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Abstract

It is shown here that Kendall's τ and Spearman's ρ are monotone with respect to the concordance ordering of pairs of discrete as well as continuous random variables. This extends and completes results of Tchen (1980) and Yanagimoto and Okamoto (1969). It is also shown that various relationships between Kendall's τ and Spearman's ρ mentioned in Nelsen (1999) remain valid for discrete variables. In particular, a result of Capéraà and Genest (1993) is extended to the case of discrete random pairs. Finally, an analytic expression is given for the most extreme values of Kendall's τ and Spearman's ρ associated with discrete uniform variates.

Key words and phrases: Concordance measures, concordance order, copulas, dependence properties, Fréchet bounds, monotone dependence.

1 Introduction

Intuitively, two random variables X and Y are concordant when large values of X tend to be associated with large values of Y. Several attempts have been made to formulate this concept precisely; see, e.g., Tchen (1980) or Scarsini (1984).

The problem of comparing the strength of monotone dependence has also attracted a lot of attention. Let (X_1, Y_1) and (X_2, Y_2) be two random pairs with identical univariate margins concentrating their mass on finitely many atoms; (X_2, Y_2) is said to be more concordant than (X_1, Y_1) if the distribution function H_2 of (X_2, Y_2) can be obtained from the distribution function H_1 of (X_1, Y_1) by a finite number of rearrangements which add mass ϵ at (x, y) and (x', y') while subtracting mass ϵ at (x', y) and (x, y'), where $x' > x$ and $y' > y$.

Tchen (1980) proved that (X_2, Y_2) is more concordant than (X_1, Y_1) if, and only if, the distribution function H_2 dominates H_1 everywhere. This yields the concept of concordance ordering. Specifically, this dependence ordering is defined as follows: given two random vectors (X_1, Y_1) and (X_2, Y_2) with identical marginals and respective cumulative distribution functions (cdf) H_1 and H_2, (X_2, Y_2) is said to be more concordant than (X_1, Y_1), denoted as $(X_1, Y_1) \prec_c (X_2, Y_2)$, if $H_1(s, t) \le H_2(s, t)$ holds for all $s, t \in \mathbb{R}$.

Since $(X_1, Y_1) \prec_c (X_2, Y_2)$ expresses the fact that (X_2, Y_2) is "more positively dependent" than (X_1, Y_1), we naturally expect that concordance measures (such as Kendall's τ or Spearman ρ) will assume a larger value for (X_2, Y_2) than for (X_1, Y_1). This is indeed the case when F and G are continuous, as can easily be deduced from the following representation of these measures:

$$\tau_{H_i} = 4\mathbb{E}_{H_i}[H_i(X_i, Y_i)] - 1 \tag{1.1}$$

and

$$\rho_{H_i} = 12\mathbb{E}_{\Pi}[H_i(X_i, Y_i)] - 3 \tag{1.2}$$

mentioned by Schweizer and Wolff (1981); here, Π corresponds to the product probability distribution. The monotonicity of τ and ρ for continuous bivariate distributions has been discussed by Joe (1997, page 37). Tchen (1980) obtained a similar monotonicity property for τ and ρ when the supports of H_1 and H_2 consist in a finite number of atoms. For a random pair (X, Y) with joint distribution H, we denote by τ_H or $\tau_{X,Y}$ the corresponding value of Kendall's τ; likewise, Spearman's ρ is denoted ρ_H or $\rho_{X,Y}$.

The main purpose of this paper is to establish the monotonicity of τ and ρ with respect to $\preceq_c$ for any type of random pairs. To this end, we will present formulas for τ and ρ that are of independent interest, in the vein of (1.1) and (1.2). Other measures for discrete data are also considered. In particular, these results complement the recent studies by Genest and Rivest (2001), where probability integral transforms are investigated for random pairs with continuous marginals. As a byproduct of these results, a first comparison between τ and ρ for general random variables is given, and using the "continuousation principle" of

Denuit and Lambert (2002), we derive a discrete version of many comparisons mentioned in Nelsen (1999), among them the result of Capéraà and Genest (1993).

Recently, several authors have studied measures of monotone dependence; Conti (1993) gives definitions of two aspects of indifference, and ties them to the concordance and discordance concepts. Kowalczyk and Niewiadomska-Bugag (1998) outline general formulas for Kendall's τ and Spearman's ρ with (1.1) and (1.2) as special cases for continuous variables. Kowalczyk and Niewiadomska-Bugag use these concordance measures for applications in correspondence analysis.

The paper is organized as follows. Section 2 shows the monotonicity with respect to the concordance ordering of Kendall's τ and Spearman's ρ for any kind of random variables. The same property of some measures that take into consideration the presence of ties for discrete random variables is also considered. In Section 3, we give a discrete version for some universal comparisons between τ and ρ mentioned in Nelsen (1999) for continuous variables. Section 4 gives extremes values of τ and ρ for uniform discrete random variables. The last section is devoted to a discussion.

2 The monotonicity of Kendall's τ, Spearman's ρ and Gamma measure γ

2.1 The monotonicity of Kendall's τ

In this section, we are going to investigate the monotonicity of Kendall's τ_H with respect to the concordance ordering. We denote $H(x^-, y^-) = \Pr[X < x, Y < y], H(x, y^-) = \Pr[X \leq x, Y < y], H(x^-, y) = \Pr[X < x, Y \leq y], F(x^-) = \Pr[X < x]$ and $G(y^-) = \Pr[Y < y]$, where H is a cumulative distribution function (*cdf*) in $\Gamma(F, G)$, the space of all *cdf* with margins F and G.

Let us begin with the following useful lemma.

Lemma 2.1. *Let (X, Y) be any random couple with joint cdf $H \in \Gamma(F, G)$. Then*

$$\mathbb{E}[H(X, Y)] = \mathbb{E}[H(X^-, Y^-)] - \mathbb{E}[F(X^-)] - \mathbb{E}[G(Y^-)] + 1. \tag{2.1}$$

Proof. Let (X, Y) and (X', Y') be independent and identically distributed pairs of random variables with cdf $H \in \Gamma(F, G)$. Then

$$\begin{aligned} \mathbb{E}[H(X, Y)] &= \Pr(X \leq X', Y \leq Y') \\ &= 1 - \Pr(X > X') - \Pr(Y > Y') + \Pr(X > X', Y > Y') \\ &= 1 - \mathbb{E}[F(X^-)] - \mathbb{E}[G(Y^-)] + \mathbb{E}[H(X^-, Y^-)]. \end{aligned}$$

□

We are now in a position to give an expression for Kendall's τ similar to (1.1) for general variables.

Proposition 2.2. *Let (X,Y) be any random couple with joint cdf $H \in \Gamma(F,G)$. Then*

$$\tau = \mathbb{E}[H(X,Y)] + \mathbb{E}[H(X^-,Y^-)] + \mathbb{E}[H(X^-,Y)] + \mathbb{E}[H(X,Y^-)] - 1. \qquad (2.2)$$

Proof. Let us start from the very definition of Kendall's τ: given an independent copy (X',Y') of (X,Y), τ is given by

$$\tau \;=\; \Pr[(X-X')(Y-Y')>0] - \Pr[(X-X')(Y-Y')<0],$$

which is the difference between the probability of concordance $P_C[H] = \Pr[(X-X')(Y-Y')>0]$ and the probability of discordance $P_D[H] = \Pr[(X-X')(Y-Y')<0]$. It is easily seen that

$$\begin{aligned} P_C[H] &= \Pr(X<X',Y<Y') + \Pr(X>X',Y>Y') \\ &= 2\mathbb{E}[H(X^-,Y^-)] \end{aligned}$$

and

$$\begin{aligned} P_D[H] &= \Pr(X<X',Y>Y') + \Pr(X>X',Y<Y') \\ &= \Pr(X<X') + \Pr(Y<Y') - \Pr(X<X',Y\le Y') - \Pr(X\le X',Y<Y') \\ &= \mathbb{E}[F(X^-)] + \mathbb{E}[G(Y^-)] - \mathbb{E}[H(X^-,Y)] - \mathbb{E}[H(X,Y^-)]. \end{aligned}$$

Combining the above information and using Lemma 2.1, we get (2.2). □

Let us now establish the monotonicity of Kendall's τ.

Proposition 2.3. *Let (X_1,Y_1) and (X_2,Y_2) be two random couples with respective distribution functions H_1 and H_2 in $\Gamma(F,G)$. Then*

$$(X_1,Y_1) \preceq_c (X_2,Y_2) \Rightarrow \tau_{H_1} \le \tau_{H_2}.$$

Proof. Let us recall the following result (see, e.g., Müller and Scarsini 2000, Theorem 2.5): $(X_1,Y_1) \preceq_c (X_2,Y_2)$ if, and only if,

$$\mathbb{E}[\phi(X_1,Y_1)] \le \mathbb{E}[\phi(X_2,Y_2)]$$

for all quasi-monotone functions ϕ, provided the expectations exist. A function ϕ is quasi-monotone if, and only if, for all x_1,x_2,y_1,y_2 such that $x_1 \le x_2$ and $y_1 \le y_2$, we have $\phi(x_2,y_2) - \phi(x_1,y_2) - \phi(x_2,y_1) + \phi(x_1,y_1) \ge 0$. Now, since $H_1 \le H_2$, and using the fact that the functions $H_1(x^-,y^-)$, $H_1(x,y^-)$ and $H_1(x^-,y)$ are quasi-monotone, because H_1 is quasi-monotone, we have

$$\begin{aligned} &\mathbb{E}_{H_1}[H_1(X_1^-,Y_1^-)] \le \mathbb{E}_{H_2}[H_1(X_2^-,Y_2^-)] \le \mathbb{E}_{H_2}[H_2(X_2^-,Y_2^-)], \\ &\mathbb{E}_{H_1}[H_1(X_1,Y_1^-)] \le \mathbb{E}_{H_2}[H_1(X_2,Y_2^-)] \le \mathbb{E}_{H_2}[H_2(X_2,Y_2^-)], \\ &\mathbb{E}_{H_1}[H_1(X_1^-,Y_1)] \le \mathbb{E}_{H_2}[H_1(X_2^-,Y_2)] \le \mathbb{E}_{H_2}[H_2(X_2^-,Y_2^-)], \end{aligned}$$

which imply from Proposition 2.2 that $\tau_{H_1} \le \tau_{H_2}$. □

2.2 The monotonicity of Spearman's ρ

The following result gives an explicit expression of Spearman's ρ for general random variables.

Proposition 2.4. *Let (X,Y) a random couple with distribution function H, and margins F and G. Spearman's ρ is then given by*

$$\rho = 3\left\{\mathbb{E}_\Pi[H(X^-,Y)] + \mathbb{E}_\Pi[H(X,Y^-)] + \mathbb{E}_\Pi[H(X^-,Y^-)] + \mathbb{E}_\Pi[H(X,Y)] - 1\right\}. \quad (2.3)$$

Proof. Let (X',Y') and (X'',Y'') be independent copies of (X,Y). Spearman's ρ is defined as

$$\rho = 3\left\{\Pr[(X-X')(Y-Y'')>0] - \Pr[(X-X')(Y-Y')<0],\right\}$$

which can be recast into

$$\begin{aligned} \rho &= 3\{\Pr[X<X',Y<Y''] + \Pr[X\le X',Y\le Y''] \\ &- \Pr[X<X',Y\le Y''] - \Pr[X\le X',Y<Y''] \\ &+ \Pr[X<X'] - \Pr[X>X'] + \Pr[Y<Y''] - \Pr[Y>Y''] - 1\}. \end{aligned} \quad (2.4)$$

Since the random vectors (X,Y), (X',Y') and (X'',Y'') are independent, we obtain that

$$\Pr[X<X',Y<Y''] = \mathbb{E}_\Pi[H(X^-,Y^-)]. \quad (2.5)$$

Similarly,

$$\begin{aligned} \Pr[X\le X',Y\le Y''] &= \mathbb{E}_\Pi[H(X,Y)], & (2.6)\\ \Pr[X<X',Y\le Y''] &= \mathbb{E}_\Pi[H(X^-,Y)], & (2.7)\\ \Pr[X\le X',Y<Y''] &= \mathbb{E}_\Pi[H(X,Y^-)], & (2.8) \end{aligned}$$

and

$$\begin{aligned} \Pr[X<X'] &= \Pr[X>X'],\\ \Pr[Y<Y''] &= \Pr[Y>Y'']. \end{aligned} \quad (2.9)$$

Finally the result is obtained by inserting (2.5), (2.6), (2.7), (2.8) and (2.9) into (2.4). □

Remark 2.5. *Formulas (2.2) and (2.3) are given in Kowalczyk and Niewiadomska-Bugag (1998), but the approach used here to derive these formulas is completely different.*

We now point out the monotonicity of Spearman's ρ, which is a direct consequence of the Proposition 2.4.

Proposition 2.6. *Let (X_1, Y_1) and (X_2, Y_2) be two random couples with respective distribution function H_1 and H_2 in $\Gamma(F, G)$. Then*

$$(X_1, Y_1) \preceq_c (X_2, Y_2) \Rightarrow \rho_{H_1} \leq \rho_{H_2}.$$

It is well known that given any bivariate distribution function H with univariate margins F and G, the inequalities

$$\max[0, F(x) + G(y) - 1] \leq H(x, y) \leq \min[F(x), G(y)] \tag{2.10}$$

are valid whatever x and y. Using Proposition 2.3, Proposition 2.6 and inequalities (2.10), we can derive bounds for the values of τ and ρ for any bivariate distributions with fixed margins. This idea is exploited in Section 4 to find explicit ranges of τ and ρ for discrete uniform random variables.

2.3 The monotonicity of gamma measure γ for discrete random variables

In this section, we discuss some measures that are specific to discrete random variables. Let H be the *cdf* of a discrete vector (X, Y), and let F and G be its marginal *cdf*. Suppose that (X_1, Y_1) and (X_2, Y_2) are two independent and identically distributed copies of (X, Y).

To allow a correction for ties, in the situation of discrete random variables, one often uses the coefficient τ_b due to Kendall (1945), defined as

$$\tau_b = \frac{\tau}{\sqrt{\Pr[X_1 \neq X_2] \Pr[Y_1 \neq Y_2]}},$$

which may be expressed also as

$$\tau_b = \frac{\tau}{\sqrt{(1 - \mathbb{E}[F(X) - F(X^-)])(1 - \mathbb{E}[G(Y) - G(Y^-)])}}. \tag{2.11}$$

Somers (1962) proposed a measure for which pairs untied on X serve as the base, rather than only those untied on both X and Y. The population version of Somers's d_X is

$$d_X = \frac{\tau}{\Pr[X_1 \neq X_2]} = \frac{\tau}{1 - \mathbb{E}[F(X) - F(X^-)]}. \tag{2.12}$$

The monotonicity with respect to the concordance ordering of these measures is an immediate consequence of the monotonicity of Kendall's τ. For continuous random variables, these measures are not considered because of the absence of ties.

The gamma measure of concordance proposed by Goodman and Kruskal (1954) is equal to the ratio of the difference between the probability of concordance $P_C[H]$ and discordance $P_D[H]$, to the sum of these probabilities:

$$\gamma = \frac{P_C[H] - P_D[H]}{P_C[H] + P_D[H]}, \tag{2.13}$$

where

$$\begin{aligned} P_C[H] &= 2\mathbb{E}[H(X^-, Y^-)], \\ P_D[H] &= \mathbb{E}[F(X^-)] + \mathbb{E}[G(Y^-)] - \mathbb{E}[H(X^-, Y)] - \mathbb{E}[H(X, Y^-)]. \end{aligned}$$

In the next proposition, we derive the monotonicity of the gamma measure with respect to the concordance ordering.

Proposition 2.7. *Let (X_1, Y_1) and (X_2, Y_2) be two discrete random vectors with respective distribution function H_1 and H_2. Then*

$$(X_1, Y_1) \preceq_c (X_2, Y_2) \Rightarrow \gamma_{H_1} \leq \gamma_{H_2}.$$

Proof. From (2.13), we have

$$\gamma_H = \frac{1 - \frac{P_D[H]}{P_C[H]}}{1 + \frac{P_D[H]}{P_C[H]}} = \frac{1 - \phi(H)}{1 + \phi(H)}. \tag{2.14}$$

Now, if $H_1 \leq H_2$ then we have that $0 \leq P_D[H_2] \leq P_D[H_1]$ and $0 < P_C[H_1] \leq P_C[H_2]$. It follows that

$$\phi(H_2) = \frac{P_D[H_2]}{P_C[H_2]} \leq \phi(H_1) = \frac{P_D[H_1]}{P_C[H_1]}. \tag{2.15}$$

Using the fact that the function $x \mapsto (1 - x)/(1 + x)$ is non-increasing in the interval $[0, \infty)$, we deduce from (2.14) and (2.15) that

$$H_1 \leq H_2 \Rightarrow \gamma_{H_1} \leq \gamma_{H_2}.$$

□

Remark 2.8. *The measure γ is the best correction for τ because it attains the optimal values -1, for a perfect discordance, and $+1$ for a perfect concordance. This is the only common measure of concordance considered so far with that property. It is always greater in absolute value than d_X, τ_b and τ:*

$$|\tau| \leq |d_X| \leq |\tau_b| \leq |\gamma|.$$

An interesting property of positive dependence is the concept of positive quadrant dependence (PQD) introduced by Lehmann (1966). This concept describes how two random variables behave together when they are simultaneously small (or large). To recall this concept, let (X, Y) be a random couple valued in $\mathbb{R} \times \mathbb{R}$ with joint cdf H, and marginals F and G. These random are said to be positively quadrant dependent if, and only if, for all $(x, y) \in \mathbb{R}^2$

$$H(x, y) \geq F(x)G(y) = \Pi(x, y). \tag{2.16}$$

From (2.16) and the results of monotonicity, we see that the PQD property implies that τ, ρ, d_X, τ_b and γ are all positive.

3 Relationship between ρ and τ for discrete random variables

Nelsen (1999), Theorem 5.2.2, shows that if X and Y are PQD continuous random variables, then

$$3\tau \geq \rho \geq 0. \tag{3.1}$$

These inequalities are in fact still true for any PQD pair of random variables. This can be viewed as a direct consequence of the general representations (2.2) and (2.3) of τ and ρ, and the fact that (2.16) implies

$$\mathbb{E}_H[\phi(X, Y)] \geq \mathbb{E}_\Pi[\phi(X, Y)]$$

for any quasi-monotone function ϕ.

Now in order to study the relationship between τ and ρ in the discrete case, let us recall the principle of "continuousation" by convolution discussed in Denuit and Lambert (2002).

3.1 Continuousation principle

Let X be a discrete random variable. Since Kendall's τ and Spearman's ρ are unaffected by strictly non-decreasing transformations of margins, we can suppose, without loss of generality, that X is valued in $\mathbb{N}$ with probability function $p_i = P[X = i]$, $i \in \mathbb{N}$. Denuit and Lambert (2002) propose to transform X into a continuous random variable X^* by defining

$$X^* = X + U,$$

where U is continuous and independent of X. For our purpose, we choose U to be uniformly distributed on $[0, 1]$. It follows that the distribution function F^* of X^* is given by

$$F_X^*(x) \;=\; \sum_{i=0}^{\infty} F_U(x-i)p_i, \quad x \;\in\; I\!R^+ \tag{3.2}$$

where F_U denotes the distribution function of U. Suppose that Y is another discrete random variable valued in $I\!N$ with probability function $q_i = P[Y = i]$, $i \in I\!N$, and let $Y^* = Y + V$ denote its continuous version, as defined above. We assume that the random vector (U, V) is independent of (X, Y). It is then easily seen that the joint distribution function of (X^*, Y^*) can be written in terms of the copula C of (U, V) as follows:

$$H^*(x,y) \;=\; \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} C(x-i, y-j)p_{ij}, \quad (x,y) \;\in\; I\!R^+ \times I\!R^+ \tag{3.3}$$

where p_{ij} denotes the joint probability function of (X, Y).

Now, let τ^* and ρ^* denote Kendall's tau and Spearman's rho of (X^*, Y^*). The next proposition shows the link between (τ, ρ) and (τ^*, ρ^*) via (τ_C, ρ_C), the values of Kendall's tau and Spearman's rho associated with the copula C.

Proposition 3.1.

$$\tau^* = \tau + \tau_C \mathbb{E}_H[S(X,Y)] \tag{3.4}$$

and

$$\rho^* = \rho + \rho_C \mathbb{E}_\Pi[S(X,Y)], \tag{3.5}$$

where

$$S(X,Y) = H(X,Y) - H(X,Y-1) - H(X-1,Y) + H(X-1,Y-1).$$

Proof. Let I_k be the unit interval $[k, k+1]$, $k \in I\!N$. From (3.3), we see for all $(x,y) \in I_i \times I_j$ that

$$H^*(x,y) \;=\; C(x-i,y-j)p_{ij} + (x-i)\sum_{k=0}^{j-1} p_{ik} + (y-j)\sum_{k=0}^{i-1} p_{kj} + H(i-1,j-1),$$

which may be rewritten in terms of C and H as follows:

$$\begin{aligned} H^*(x,y) \;&=\; C(x-i,y-j)p_{ij} + (x-i)\,[H(i,j-1) - H(i-1,j-1)] \\ &+\; (y-j)\,[H(i-1,j) - H(i-1,j-1)] + H(i-1,j-1), \end{aligned}$$

so that

$$dH^* = p_{ij}\,\frac{\partial^2 C(x-i,y-j)}{\partial x \partial y}\,dxdy.$$

Now, combining the previous formulas we get

$$\begin{aligned}
\tau^* &= 4E_{H^*}[H^*(X,Y)] - 1 \\
&= 4\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\int_{I_i}\int_{I_j} H^*(x,y)dH^* - 1 \\
&= (\tau_C + 1)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} p_{ij}^2 + 2E_H\left[H(X-1,Y) + H(X,Y-1)\right] - 1. \qquad (3.6)
\end{aligned}$$

Noting that

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} p_{ij}^2 = E_H[S(X,Y)]. \qquad (3.7)$$

Inserting (3.7) in (3.6), then the result follows from (2.2). A similar argument implies that

$$\begin{aligned}
\rho^* &= 12E_{\Pi}[H^*(X,Y)] - 3 \\
&= 12\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\int_{I_i}\int_{I_j} H^*(x,y)dF_X^*(x)dF_Y^*(y) - 3 \\
&= (\rho_C + 3)E_{\Pi}[S(X,Y)] + 6E_{\Pi}\left[H(X-1,Y) + H(X,Y-1)\right] - 3 \\
&= \rho + \rho_C\mathbb{E}_{\Pi}[S(X,Y)].
\end{aligned}$$

□

The quantities $\tau_C E_H[S(X,Y)]$ and $\tau_C E_{\Pi}[S(X,Y)]$ in Proposition 3.1 can be interpreted as an interaction term between the continuous and discrete effects induced by C and H, respectively. Moreover, from formulas (3.4) and (3.5), we see that Kendall's τ and Spearman's ρ are preserved by continuousation when U and V are chosen to be independent, since $\tau_C = \rho_C = 0$ in that case.

The following corollary is useful.

Corollary 3.2. *Let X and Y be discrete random variables, and let τ and ρ denote Kendall's τ and Spearman's ρ respectively. Then, there exists a continuous random vector (X^*, Y^*) whose values of Kendall's tau and Spearman's rho are equal to τ and ρ, respectively.*

As a consequence of corollary 3.2, many universal inequalities for Kendall's τ and Spearman's ρ derived in the continuous case remain true also in the discrete case; examples are provided by Theorem (5.1.9), Theorem (5.1.10) and Corollary (5.1.11) in Nelsen (1999).

3.2 ρ is larger that τ for positively discrete dependent random variables

Following Lehmann (1966) and Esary and Proschan (1972), a random variable Y is said to be *left-tail decreasing* in another variable X, denoted $LTD(Y|X)$ if, and only if,

$$\Pr[Y \leq y | X \leq x]$$

is a non-increasing function of x for all possible values of y. Likewise, Y is declared to be *right-tail increasing* in another variable X, denoted $RTI(Y|X)$ if, and only if,

$$\Pr[Y > y | X > x]$$

is a non-decreasing function of x for all possible values of y. Similarly, the concept of *left-tail increasing* and *right-tail decreasing*, denoted *LTI* and *RTD* respectively, are defined analogously by exchanging the words "increasing" and "decreasing" in the above definitions.

From Lehmann (1966), it is seen that Spearman's ρ and Kendall's τ must be non-negative when Y is *left-tail decreasing* or *right-tail increasing* in X, or vice versa. Moreover, Capéraà and Genest (1993), Proposition 2.3, states that when X and Y are continuous real-valued random variables, then

$$RTI(Y|X) \ \text{ and } \ LTD(Y|X) \quad \Longrightarrow \quad \rho_{X,Y} \geq \tau_{X,Y} \geq 0. \tag{3.8}$$

Hereafter, we are going to show that the preceding result remains valid when both of X and Y are discrete random variables. The next result points out that the concepts of the *left-tail decreasingness* and *right-tail increasingness* are preserved through an application of the "continuouosation principle."

Proposition 3.3. *Let X and Y be discrete random variables valued in $\mathbb{N}$ with joint cumulative distribution function $H(i,j)$ and marginals $F(i)$ and $G(j)$, respectively. Let $X^* = X + U$ and $Y^* = Y + V$ be their continuous versions such that U and V are independent uniform variables valued in $[0,1]$. Then*

$$LTD(Y|X) \iff LTD(Y^*|X^*), \tag{3.9}$$

$$RTI(Y|X) \iff RTI(Y^*|X^*). \tag{3.10}$$

Proof. To show (3.9), let $(x,y) \in I_i \times I_j$ with $I_k = [k, k+1]$ and $k \in \mathbb{N}$. From (3.2) and (3.3), we have

$$\Pr[Y^* \leq y | X^* \leq x] \quad = \quad \frac{(x-i)\delta_1(y) + \delta_2(y)}{(x-i)p_i + F(i-1)} \, ,$$

where

$$\delta_1(y) = (y-j)p_{ij} + H(i, j-1) - H(i-1, j-1)$$

and

$$\delta_2(y) = (y-j)\left[H(i-1,j) - H(i-1,j-1)\right] + H(i-1,j-1).$$

Now, Y^* is left-tail decreasing in X^* if, and only if, for all $i,j \in I\!N$

$$\delta_1(y)F(i-1) - p_i\delta_2(y) \le 0, \quad \text{for all} \quad y \in I_j$$

which is equivalent to

$$(y-j)A_{ij} + B_{ij} \le 0 \quad \text{for all} \quad y \in I_j.$$

This, in turn, holds true if, and only if,

$$B_{ij} \le 0 \quad \text{and} \quad A_{ij} + B_{ij} \le 0, \quad \text{for all} \quad i,j \in I\!N$$

where

$$A_{ij} = p_{ij}F(i-1) - p_i\left[H(i-1,j) - H(i-1,j-1)\right]$$

and

$$B_{ij} = F(i-1)\left[H(i,j-1) - H(i-1,j-1)\right] - p_iH(i-1,j-1).$$

Now both of $B_{ij} \le 0$ and $A_{ij} + B_{ij} \le 0$ are equivalent to $LTD(X|Y)$. Indeed, for all $i,j \in I\!N$

$$\begin{aligned} B_{ij} \le 0 \quad &\Longleftrightarrow \quad F(i-1)H(i,j-1) \le F(i)H(i-1,j-1) \\ &\Longleftrightarrow \quad LTD(Y|X). \end{aligned}$$

Similarly,

$$\begin{aligned} A_{ij} + B_{ij} \le 0 \quad &\Longleftrightarrow \quad F(i-1)H(i,j) \le F(i)H(i-1,j) \\ &\Longleftrightarrow \quad LTD(Y|X). \end{aligned}$$

In order to prove (3.10), one can see that

$$\Pr[Y^* \ge y | X^* \ge x] = \frac{(x-i)\bar{\delta}_1(y) + \bar{\delta}_2(y)}{\bar{F}(i-1) - (x-i)p_i}\,, \tag{3.11}$$

where $\bar{F}(i-1) = \Pr[X > i-1]$, $\bar{H}(i,j) = \Pr[X > i, Y > j]$ and

$$\begin{aligned} \bar{\delta}_1(y) &= (y-j)p_{ij} + \bar{H}(i,j-1) - \bar{H}(i-1,j-1), \\ \bar{\delta}_2(y) &= (y-j)[\bar{H}(i-1,j) - \bar{H}(i-1,j-1)] + \bar{H}(i-1,j-1). \end{aligned}$$

It follows that $RTI(Y^*|X^*)$ if and only if for all $i,j \in I\!N$

$$\bar{\delta}_1(y)\bar{F}(i-1) + p_i\bar{\delta}_2(y) \ge 0, \quad \text{for all} \quad y \in I_j$$

which can be rewritten as

$$(y-j)\bar{A}_{ij}+\bar{B}_{ij}\geq 0 \quad \text{for all} \quad y\in I_j.$$

The latter inequality is true if, and only if,

$$\bar{B}_{ij}\geq 0 \quad \text{and} \quad \bar{A}_{ij}+\bar{B}_{ij}\geq 0 \quad \text{for all} \quad i,j\in I\!N \tag{3.12}$$

with

$$\begin{aligned}\bar{A}_{ij} &= \bar{F}(i-1)p_{ij}+p_i[\bar{H}(i-1,j)-\bar{H}(i-1,j-1)],\\ \bar{B}_{ij} &= \bar{F}(i-1)[\bar{H}(i,j-1)-\bar{H}(i-1,j-1)]+p_i\bar{H}(i-1,j-1).\end{aligned}$$

As previously, one can get that for all $i,j\in I\!N$

$$\begin{aligned}\bar{B}_{ij}\geq 0 \quad &\Longleftrightarrow \quad \bar{F}(i-1)\bar{H}(i,j-1)\geq \bar{F}(i)\bar{H}(i-1,j-1)\\ &\Longleftrightarrow \quad RTI(Y|X)\end{aligned}$$

and

$$\begin{aligned}\bar{A}_{ij}+\bar{B}_{ij}\geq 0 \quad &\Longleftrightarrow \quad \bar{F}(i-1)\bar{H}(i,j)\geq \bar{F}(i)\bar{H}(i-1,j)\\ &\Longleftrightarrow \quad RTI(Y|X).\end{aligned}$$

□

The following corollary gives the discrete version of the result of Capéraà and Genest (1993). It can be viewed as a direct consequence of Proposition 3.3 and Corollary 3.2.

Corollary 3.4. *Let X and Y be discrete random variables valued in $I\!N$. Then*

$$LTD(Y|X) \quad \textit{and} \quad RTI(Y|X) \quad \Rightarrow \quad \rho_{X,Y}\geq \tau_{X,Y}\geq 0.$$

Remark 3.5. *A similar reasoning as in the proof of Proposition 3.3, allows to conclude that*

$$LTI(Y|X) \iff LTI(Y^*|X^*), \tag{3.13}$$
$$RTD(Y|X) \iff RTD(Y^*|X^*) \tag{3.14}$$

so that, from (3.13), (3.14) and the Remark 2.5 in Capéraà and Genest (1993) we conclude that

$$LTI(Y|X) \quad \textit{and} \quad RTD(Y|X) \quad \Rightarrow \quad \rho_{X,Y}\leq \tau_{X,Y}\leq 0.$$

Corollary 3.4 and Remark 3.5 are in fact true for any discrete random vector (X,Y) because τ and ρ are unaffected by strictly monotone transformations of the margins.

4 Illustration

In this section, we investigate the range of Kendall's τ and Spearman's ρ in the Fréchet space $\Gamma(F_n, F_n)$, with F_n an ordinal discrete uniform distribution with n atoms. Let τ_n and ρ_n be Kendall's τ and Spearman's ρ associated to $H_n \in \Gamma(F_n, F_n)$. From the representations (2.2) and (2.3), one can see that

$$\tau_n = \sum_{i=1}^{n}\sum_{j=1}^{n} T_{ij}^{(n)} S_{ij}^{(n)} - 1, \tag{4.1}$$

$$\rho_n = 3\sum_{i=1}^{n}\sum_{j=1}^{n} T_{ij}^{(n)} [F_n(i) - F_n(i-1)]^2 - 3, \tag{4.2}$$

where $T_{ij}^{(n)}$, $S_{ij}^{(n)}$ are given by

$$T_{ij}^{(n)} = H_n(i,j) + H_n(i-1, j-1) + H_n(i, j-1) + H_n(i-1, j), \tag{4.3}$$

$$S_{ij}^{(n)} = H_n(i,j) - H_n(i, j-1) - H_n(i-1, j) + H_n(i-1, j-1). \tag{4.4}$$

Now let $\tau_{\min}^{(n)}$, $\rho_{\min}^{(n)}$, and $\tau_{\max}^{(n)}$, $\rho_{\max}^{(n)}$ denote the values of Kendall's τ and Spearman's ρ corresponding to the lower and upper Fréchet bounds in $\Gamma(F_n, F_n)$. The following proposition provides an explicit expressions for $\tau_{\min}^{(n)}$ and $\rho_{\min}^{(n)}$.

Proposition 4.1.

$$\tau_{\min}^{(n)} = -\frac{n-1}{n} \quad \textit{and} \quad \rho_{\min}^{(n)} = -\frac{n^2-1}{n^2}. \tag{4.5}$$

Proof. For $H_n(i,j) = \max[i/n + j/n - 1, 0]$, we observe from (4.3) and (4.4) that

$$T_{ij}^{(n)} = \frac{1}{n}\,\mathbb{1}_{[i+j=n+1]} + \frac{4}{n}\,(i+j-n-1)\mathbb{1}_{[i+j\geq n+2]} \quad \text{and} \quad T_{ij}^{(n)} S_{ij}^{(n)} = \frac{1}{n}\,\mathbb{1}_{[i+j=n+1]},$$

where $\mathbb{1}_A$ is the indicator function of A. Thus (4.1) and (4.2) imply that

$$\tau_{\min}^{(n)} = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{1}_{[i+j=n+1]} - 1 = -\frac{n-1}{n}$$

and

$$\begin{aligned}
\rho_{\min}^{(n)} &= \sum_{i+j=n+1} \frac{3}{n^3} + \frac{12}{n^3} \sum_{i+j \geq n+2} (i+j-n-1) - 3 \\
&= \frac{3}{n^2} + \frac{12}{n^3} \sum_{k=2}^{n} \sum_{i+j=k+n} (k-1) - 3 \\
&= \frac{3}{n^2} + \frac{12}{n^3} \sum_{k=1}^{n-1} k(n-k) - 3 \\
&= -\frac{n^2-1}{n^2} .
\end{aligned}$$

□

Now, let us examine the upper bounds $\tau_{\max}^{(n)}$ and $\rho_{\max}^{(n)}$.

Proposition 4.2.

$$\tau_{\max}^{(n)} = \frac{n-1}{n} \quad \textit{and} \quad \rho_{\max}^{(n)} = \frac{n^2-1}{n^2} . \tag{4.6}$$

Proof. For $H_n(i,j) = \min(i/n, j/n)$, we get from (4.3) and (4.4) that

$$T_{ij}^{(n)} = \frac{4i-2}{n} \mathbb{1}_{[i<j]} + \frac{4j-2}{n} \mathbb{1}_{[i>j]} + \frac{4i-3}{n} \mathbb{1}_{[i=j]} \quad \text{and} \quad S_{ij}^{(n)} T_{ij}^{(n)} = \frac{4i-3}{n^2} \mathbb{1}_{[i=j]}.$$

It follows from (4.1) and (4.2) that

$$\tau_{\max}^{(n)} = \sum_{i=1}^{n} S_{ii}^{(n)} T_{ii}^{(n)} - 1 = \frac{n-1}{n}$$

and

$$\begin{aligned}
\rho_{\max}^{(n)} &= \frac{6}{n^2} \sum_{i=1}^{n} \sum_{j<i} \frac{4j-2}{n} + \frac{3}{n^2} \sum_{i=1}^{n} \frac{4i-3}{n} - 3 \\
&= \frac{12}{n^3} \sum_{i=1}^{n} (i-1)^2 + \frac{3}{n^3} \sum_{i=1}^{n} (4i-3) - 3 = \frac{n^2-1}{n^2} .
\end{aligned}$$

□

As an immediate consequence of the above results, one can conclude that the ranges of τ_n and ρ_n for any uniform discrete random variables X and Y with the same number of atoms n are given by

$$I_n = \left[-\frac{n-1}{n}, \frac{n-1}{n}\right] \quad \text{and} \quad J_n = \left[-\frac{n^2-1}{n^2}, \frac{n^2-1}{n^2}\right],$$

respectively. These ranges are increasing with n and are linked by the relation

$$J_n = \frac{n+1}{n}\, I_n,$$

in the sense that the range J_n is larger than I_n with the factor $(n+1)/n$.

5 Conclusion

Using general expressions for Kendall's τ and Spearman's ρ, the monotonicity of these measures with respect to the concordance ordering has been shown. The monotonicity of some measures of concordance for discrete variables was also considered. To the best of the authors' knowledge, the monotonicity property of these measures for arbitrary pairs of variables has never been explicitly investigated in the literature before. Comparisons between Kendall's τ and Spearman's ρ were also given. It was mentioned that using this approach, it is possible to obtain a discrete version of many relationships between Kendall's τ and Spearman's ρ, such as those reported in Nelsen (1999). As mentioned by Capéraà and Genest (1993), the result stated in Corollary 3.4 may help to explain why the usual estimate of Spearman's ρ is larger than the observed value of Kendall's τ statistic in moderate to large samples from bivariate discrete populations. Such behaviour should be expected in situations where the parent distribution falls under the cover of Corollary 3.4.

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