right) \leq C_2 e^{-n\varepsilon^2/m^2(n)}$$

for all $n \geq 1$. Since $\lim_{n\to\infty} \left(m^2(n)\log n\right)/n = 0$, we have $\left(m^2(n)\log n\right)/n \leq \varepsilon^2/2$, for n large enough. Hence $\exp(-n\varepsilon^2/m^2(n)) \leq n^{-2}$, for all n sufficiently large. This implies

$$\sum_{n=1}^{\infty} P\left(m(n)||\widehat{F}_n - F|| > \varepsilon\right) < \infty, \quad \sum_{n=1}^{\infty} P\left(m(n)||\widehat{F}_{X,n} - F_X|| > \varepsilon\right) < \infty$$

showing therefore that $m(n)||\widehat{F}_n - F||$ and $m(n)||\widehat{F}_{X,n} - F_X||$ converge completely to 0. Thus $\widehat{\varphi}_{m(n),n}$ converges completely and uniformly to $\varphi_{m(n)}$ in view of (2) and (3). Since $\lim_{n\to\infty} m(n) = \infty$ and $\lim_{m\to\infty} ||\varphi_m - \varphi|| = 0$, we have $\lim_{n\to\infty} ||\varphi_{m(n)} - \varphi|| = 0$. Finally, we obtain the desired result by using the following inequalities,

$$||\widehat{\varphi}_{m(n),n}^{\#} - \varphi|| \leq ||\widehat{\varphi}_{m(n),n} - \varphi|| \leq ||\widehat{\varphi}_{m(n),n} - \varphi_{m(n)}|| + ||\varphi_{m(n)} - \varphi||.$$

This completes the proof. $\square$

Proof of Theorem 2.4. We only need to show that $m(n)||\widehat{F}_n - F||$ and $m(n)||\widehat{F}_{X,n} - F_X||$ converge almost surely to 0, and then we follow the same setup used to prove the last result of Theorem 2.3. We have from the law of the iterated logarithm ([12], Theorem B, p. 62),

$$||\widehat{F}_{X,n} - F_X|| \leq 2C(F_X)\left(\log\log n/n\right)^{1/2}, \quad ||\widehat{F}_n - F|| \leq 2C(F)\left(\log\log n/n\right)^{1/2}$$

for all n large enough, with probability 1, where $C(F_X)$ and $C(F)$ are two finite positive constants. Since $\lim_{n\to\infty} m(n)\left(\log\log n/n\right)^{1/2} = 0$, the conclusion follows directly from the above inequalities. □

Proof of Lemma 2.5. Let $\varepsilon > 0$ and $n \geq 1$. Since φ_m converges uniformly to φ as $m \to \infty$, we have

$$\exists m_\varepsilon \text{ such that } ||\varphi_{m_\varepsilon} - \varphi|| < \varepsilon/2. \tag{4}$$

We also have in view of Lemma 2.2,

$$\sum_{n=1}^{\infty} P\left(||\widehat{\varphi}_{m_\varepsilon,n} - \varphi_{m_\varepsilon}|| > \varepsilon/2\right) < \infty. \tag{5}$$

We know that $\widehat{\varphi}_{m_\varepsilon,n}(x) \leq \widehat{\varphi}_n(x) \leq \varphi(x)$ with probability 1, for any $x \in D$. Here, we need to extend this result to show that

$$\forall x \in D,\ \widehat{\varphi}_{m_\varepsilon,n}(x) \leq \widehat{\varphi}_n(x) \leq \varphi(x) \tag{6}$$

with probability 1. We know that $y \leq \varphi(x)$ for any $(x, y) \in \Psi$ such that $F_X(x) > 0$. Since the random variable $F_X(X_i)$ is uniform on $(0, 1)$, it is almost surely strictly positive, and since $(X_i, Y_i) \in \Psi$ almost surely, we have $Y_i \leq \varphi(X_i)$ with probability 1. Put $\Omega_i = \{Y_i \leq \varphi(X_i)\}$, $i = 1, \cdots, n$. We have $P(\Omega_i)$=1, for $i = 1, \cdots, n$. Let $\Omega_0 = \cap_{i=1}^n \Omega_i$. Then $P(\Omega_0) = 1$. To prove (6), it is sufficient to show that $\Omega_0 \subset \{\forall x \in D,\ \max_{i|X_i \leq x} Y_i \leq \varphi(x)\}$. If $\omega \in \Omega_0$, then $Y_i(\omega) \leq \varphi(X_i(\omega))$, for all $i = 1, \cdots, n$. In particular, we obtain by using the monotonicity of φ,

$$\forall x \in D,\ \forall i \text{ such that } X_i(\omega) \leq x\ :\ Y_i(\omega) \leq \varphi(X_i(\omega)) \leq \varphi(x).$$

Hence, $\max_{i|X_i(\omega)\leq x} Y_i(\omega) \leq \varphi(x)$ for any $x \in D$, and thus, $\omega \in \{\forall x \in D,\ \max_{i|X_i \leq x} Y_i \leq \varphi(x)\}$. This ends the proof of (6). Now we obtain by using (6),

$$||\widehat{\varphi}_n - \varphi|| \leq ||\widehat{\varphi}_{m_\varepsilon,n} - \varphi|| \leq ||\widehat{\varphi}_{m_\varepsilon,n} - \varphi_{m_\varepsilon}|| + ||\varphi_{m_\varepsilon} - \varphi||$$

with probability 1. Combining with (4), we get

$$\begin{aligned} P\left(||\widehat{\varphi}_n - \varphi|| > \varepsilon\right) &\leq P\left(||\widehat{\varphi}_{m_\varepsilon,n} - \varphi_{m_\varepsilon}|| + \varepsilon/2 > ||\widehat{\varphi}_{m_\varepsilon,n} - \varphi_{m_\varepsilon}|| + ||\varphi_{m_\varepsilon} - \varphi|| > \varepsilon\right) \\ &\leq P\left(||\widehat{\varphi}_{m_\varepsilon,n} - \varphi_{m_\varepsilon}|| > \varepsilon/2\right). \end{aligned}$$

Thus $\sum_{n=1}^{\infty} P\left(||\widehat{\varphi}_n - \varphi|| > \varepsilon\right) < \infty$, in view of (5). □

Proof of Theorem 2.6. We know from Aragon et al. ([1], Proposition 2.4) that q_α converges uniformly to φ as $\alpha \nearrow 1$ and so, we obtain the first result by using Lemma 2.1,

$$||q_\alpha^{\#} - \varphi|| \leq ||q_\alpha - \varphi|| \longrightarrow 0 \quad \text{as} \quad \alpha \nearrow 1.$$

It follows from [1] (see the appendix : last inequality of the proof of Theorem 4.3) that, for any $\alpha > 0$ and all $x \in D$,

$$0 \leq \widehat{\varphi}_n(x) - \widehat{q}_{\alpha,n}(x) \leq n(1-\alpha)\nu \widehat{F}_{X,n}(x)$$

with probability 1, where $\nu < \infty$ denotes the upper boundary of the support of Y. This implies, for any $\alpha > 0$,

$$||\widehat{\varphi}_n - \widehat{q}_{\alpha,n}|| \leq n(1-\alpha)\nu \left(||\widehat{F}_{X,n} - F_X|| + ||F_X||\right)$$

with probability 1. Therefore, by choosing α as a function of n such that $n\left(1 - \alpha(n)\right) \to 0$ as $n \to \infty$, we obtain by using Glivenko-Cantelli Theorem and the continuity of F_X on D ($||F_X|| < \infty$),

$$||\widehat{\varphi}_n - \widehat{q}_{\alpha(n),n}|| \xrightarrow{co.} 0 \quad \text{as} \quad n \to \infty. \tag{7}$$

Thus, we get by applying Lemma 2.1,

$$||\widehat{q}^{\#}_{\alpha(n),n} - \varphi|| \leq ||\widehat{q}_{\alpha(n),n} - \varphi|| \leq ||\widehat{q}_{\alpha(n),n} - \widehat{\varphi}_n|| + ||\widehat{\varphi}_n - \varphi||$$

which converges completely to 0 as $n \to \infty$, in view of (7) and Lemma 2.5. □

4 Numerical Illustration

In this section, we illustrate our concept of monotone partial frontiers through two examples, one with a simulated sample and one with a real data set.

4.1 Example 1

First we simulate a sample of $n = 100$ observations (x_i, y_i) according the data generating process $Y = \exp(-5 + 10X)/(1 + \exp(-5 + 10X)) \exp(-U)$, where X is uniform on $(0, 1)$ and U is exponential with mean 1/3.

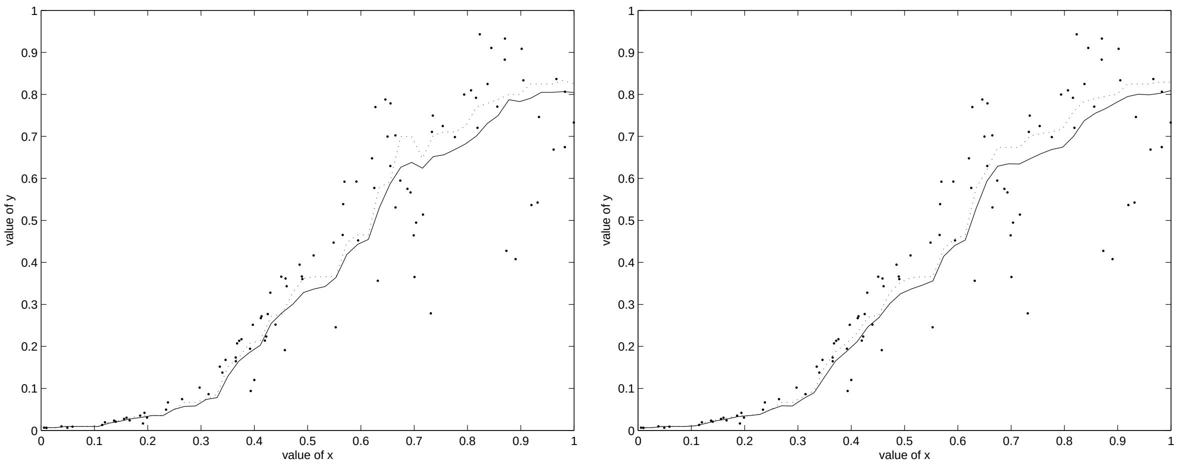

Figure 1: $n = 100$: The initial estimators $\widehat{q}_{.93,n}$ and $\widehat{\varphi}_{8,n}$ on the left and their isotonized versions $\widehat{q}^{\#}_{.93,n}$ and $\widehat{\varphi}^{\#}_{8,n}$ on the right. Solid line is for $m = 8$ and dotted line for $\alpha = .93$.

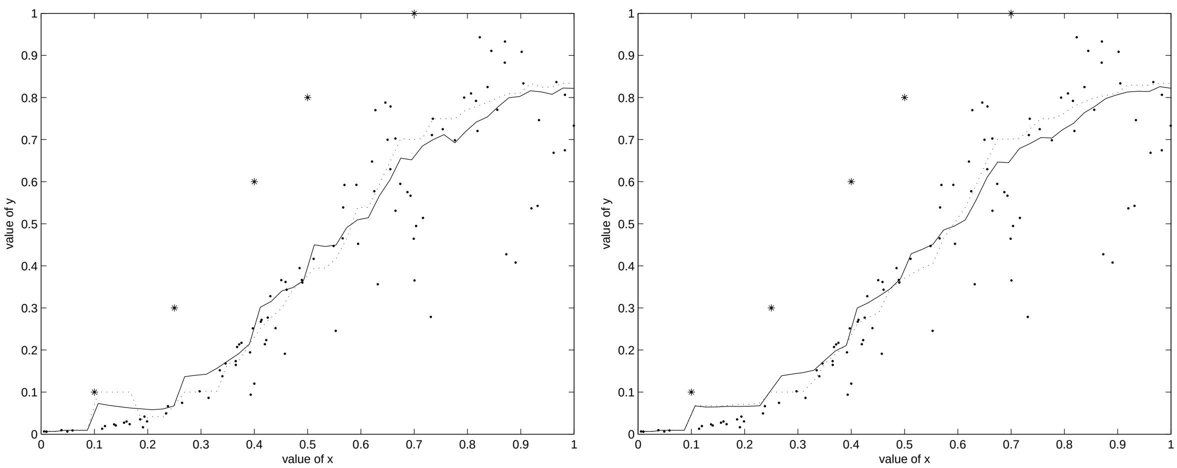

Figure 2: n=105: Same as above with 5 outliers included.

On the left-hand side of Figure 1, we plot in dotted line the quantile frontier $\widehat{q}_{\alpha,n}$ of order $\alpha = .93$, and in solid line the frontier $\widehat{\varphi}_{m,n}$ of order $m = 8$ (computed with $B = 500$ Monte-Carlo draws). Here, the quantile frontier $\widehat{q}_{.93,n}$ is everywhere above the frontier $\widehat{\varphi}_{8,n}$.

The isotonized versions of these frontiers are displayed on the right-hand side. For the computations, we simply define D as a discrete grid of n points equispaced between the min and the max of the observations.

In order to test the robustness of the isotonic estimators $\widehat{q}^{\#}_{.93,n}$ and $\widehat{\varphi}^{\#}_{8,n}$ with respect to the initial ones, we add in the data set five outliers, indicated by "*" in Figure 2, and we plot the same frontier estimators as above. We remark that both isotone frontiers are more resistant to the five outliers than the initial ones. This is natural since, by construction (see Equation (1)), the monotone function $r^{\#}$ is everywhere below the monotone upper boundary r^u of the initial function r.

4.2 Example 2

We examine here real data in an univariate case: the data are reported by Cazals et al. [4] and Aragon et al. [1] on frontier analysis of 9521 French post offices observed in 1994, with X as the quantity of labor and Y as the volume of delivered mail.

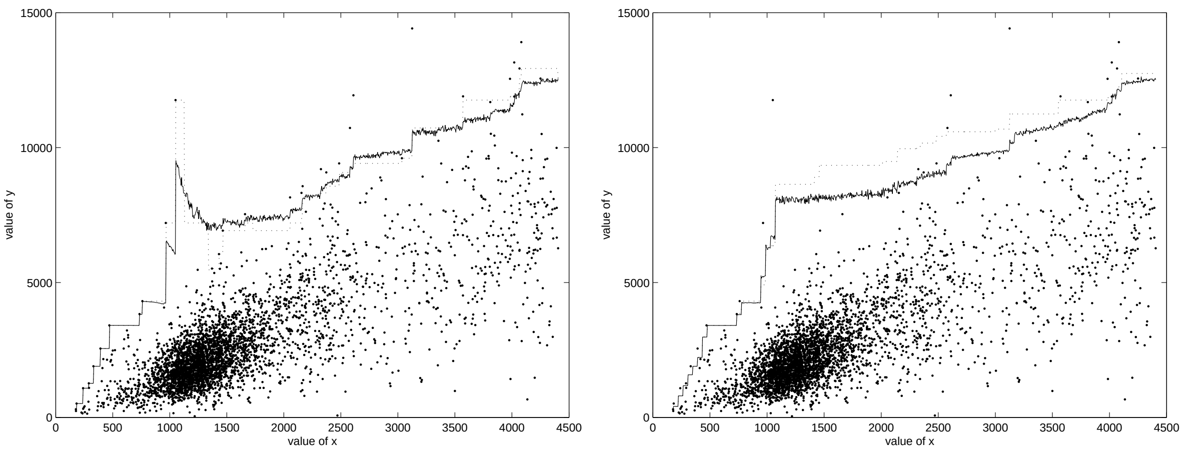

Figure 3: $n = 4000$: On the left, the frontiers $\widehat{q}_{.999,n}$ in dotted line and $\widehat{\varphi}_{600,n}$ in solid line. On the right, the monotone frontiers $\widehat{q}^{\#}_{.999,n}$ in dotted line and $\widehat{\varphi}^{\#}_{600,n}$ in solid line.

In this illustration, we only consider the $n = 4000$ observed post offices with the smallest levels x_i plotted in Figure 3 on the left-hand side, along with the quantile frontier $\widehat{q}_{\alpha,n}$ of order $\alpha = .999$ in dotted line, and the frontier $\widehat{\varphi}_{m,n}$ of order $m = 600$ in solid line ($B = 500$). The isotonized versions of these extreme frontiers are displayed on the right-hand side of

Figure 3. Here also, we use a discrete grid of only 100 points equispaced between the min and the max of the first 4000 observations.

It is clear that the monotone estimators $\widehat{q}^{\#}_{.999,n}$ and $\widehat{\varphi}^{\#}_{600,n}$ are more resistant to the super-efficient post offices than their initial versions $\widehat{q}_{.999}$ and $\widehat{\varphi}_{600,n}$.

More generally, for any orders α and m, the isotonized partial frontiers $\widehat{q}^{\#}_{\alpha}$ and $\widehat{\varphi}^{\#}_{m,n}$ are more robust to extreme values than the initial versions $\widehat{q}_{\alpha}$ and $\widehat{\varphi}_{m,n}$ introduced by Aragon et al. [1] and Cazals et al. [4], respectively, due to the average in the definition of the # operator.

5 Conclusions

Order-m frontier and order-α quantile frontier functions are very useful to provide non parametric estimators of boundaries which are more robust to outliers and/or extreme values than the usual envelopment estimators (FDH/DEA).

Their monotonized versions proposed in this paper are very easy to compute and provide estimators sharing the same properties as the original ones.

These new estimators appear to be even more robust to outliers than their original versions.

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