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ON THE MONOTONICITY OF SOME NONPARAMETRIC DEPENDENCE MEASURES WITH RESPECT TO CONCORDANCE ORDERING

A. TAJAR, M. MESFIOUI and M. DENUIT

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Abdelouahid Tajar
Institut de Statistique
Université Catholique de Louvain
Voie du Roman Pays, 20
B-1348 Louvain-la-Neuve, Belgium
tajar@stat.ucl.ac.be

Mhamed Mesfioui
Département de mathématiques et d'informatique
Université du Québec à Trois-Rivières
C.P. 500 Trois-Rivières, Québec, Canada G9A 5H7
Mhamed_Mesfioui@UQTR.UQuebec.CA

Michel Denuit
Institut de Statistique
Université Catholique de Louvain
Voie du Roman Pays, 20
B-1348 Louvain-la-Neuve, Belgium
denuit@stat.ucl.ac.be

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Abstract

This paper explores the monotonicity with respect to concordance ordering of Kendall's τ and Spearman's ρ for counting random variables with an unbounded support. This generalizes some results of Schweizer and Wolff (1981) and Tchen (1980).

Key words and phrases: Concordance order, Kendall's τ, Spearman's ρ, Fréchet bounds.

1 Introduction

Intuitively, two random variables X and Y are concordant when large values of X tend to be associated with large values of Y. Several attempts have been made to formulate this concept precisely; see e.g. Tchen (1980) or Scarsini (1984).

The problem of comparing the strength of dependence has also attracted a lot of attention. Let (X_1, Y_1) and (X_2, Y_2) be two random couples with identical univariate marginals concentrating their mass on finitely many atoms; (X_2, Y_2) is said to be more concordant than (X_1, Y_1) if the distribution function H_2 of (X_2, Y_2) can be obtained from the distribution function H_1 of (X_1, Y_1) by a finite number of repairings which add mass ϵ at (x, y) and (x', y') while substracting mass ϵ at (x', y) and (x, y') where $x' > x$ and $y' > y$.

Tchen (1980) proved that (X_2, Y_2) is more concordant than (X_1, Y_1) if, and only if, the distribution function H_2 dominates H_1 everywhere. This yields the concept of concordance ordering. Specifically, this dependence ordering is defined as follows: given two random vectors (X_1, Y_1) and (X_2, Y_2) with identical marginals and respective distribution functions H_1 and H_2; (X_2, Y_2) is said to be more concordant than (X_1, Y_1), denoted as $(X_1, Y_1) \prec_c (X_2, Y_2)$, if $H_1(s, t) \leq H_2(s, t)$ holds for all $s, t \in \mathbb{R}$.

Since $(X_1, Y_1) \prec_c (X_2, Y_2)$ expresses the fact that (X_2, Y_2) is "more positively dependent" than (X_1, Y_1), we naturally expect that dependence measures (as Kendall's τ or Spearman ρ) will assume a larger value for (X_2, Y_2). This is indeed the case when F and G are continuous, and straightly results from the following representations of these dependence measures:

$$\tau_{H_i} = 4\mathbb{E}[H_i(X_i, Y_i)] - 1 \tag{1.1}$$

and

$$\rho_{H_i} = 12\mathbb{E}[H_i(X_i, Y_i)] - 3 \tag{1.2}$$

derived by Schweizer and Wolff (1981). The monotonicity of τ and ρ for continuous bivariate distributions has been discussed by Joe (1997, page 37). Tchen (1980) obtained a similar monotonicity property for τ and ρ when the supports of H_1 and H_2 consist in a finite number of atoms.

The aim of this short note is to establish the monotonicity of τ and ρ with respect to $\preceq_c$ for random couples valued in $\mathbb{N} \times \mathbb{N}$, with $\mathbb{N} = \{0, 1, 2, \ldots\}$. For this purpose, we will derive representations for τ and ρ that are of independent interest, in the vein of (1.1) and (1.2). In particular, these results complement the recent studies by Genest and Rivest (2001) and Nelsen et al. (2001) where probability integral transforms are investigated for random couples with continuous marginals. As a byproduct of our results, bounds for τ and ρ are easily obtained by means of Fréchet bounds. As already pointed out in the literature (e.g. by Tajar, Denuit and Lambert (2001)), when discrete marginals are involved, the largest (resp. smallest) value for τ and ρ is strictly smaller (resp. larger) than 1 (resp. -1).

2 Kendall's τ for bivariate counting distributions

In this section, we are going to investigate the monotonocity of Kendall's τ with respect to concordance ordering. Let us begin with some useful lemmatas.

Lemma 2.1. *Let (X,Y) be a random couple valued in $\mathbb{N}\times\mathbb{N}$ with joint cdf H, and marginals F and G. Then,*

$$\mathbb{E}[H(X,Y)] = \mathbb{E}[H(X-1,Y-1)] - \mathbb{E}[F(X-1)] - \mathbb{E}[G(Y-1)] + 1. \tag{2.1}$$

Proof. In virtue of Fubini's theorem, we have that,

$$\begin{aligned}
\mathbb{E}[H(X,Y)] &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} H(i,j)\Pr[X=i,Y=j]. \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{i}\sum_{k=0}^{j} \Pr[X=l,Y=k]\Pr[X=i,Y=j] \\
&= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\left\{\sum_{i=l}^{\infty}\sum_{j=k}^{\infty}\Pr[X=i,Y=j]\right\}\Pr[X=l,Y=k] \\
&= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\Pr[X\geq l,Y\geq k]\Pr[X=l,Y=k]. \qquad (2.2)
\end{aligned}$$

Now, since

$$\Pr[X\geq l, Y\geq k] = 1 - F(l-1) - G(k-1) + H(l-1,k-1) \tag{2.3}$$

we get (2.1) from (2.3) and (2.2). □

Lemma 2.2. *Let (X,Y) be a random couple valued in $\mathbb{N}\times\mathbb{N}$ with joint cdf H, and marginals F and G. Then,*

$$\begin{aligned}
\sum_{i,j\in\mathbb{N}}\left\{\Pr[X=i,Y=j]\right\}^2 &= \mathbb{E}[H(X,Y)] - \mathbb{E}[H(X-1,Y)] \\
&\quad -\mathbb{E}[H(X,Y-1)] + \mathbb{E}[H(X-1,Y-1)].
\end{aligned}$$

Proof. It suffices to write

$$\begin{aligned}
&\sum_{i,j\in\mathbb{N}}\left\{\Pr[X=i,Y=j]\right\}^2 \\
=\ &\sum_{i,j\in\mathbb{N}}\Pr[X=i,Y=j]\Pr[X=i,Y=j] \\
=\ &\sum_{i,j\in\mathbb{N}}\Big\{\Pr[X\leq i,Y\leq j] - \Pr[X\leq i-1,Y\leq j] \\
&\quad - \Pr[X\leq i,Y\leq j-1] + \Pr[X\leq i-1,Y\leq i-1]\Big\}\Pr[X=i,Y=j] \\
=\ &\mathbb{E}[H(X,Y)] - \mathbb{E}[H(X-1,Y)] - \mathbb{E}[H(X,Y-1)] \\
&\quad +\mathbb{E}[H(X-1,Y-1)],
\end{aligned}$$

which ends the proof. □

We are now in a position to give an expression for Kendall's τ similar to (1.1).

Proposition 2.3. *Let (X, Y) be a random couple valued in $\mathbb{N} \times \mathbb{N}$ with joint cdf H, and marginals F and G. Then,*

$$\begin{aligned} \tau_H &= 2\mathbb{E}[H(X-1, Y-1)] + \mathbb{E}[H(X-1, Y)] + \mathbb{E}[H(X, Y-1)] \\ &\qquad + \mathbb{E}[F(X)] + \mathbb{E}[G(Y)] - 2. \end{aligned} \tag{2.4}$$

Proof. Let us start from the very definition of Kendall's τ: given an independent copy (X', Y') of (X, Y), τ_H is given by

$$\tau_H = \Pr[(X - X')(Y - Y') > 0] - \Pr[(X - X')(Y - Y') < 0].$$

It is easily seen that τ_H can be cast into

$$\tau_H = 4 \Pr[X > X', Y > Y'] + \Pr[X = X' \cup Y = Y'] - 1.$$

Now, it's clear that

$$\begin{aligned} \Pr[X > X', Y > Y'] &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \Pr[X < i, Y < j] \Pr[X = i, Y = j] \\ &= \mathbb{E}[H(X-1, Y-1)] \end{aligned}$$

and

$$\begin{aligned} \Pr[X = X' \cup Y = Y'] &= \sum_{i=0}^{\infty} \{\Pr[X = i]\}^2 + \sum_{i=0}^{\infty} \{\Pr[Y = i]\}^2 \\ &\quad - \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \{\Pr[X = i, Y = j]\}^2 \\ &= \mathbb{E}[F(X) - F(X-1)] + \mathbb{E}[G(Y) - G(Y-1)] \\ &\quad - \mathbb{E}[H(X, Y)] + \mathbb{E}[H(X-1, Y)] \\ &\quad + \mathbb{E}[H(X, Y-1)] - \mathbb{E}[H(X-1, Y-1)] \end{aligned}$$

where the last equality is obtained from Lemma 2.2. It follows then that

$$\begin{aligned} \tau &= 4\mathbb{E}[H(X-1, Y-1)] - \mathbb{E}[H(X, Y)] + \mathbb{E}[H(X-1, Y)] \\ &\quad + \mathbb{E}[H(X, Y-1)] - \mathbb{E}[H(X-1, Y-1)] \\ &\quad + \mathbb{E}[F(X) - F(X-1)] + \mathbb{E}[G(Y) - G(Y-1)] - 1. \end{aligned}$$

Lemma 2.1 then yields the announced result. □

Let us now establish the monotonicity of Kendall's τ.

Proposition 2.4. *Let (X_1, Y_1) and (X_2, Y_2) be two random couples valued in $\mathbb{N} \times \mathbb{N}$ with respective distribution function H_1 and H_2, and identical marginals F and G. Then,*

$$(X_1, Y_1) \preceq_c (X_2, Y_2) \Rightarrow \tau_{H_1} \leq \tau_{H_2}.$$

Proof. Let us recall the following result (see e.g. Müller and Scarsini (2000), Theorem 2.5): $(X_1, Y_1) \preceq_c (X_2, Y_2)$ if, and only if,

$$\mathbb{E}[\phi(X_1, Y_1)] \leq \mathbb{E}[\phi(X_2, Y_2)] \text{ for all quasi-monotone functions } \phi$$

provided the expectations exist. Since $H_1 \leq H_2$ and using the fact that the functions $H_1(i-1, j-1)$, $H_1(i, j-1)$ and $H_1(i-1, j)$ are quasi-monotone, because H_1 is quasi monotone, we have

$$\begin{aligned}
&\mathbb{E}[H_1(X_1 - 1, Y_1 - 1)] \leq \mathbb{E}[H_1(X_2 - 1, Y_2 - 1)] \leq \mathbb{E}[H_2(X_2 - 1, Y_2 - 1)], \\
&\mathbb{E}[H_1(X_1, Y_1 - 1)] \leq \mathbb{E}[H_1(X_2, Y_2 - 1)] \leq \mathbb{E}[H_2(X_2, Y_2 - 1)], \\
&\mathbb{E}[H_1(X_1 - 1, Y_1)] \leq \mathbb{E}_{H_2}[H_1(X_2 - 1, Y_2)] \leq \mathbb{E}[H_2(X_2 - 1, Y_2 - 1)]
\end{aligned}$$

which imply from Proposition 2.3 that $\tau_{H_1} \leq \tau_{H_2}$. □

It is well-known that given any bivariate distribution function H with univariate margins F and G, the inequalities

$$\max[0, F(x) + G(y) - 1] \leq H(x, y) \leq \min[F(x), G(y)] \tag{2.5}$$

are valid whatever x and y. Considering Proposition 2.4, this yields bounds on the value of τ for bivariate counting distributions once the margins F and G are given.

3 Spearman's ρ for bivariate counting distributions

The following result gives an explicit expression of Spearman's ρ in the situation of ordinal discrete random variables.

Proposition 3.1. *Let (X, Y) be valued in $\mathbb{N} \times \mathbb{N}$ with distribution function H, and margins F and G, and let $(X^\perp, Y^\perp)$ be a random couple with joint distribution function FG. Spearman's ρ_H is then given by*

$$\begin{aligned}
\rho_H \;=\; 3\Big\{&\mathbb{E}[H(X^\perp, Y^\perp)] + \mathbb{E}[H(X^\perp - 1, Y^\perp)] \\
&+\mathbb{E}[H(X^\perp, Y^\perp - 1)] + \mathbb{E}[H(X^\perp - 1, Y^\perp - 1)] - 1\Big\}.
\end{aligned} \tag{3.1}$$

Proof. Let (X', Y') and (X'', Y'') be independent copies of (X, Y). Spearman's ρ is defined as

$$\rho_H = 3\Big\{ \Pr[(X - X')(Y - Y'') > 0] - \Pr[(X - X')(Y - Y') < 0] \Big\}$$

which can be cast into

$$\begin{aligned}
\rho_H \;=\; 3\{ &\Pr[X < X', Y < Y''] + \Pr[X \leq X', Y \leq Y''] \\
&- \Pr[X < X', Y \leq Y''] - \Pr[X \leq X', Y < Y''] \\
&+ \Pr[X < X'] - \Pr[X > X'] + \Pr[Y < Y''] - \Pr[Y > Y''] - 1\}.
\end{aligned} \tag{3.2}$$

Since the random vectors $(X,Y),(X',Y')$ and (X'',Y'') are independent, we have

$$\begin{aligned}\Pr[X<X',Y<Y''] &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} H(i-1,j-1)\Pr[X'=i]\Pr[Y''=j] \\ &= \mathbb{E}[H(X^{\perp}-1,Y^{\perp}-1)]. \qquad (3.3)\end{aligned}$$

Similarly,

$$\begin{aligned}\Pr[X\le X',Y\le Y''] &= \mathbb{E}[H(X^{\perp},Y^{\perp})], \qquad (3.4)\\ \Pr[X< X',Y\le Y''] &= \mathbb{E}[H(X^{\perp}-1,Y^{\perp})], \qquad (3.5)\\ \Pr[X\le X',Y< Y''] &= \mathbb{E}[H(X^{\perp},Y^{\perp}-1)] \qquad (3.6)\end{aligned}$$

and

$$\Pr[X<X']=\Pr[X>X'] \quad \text{and} \quad \Pr[Y<Y'']=\Pr[Y>Y'']. \qquad (3.7)$$

Finally the result is obtained by inserting (3.3), (3.4), (3.5), (3.6) and (3.7) in (3.2). □

Now, we point out the following result concerning the monotonicity of Spearman's ρ. The proof is a direct consequence of the Proposition 3.1.

Proposition 3.2. *Let (X_1,Y_1) and (X_2,Y_2) be two random couples valued in $\mathbb{N}\times\mathbb{N}$ with respective distribution function H_1 and H_2, and identical marginals F and G. Then,*

$$(X_1,Y_1)\preceq_c (X_2,Y_2) \Rightarrow \rho_{H_1}\le\rho_{H_2}.$$

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