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ON THE PRICING OF TOP & DROP EXCESS OF LOSS COVERS

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On the Pricing of Top & Drop Excess of Loss Covers

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Abstract

A Top & Drop cover is a reinsurance cover that may be found on the retrocession world. It offers some capacity that may be used either for a top layer or for a working layer. In the latter case one speaks of a drop. Within the collective risk model we show in this paper how to use the multivariate version of the Panjer's algorithm in order to price this cover. We also compare the exact price with the prices obtained with the Fréchet bounds, and with the price obtained with the incorrectly assumed hypothesis of independence. Generalization of the results in dimension higher than 2 is also provided.

Keywords

Multivariate Panjer's algorithm, excess of loss pricing, dependence, correlation order, stop-loss order, comonotonic risks, Fréchet bounds, supermodular order.

1 Introduction

In classical risk theory, it is generally assumed that all random variables are mutually independent. Often this is quite plausible and it has the great advantage to let the mathematics easy.

Recently the notion of dependence in actuarial science has been studied by many authors. Some techniques have been proposed to relax the notion of dependence.
On the one hand, some authors have tackled the problem with the aim of producing (upper and lower) bounds on the quantities of interest when some stochastic dependence is present. See the review paper by Dhaene et al. (2001) and the references therein.
On the other hand, some authors have concentrated on exact models with possible stochastic dependence. These models are classical extensions of the univariate collective risk model. We distinguish between two kinds of models :

a) Assume independent and identically distributed (iid in short) realizations of the random vector $(X_1, X_2, \ldots, X_n)$ independent of the counting random variable N. Then it may be interesting to obtain the distribution of

$$(S_1, \ldots, S_n) = (\sum_{i=1}^{N} X_{1i}, \ldots, \sum_{i=1}^{N} X_{ni}).$$

In the framework of the Panjer's algorithm, Sundt (1999) published a multivariate extension of the latter, allowing for practical calculations within this multivariate collective risk model.

b) Assume iid realizations of the random variable X and iid realizations of the random variable Y. Assume mutual independence between X, Y, and the counting variables N and M. N and M may be dependent. Then it may be interesting to obtain the distribution of

$$(S, T) = (\sum_{i=1}^{N} X_i, \sum_{i=1}^{M} Y_i).$$

Some types of dependence between N and M may be modelled, e.g. by using the trivariate reduction method, or mixing the bivariate Poisson distribution. Walhin and Paris (2000b) and Walhin and Paris (2000c) provide some sophisticated bivariate counting models allowing for these calculations. Multivariate extensions are also possible (see Ambagaspitya (1998) for multivariate models).

In this paper we will work with a known dependence structure within model a). This will allow us to make the exact calculations, the calculations with the uncorrectly assumed hypothesis of independence between the random variables $S_1, \ldots, S_n$ and we will be able to compare the results with some upper and lower bounds.
It is remarkable that reinsurance introduces artificially some dependencies within some of its treaties. Think e.g. of the ECOMOR-type treaties where order statistics are necessary. Recently Walhin (2001) described a very particular excess of loss treaty where some layers inure to the benefit of other layers. The pricing of this treaty necessitated the use of the multivariate Panjer's algorithm. In Walhin and Paris (2000a) the multivariate version of the Panjer's algorithm is used to study the retained risk of the Cedent when it buys excess of loss reinsurance with paid reinstatements.
In the present paper we propose a new application of the multivariate extension of the Panjer's algorithm to price the so-called "Top & Drop" cover. This reinsurance treaty, that

would be found primarily on the retrocession world, covers both a top layer and a working layer. One immediately feels the presence of stochastic dependence in the model as large claims (hitting the top layer) necessarily hit the working layer.
The rest of the paper will be organized as follows. Section 2 describes two types of Top & Drop covers within a relatively general collective risk model. Section 3 recalls the multivariate Panjer's algorithm. Section 4 recalls and applies some results related to the correlation order, the stop-loss order, and comonotonicity. Section 5 provides the numerical results and compares them with the case where independence would be uncorrectly assumed and with the corresponding Fréchet bounds. Section 6 provides a generalization of some theoretical results in dimension higher than 2. Section 7 concludes.

2 Top & Drop Cover

Recently the first author has been given the opportunity to look at the following excess of loss cover which we call example 1 :

200 XS 800
OR
200 XS 200 in the aggregate for each loss exceeding 20 (losses to be aggregated from groud up (fgu in short) but with a maximum of 100 each and every loss occurrence).
No reinstatement, that is, the maximal annual amount to be paid by the reinsurer is 200.
The aim of this treaty is to cover a top layer (200 XS 800) which has a small chance to be hit and, simultaneously, a potential high frequency of small claims.
We also suggest to study another example of Top and Drop cover, let us say example 2 :

200 XS 800
OR
200 XS 200
with a global annual aggregate deductible of 400 and unlimited free reinstatements.
The aim of the treaty is clearly to cover an extra reinstatement on the low layer (which typically would be protected by a classical 200 XS 200 with one reinstatement) and/or a top layer.

Let us propose an actuarial model for this treaty.
Let X be the random variable representing the large claims. Let N be the annual number of large claims. Let us assume that N is Poisson distributed with mean λ and X is limited Pareto distributed with parameters A_l, B_l and α_l ($X \sim Pa(A_l, B_l, \alpha_l)$) :

$$\begin{aligned} F_X(x) &= 0 \quad \text{if } x < A_l \\ &= \frac{A_l^{-\alpha_l} - x^{-\alpha_l}}{A_l^{-\alpha_l} - B_l^{-\alpha_l}} \quad \text{if } A_l \leq x < B_l \\ &= 1 \quad \text{if } x \geq B_l, \end{aligned}$$

where $F_X(x) = \mathbb{P}[X \leq x]$ denotes the cumulative density function of X.
Let Y be the random variable representing the small claims. Let M be the annual number

of small claims. Let us assume that M is Poisson distributed with mean λ_s and Y is limited Pareto distributed with parameters A_s, B_s and α_s. If one assumes only two types of claims (large and small) then $B_s = A_l$.

Generally one assumes one type of claims in the collective risk model. Here we propose to assume two types of claims (the extension to more types of claims is trivial) because the comportment of small claims may be very different than the comportment of large claims. Therefore we prefer to work in a generalised model.

We will assume mutual independence between the random variables N, M, X and Y.

In mathematical terms, the cover offered by the reinsurer in example 1 reads :

$$\begin{aligned}
X_i^{Re-top} &= \min(200, \max(0, X_i - 800)), \\
X_i^{Re-drop} &= \min(100, X_i I_{X_i \geq 20}), \\
Y_i^{Re-drop} &= \min(100, Y_i I_{Y_i \geq 20}), \\
S &= X_1^{Re-top} + \cdots + X_N^{Re-top}, \\
T &= X_1^{Re-drop} + \cdots + X_N^{Re-drop}, \\
U &= Y_1^{Re-drop} + \cdots + Y_M^{Re-drop}, \\
\text{Cover} &= \min(200, S + \max(0, T + U - 200)).
\end{aligned}$$

where I_A denotes the indicator function : $I_A = 1$ if A is true, $I_A = 0$ if A is false.

The cover offered by the reinsurer in example 2 reads :

$$\begin{aligned}
X_i^{Re-top} &= \min(200, \max(0, X_i - 800)), \\
X_i^{Re-drop} &= \min(200, \max(0, X_i - 200)), \\
Y_i^{Re-drop} &= \min(200, \max(0, Y_i - 200)), \\
S &= X_1^{Re-top} + \cdots + X_N^{Re-top}, \\
T &= X_1^{Re-drop} + \cdots + X_N^{Re-drop}, \\
U &= Y_1^{Re-drop} + \cdots + Y_M^{Re-drop}, \\
\text{Cover} &= \max(0, S + T + U - 400).
\end{aligned}$$

Clearly, in both examples, S and T are correlated. Indeed we have

- S and T are random sums of non-negative random variables with identical number of terms.
- the summands, X_i^{Re-top} and $X_i^{Re-drop}$ are themselves correlated.

This means that even the expectation of the Cover, $\mathbb{E}Cover$ needs the joint distribution of (S, T). As explained in the introduction, it is possible to obtain this joint distribution with the multivariate version of the Panjer's algorithm. Section 3 gives some details.

3 Multivariate Panjer's algorithm

As we want to apply Panjer's type algorithms, it is needed to work with lattice distributions. Therefore we first need to obtain a discretization of the distributions of X and Y. The local one moment matching method (see Gerber (1982)) is a good choice in the sense that it conserves the first moment and is stop-loss conservative. Furthermore, in the case of the limited Pareto distribution ($X \sim Pa(A, B, \alpha)$), it is not difficult to obtain closed-form of the corresponding lattice distribution. Let us choose a span h. It is not difficult to show that the probabilities of the lattice version of X, which we denote by X_{dis}, are given by :

$$
\begin{aligned}
f_{X_{dis}}(A) &= 1 - \frac{\frac{(A+h)^{1-\alpha}}{1-\alpha} - \frac{A^{1-\alpha}}{1-\alpha} - B^{-\alpha}h}{h(A^{-\alpha} - B^{-\alpha})}, \\
f_{X_{dis}}(A+jh) &= \frac{2(A+jh)^{1-\alpha} - (A+(j-1)h)^{1-\alpha} - (A+(j+1)h)^{1-\alpha}}{h(1-\alpha)(A^{-\alpha} - B^{-\alpha})} \quad , \quad j = 1, \ldots, \frac{B-A}{h} - 1, \\
f_{X_{dis}}(B) &= 1 - f_{X_{dis}}(A) - f_{X_{dis}}(A+h) - \cdots - f_{X_{dis}}(B-h).
\end{aligned}
$$

Now let us assume that N belongs to the Panjer's class of counting distributions (see Panjer (1981)), i.e. N has the following property :

$$
\frac{\mathbb{P}[N=n]}{\mathbb{P}[N=n-1]} = a + \frac{b}{n} \quad , \quad n \geq 1.
$$

Let us recall that the Panjer's class of counting distributions contains the following members : Poisson ($a = 0$), Negative Binomial ($a > 0$) and Binomial ($a < 0$). We are interested in the joint distribution of (S, T) :

$$
(S, T) = (\sum_{i=1}^{N} X_i^{Re-top}, \sum_{i=1}^{N} X_i^{Re-drop}),
$$

where $(X_i^{Re-top}, X_i^{Re-drop})$ are iid realizations of the lattice type random vector $(X^{Re-top}, X^{Re-drop})$.
Sundt (1999) published the multivariate version of the Panjer's algorithm. We here give the bivariate version :

$$
\begin{aligned}
f_{S,T}(0,0) &= \Psi_N(f_{X^{Re-top},X^{Re-drop}}(0,0)), \\
f_{S,T}(s,t) &= \sum_{x,y}^{s,t} \frac{as+bx}{s(1-af_{X^{Re-top},X^{Re-drop}}(0,0))} f_{S,T}(s-x,t-y) f_{X^{Re-top},Y^{Re-drop}}(x,y) \quad , \quad s \geq 1, \\
f_{S,T}(s,t) &= \sum_{x,y}^{s,t} \frac{at+by}{t(1-af_{X^{Re-top},X^{Re-drop}}(0,0))} f_{S,T}(s-x,t-y) f_{X^{Re-top},X^{Re-drop}}(x,y) \quad , \quad t \geq 1,
\end{aligned}
$$

where we use the following notation :

$$
\sum_{x,y}^{s,t} g(x,y) = \sum_{x=0}^{s} \sum_{y=0}^{t} g(x,y) - g(0,0),
$$

and $\Psi_N(u)$ denotes the probability generating function of N :

$$\Psi_N(u) = \mathbb{E}[u^N].$$

In our case the recursions are easier beacuse $a = 0$ in the Poisson case.
As explained in Walhin (2001), it may be interesting to benefit of the particular structure of the random vector $(X^{Re-top}, X^{Re-drop})$. In particular, if only a few points have a non-negative probability, it may be interesting, from a computing-time point of view, to organize the calculations in function of these points with a strictly positive mass.
With this tool we are now able to obtain the joint distribution of (S, T).

4 Comonotonicity, correlation order and stop-loss order

In this section we will use extensively the seminal paper of Dhaene and Goovaerts (1996) on dependency of risks applied in actuarial science.

Definition 4.1. Let $R(F_1, F_2)$ be a bivariate Fréchet space, i.e. the class of all bivariate distributed random variables with given marginals F_1 and F_2. For any $(X_1, X_2) \in R(F_1, F_2)$, we have

$$\begin{aligned} F_1(x) &= \mathbb{P}[X_1 \leq x], \\ F_2(x) &= \mathbb{P}[X_2 \leq x], \\ F_{X_1,X_2}(x_1, x_2) &= \mathbb{P}[X_1 \leq x_1, X_2 \leq x_2]. \end{aligned}$$

Definition 4.2. Let (X_1, X_2) and (Y_1, Y_2) be elements of $R(F_1, F_2)$. Then we say that (X_1, X_2) is less correlated than (Y_1, Y_2), written $(X_1, X_2) \leq_c (Y_1, Y_2)$, if

$$\mathbb{C}ov(f(X_1), g(X_2)) \leq \mathbb{C}ov(f(Y_1), g(Y_2)),$$

for all non-decreasing functions f and g for which the covariances exist.

Dhaene and Goovaerts (1996) proved the following characterization theorem :

Theorem 4.3. *Let (X_1, X_2) and (Y_1, Y_2) be elements of $R(F_1, F_2)$. Then the following statements are equivalent :*

- $(X_1, X_2) \leq_c (Y_1, Y_2)$
- $F_{X_1,X_2}(x_1, x_2) \leq F_{Y_1,Y_2}(x_1, x_2) \quad , \quad \forall x_1, x_2 \geq 0.$

We will need the following lemma before proving the next theorems :

Lemma 4.4. *Let (U_1, U_2) and (V_1, V_2) be elements of $R(F_1, F_2)$. Let (R_1, R_2) be a random vector independent of (U_1, U_2) and (V_1, V_2). Then we have that*

$$(U_1, U_2) \leq_c (V_1, V_2)$$

implies

$$(U_1 + R_1, U_2 + R_2) \leq_c (V_1 + R_1, V_2 + R_2).$$

Proof
Using Theorem 4.3, we just have to prove that

$$F_{U_1+R_1,U_2+R_2}(x_1,x_2) \leq F_{V_1+R_1,V_2+R_2}(x_1,x_2) \quad , \quad \forall x_1,x_2 \geq 0.$$

This result is immediate if one considers the convolution formula :

$$\begin{aligned}
F_{U_1+R_1,U_2+R_2}(x_1,x_2) &= \int_{z_1=0}^{x_1}\int_{z_2=0}^{x_2} F_{U_1,U_2}(x_1-z_1,x_2-z_2)dF_{R_1,R_2}(z_1,z_2) \\
&\leq \int_{z_1=0}^{x_1}\int_{z_2=0}^{x_2} F_{V_1,V_2}(x_1-z_1,x_2-z)dF_{R_1,R_2}(z_1,z_2) \\
&= F_{V_1+R_1,V_2+R_2}(x_1,x_2).
\end{aligned}$$

■

Theorem 4.5. *Let (W_i,X_i) and (Y_i,Z_i) be independent elements of $R(F_1,F_2)$ such that*

$$(W_i,X_i) \leq_c (Y_i,Z_i) \quad \forall i=1,2,\ldots,n.$$

Then we have that

$$(W_1+\cdots+W_n, X_1+\cdots+X_n) \leq_c (Y_1+\cdots+Y_n, Z_1+\cdots+Z_n).$$

Proof
By hypothesis, we have

$$(W_1,X_1) \leq_c (Y_1,Z_1).$$

Let us assume that the result is true for n. Let us show that it is true for $n+1$.
Let us apply Lemma 4.4 with

$$\begin{aligned}
(U_1,U_2) &= (W_1+\cdots+W_n, X_1+\cdots+X_n), \\
(V_1,V_2) &= (Y_1+\cdots+Y_n, Z_1+\cdots+Z_n), \\
(R_1,R_2) &= (W_{n+1},X_{n+1}).
\end{aligned}$$

Then we have

$$(W_1+\cdots+W_{n+1}, X_1+\cdots+X_{n+1}) \leq_c (Y_1+\cdots+Y_n+W_{n+1}, Z_1+\cdots+Z_n+W_{n+1}).$$

Let us apply Theorem 4.5 with

$$\begin{aligned}
(X_1,X_2) &= (W_{n+1},X_{n+1}), \\
(Y_1,Y_2) &= (Y_{n+1},Z_{n+1}), \\
(Z_1,Z_2) &= (Y_1+\cdots+Y_n, Z_1+\cdots+Z_n).
\end{aligned}$$

Then we have

$$(W_{n+1}+Y_1\cdots+Y_n, X_{n+1}+Z_1\cdots+Z_n) \leq_c (Y_1+\cdots+Y_{n+1}, Z_1+\cdots+Z_{n+1}).$$

By transitivity, we then have

$$(W_1+\cdots+W_{n+1}, X_1+\cdots+X_{n+1}) \leq_c (Y_1+\cdots+Y_{n+1}, Z_1+\cdots+Z_{n+1}),$$

which is the annouced result. ■

Theorem 4.6. *Let (W_i, X_i) and (Y_i, Z_i) be iid realizations of (W, X) and (Y, Z). Let us assume that (W, X) and (Y, Z) are elements of $R(F_1, F_2)$. Let N be a random variable independent of the realizations of (W, X) and (Y, Z) with $(W, X) <_c (Y, Z)$. Let us define*

$$\begin{aligned} (S_W, S_X) &= (\sum_{i=1}^{N} W_i, \sum_{i=1}^{N} X_i), \\ (S_Y, S_Z) &= (\sum_{i=1}^{N} Y_i, \sum_{i=1}^{N} Z_i). \end{aligned}$$

Then we have that

$$(S_W, S_X) \leq_c (S_Y, S_Z).$$

Proof
Using Theorem 4.3, we just have to prove that

$$F_{S_W,S_X}(x_1, x_2) \leq F_{S_Y,S_Z}(x_1, x_2) \quad , \quad \forall x_1, x_2 \geq 0.$$

or equivalently

$$\sum_{n=0}^{\infty} \mathbb{P}[N = n] F_{W,X}^{*n}(x_1, x_2) \leq \sum_{n=0}^{\infty} \mathbb{P}[N = n] F_{Y,Z}^{*n}(x_1, x_2) \quad , \quad \forall x_1, x_2 \geq 0.$$

where $F_{W,X}^{*n}$ denotes the distribution the n-fold convolution of the random vector (W, X) :

$$(W_1 + \cdots + W_n, X_1 + \cdots + X_n).$$

Using Theorem 4.5 in conjuction with Theorem 4.3, we immediately have the announced result. ■

We will need the following definition of positively quadrant dependency (see Dhaene and Goovaerts (1996) for the definition and the proof of the characterization).

Definition 4.7. Let (X_1, X_2) be an element of $R(F_1, F_2)$. Let $(X_1^{\perp}, X_2^{\perp})$ be the independent version of (X_1, X_2), i.e. $(X_1^{\perp}, X_2^{\perp}) \in R(F_1, F_2)$ and $X_1^{\perp}$ is independent of $X_2^{\perp}$. One says that (X_1, X_2) is positively quadrant dependent (PDQ in short) if one of the following equivalent statements holds :

- $F_{X_1^{\perp}, X_2^{\perp}}(x_1, x_2) \leq F_{X_1,X_2}(x_1, x_2) \quad , \quad x_1, x_2 \geq 0.$
- $F_{X_1}(x_1) F_{X_2}(x_2) \leq F_{X_1,X_2}(x_1, x_2) \quad , \quad x_1, x_2 \geq 0.$
- $(X_1^{\perp}, X_2^{\perp}) \leq_c (X_1, X_2).$
- $\mathbb{C}ov(f(X_1), g(X_2)) \geq 0 \quad , \quad \forall$ non-decreasing functions f and g.

The following result will also prove to be useful :

Theorem 4.8. *Let X_i and Y_i be iid realizations of X and Y. Let us assume that X, Y and N are mutually independent. Let us further assume that X and Y are non-negative. Let*

$$\begin{aligned} S &= X_1 + \cdots + X_N, \\ T &= Y_1 + \cdots + Y_N. \end{aligned}$$

Let $S^{\perp}$ and $T^{\perp}$ be the independent versions of S and T, i.e. $S^{\perp}$ and $T^{\perp}$ are independent and have the same distributions as S and T respectively.
Then we have that

$$(S^{\perp}, T^{\perp}) \leq_c (S, T).$$

Proof
Using Definition 4.7, we have to prove that (S, T) is PDQ. We will prove the covariance relation.
Let f and g be non-decreasing functions.
By the decomposition formula of the covariance, we have

$$\mathbb{C}ov(f(S), g(T)) = \mathbb{E}(\mathbb{C}ov(f(S), g(T)|N)) + \mathbb{C}ov(\mathbb{E}(f(S)|N), \mathbb{E}(g(T)|N)).$$

The first term of the right hand side of the equality vanishes because the covariance between independent random variables is 0. For the second term, it is clear that the expectations are increasing functions of N and therefore the second term may be rewritten

$$\mathbb{C}ov(u(N), v(N)),$$

where u and v are non-decreasing functions. This covariance is clearly non-negative, which closes the proof. ■

We will use the stop-loss order for ordering random variables in function of their risk.

Definition 4.9. Let X and Y be two non-negative random variables. We say that X is smaller in the stop-loss order than Y (written $X \leq_{sl} Y$) whenever we have one of the following equivalent statements

- $\mathbb{E}[\max(0, X - d)] \leq \mathbb{E}[\max(0, Y - d)] \quad , \quad \forall d \geq 0.$
- $\mathbb{E}u(X) \leq \mathbb{E}v(Y) \quad , \quad \forall$ increasing convex functions u, v.

Dhaene and Goovaerts (1996) proved the following dependency result :

Theorem 4.10. *Let (X_1, X_2) and (Y_1, Y_2) be two elements of $R(F_1, F_2)$. Then*

$$(X_1, X_2) \leq_c (Y_1, Y_2)$$

implies

$$X_1 + X_2 \leq_{sl} Y_1 + Y_2.$$

We will now prove an extension of Theorem 4.10 which will be necessary to obtain bounds for the numerical example 1.

Theorem 4.11. *Let* (X_1, X_2) *and* (Y_1, Y_2) *be two elements of* $R(F_1, F_2)$*. Then*

$$(X_1, X_2) \leq_c (Y_1, Y_2)$$

implies

$$\max(0, X_1 - a) + \max(0, X_2 - b) \leq_{sl} \max(0, Y_1 - a) + \max(0, Y_2 - b) \quad , \quad \forall a, b \geq 0.$$

Proof
The proof follows the lines of Dhaene and Goovaerts (1996). We use the notation

$$(X - d)_+ = \max(0, X - d).$$

We have that

$$\mathbb{E}((X_1-a)_+ + (X_2-b)_+ - d)_+ = \mathbb{E}(X_1-a)_+ + \mathbb{E}(X_2-b)_+ - d + \mathbb{E}(d-(X_1-a)_+ - (X_2-b)_+)_+.$$

For non-negative real numbers x_1 and x_2 the following equality holds

$$(d - x_1 - x_2)_+ = \int_0^d I(x_1 \leq a + x, x_2 \leq b + x - d)dx,$$

so that

$$\begin{aligned} E(d - X_1 - X_2)_+ &= \int_0^d \mathbb{E}(I(X_1 \leq a + x, X_2 \leq b + x - d)dx \\ &= \int_0^d F_{X_1,X_2}(a + x, b + x - d)dx. \end{aligned}$$

Using Theorem 4.3 in connection with the above equality and the fact that $\mathbb{E}(X_1 - a)_+ = \mathbb{E}(Y_1 - a)_+$ and $\mathbb{E}(X_2 - a)_+ = \mathbb{E}(Y_2 - a)_+$, we have the announced result. ∎

In the particular case $a = 0 = b$ we obtain Theorem 4.10.

We will need the notion of comonotonic random variables (see e.g. Wang and Dhaene (1998)).

Definition 4.12. Two risks X and Y are said to be comonotonic

- if their bivariate cumulative density function satisfies

$$F_{X,Y}(x, y) = \min(F_X(x), F_Y(y)) \quad , \quad \forall x, y \geq 0,$$

  or, equivalently,

- if there exists a random variable Z and non-decreasing functions u and v on $\mathbb{R}$ such that

$$(X, Y) \text{ is distributed as } (u(Z), v(Z)).$$

An interesting result, related to the concept of comonotonicity is the following theorem, due to Fréchet (1951) and Hoeffding (1940) :

Theorem 4.13. *Let* (X_1, X_2) *be an element of* $R(F_1, F_2)$*. Then we have*

$$\max[F_1(x_1) + F_2(x_2) - 1; 0] \leq F_{X_1,X_2}(x_1, x_2) \leq \min[F_1(x_1); F_2(x_2)].$$

We will be interested in calculating premiums with the PH-transform premium principle, introduced by Wang (1996).

Definition 4.14. Let X be a risk. The premium, according to the PH-transform premium principle is

$$\text{Premium} = \Pi_\rho(X) = \int_0^\infty (1 - F_X(x))^\rho dx,$$

where $0 \leq \rho \leq 1$.

In particular when $\rho = 0$ we obtain the pure premium.
The following result, which is due to Wang (1996) will be useful.

Theorem 4.15. *Let* X *and* Y *be two risks ordered in the stop-loss order :* $X \leq_{sl} Y$*. Then we have*

$$\Pi_\rho(X) \leq \Pi_\rho(Y).$$

5 Numerical results

We will work with the following particular data :

	λ	α	A	B
large	0.3	0.9	400	1000
small	2.5	1.4	20	400

Table 5.1: Poisson and Pareto parameters

As announced in section 3, all continuous random variables are discretized using the local one moment matching method. It will be tacitely assumed that all random variables in this section should be understood as being the discrete version of the original random variable. We will work with a discretisation step $h = 10$.

5.1 Numerical example 1

Some preliminary statistics may be interesting :

	$X^{Re-drop}$	X^{Re-top}	$Y^{Re-drop}$
$\mathbb{E}$	100	16.14	42.87
$\mathbb{V}ar$	0	1817	631.72

Table 5.2: Example 1 : summands

	$S^{Re-drop}$	T^{Re-top}	$U^{Re-drop}$
$\mathbb{E}$	30	4.84	107.17
$\mathbb{V}ar$	3000	623.4	6173.89

Table 5.3: Example 1 : aggregate claims

The correlation between S^{Re-top} and $T^{Re-drop}$ is 0.35.
Let us compute the pure premium for our cover :

$$\mathbb{E}Cover = \sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{u=0}^{\infty} f_{S,T}(s,t)f_U(u)\min(200, s+\max(0,t+u-200)) = 20.519.$$

Now we are able to obtain easy upper and lower bounds. Indeed, let

$$\begin{aligned} F_{S^{min},T^{min}} &= \max[F_S(x_1)+F_T(x_2)-1;0], \\ F_{S^{max},T^{max}} &= \min[F_S(x_1),F_T(x_2)]. \end{aligned}$$

Using Theorem 4.13 in connection with Theorem 4.3 we have that

$$(S^{min},T^{min}) \leq_c (S,T) \leq_c (S^{max},T^{max}).$$

Using Lemma 4.4, we have that

$$(S^{min},T^{min}+U) \leq_c (S,T+U) \leq_c (S^{max},T^{max}+U).$$

Using Theorem 4.11, we have that

$$\mathbb{E}[\max(0;S^{min}+\max(0;T^{min}+U-200)-200)]$$

$$\leq$$

$$\mathbb{E}[\max(0;S+\max(0;T+U-200)-200)]$$

$$\leq$$

$$\mathbb{E}[\max(0;S^{max}+\max(0;T^{max}+U-200)-200)],$$

which is equivalent to

$$\mathbb{E}[\min(200; S^{max} + \max(0; T^{max} + U - 200))]$$
$$\leq$$
$$\mathbb{E}[\min(200; S + \max(0; T + U - 200))]$$
$$\leq$$
$$\mathbb{E}[\min(200; S^{min} + \max(0; T^{min} + U - 200))].$$

The numerical bounds are

$$19.469 \leq 20.519 \leq 21.279.$$

Let us observe that it is possible to improve the upper bound. Let $X^{Re-top,\perp}$ and $X^{Re-drop,\perp}$ be the independent versions of X^{Re-top} and $X^{Re-drop}$, that is $X^{Re-top,\perp}$ and $X^{Re-drop,\perp}$ are independent and have the same distributions as X^{Re-top} and $X^{Re-drop}$, respectively.
Let us define

$$\begin{array}{rcl} S^{\perp} & = & X_1^{Re-top,\perp} + \cdots + X_N^{Re-top,\perp}, \\ T^{\perp} & = & X_1^{Re-drop,\perp} + \cdots + X_N^{Re-drop,\perp}. \end{array}$$

It is clear, from their construction, that X^{Re-top} and $X^{Re-drop}$ are comonotonic random variables. We then have that

$$\begin{array}{rcl} F_{X^{Re-top,\perp},X^{Re-drop,\perp}}(x_1,x_2) & = & F_{X^{Re-top,\perp}} F_{X^{Re-drop,\perp}}(x_1,x_2) \\ & \leq & \min(F_{X^{Re-top}}(x_1), F_{X^{Re-drop}}(x_2)) \\ & = & F_{X^{Re-top},X^{Re-drop}}(x_1,x_2) \quad , \quad \forall x_1, x_2 \geq 0. \end{array}$$

Therefore

$$(X^{Re-top,\perp}, X^{Re-drop,\perp}) \leq_c (X^{Re-top}, X^{Re-drop}).$$

Using theorem 4.6 we have that

$$(S^{\perp}, T^{\perp}) \leq_c (S, T).$$

However $S^{\perp}$ and $T^{\perp}$ are not independent and there is no interest to work with the random vector $(S^{\perp}, T^{\perp})$. Then, let us define the independent versions of $S^{\perp}$ and $T^{\perp}$: $S^{\perp\perp}$ and $T^{\perp\perp}$. Using Theorem 4.11 we have that

$$(S^{\perp\perp}, T^{\perp\perp}) \leq_c (S^{\perp}, T^{\perp}).$$

By transitivity we then obtain

$$(S^{\perp\perp}, T^{\perp\perp}) \leq_c (S, T).$$

Using Lemma 4.4 we have that

$$(S^{\perp\perp}, T^{\perp\perp} + U) \leq_c (S, T + U).$$

Using theorem 4.8 we have that

$$\mathbb{E}[\max(0; S^{\perp\perp} + \max(0; T^{\perp\perp} + U - 200) - 200)] \leq \mathbb{E}[\max(0; S + \max(0; T + U - 200) - 200)],$$

which is equivalent to

$$\mathbb{E}[\min(200; S + \max(0; T + U - 200))] \leq \mathbb{E}[\min(200; S^{\perp\perp} + \max(0; T^{\perp\perp} + U - 200))].$$

Numerically we have that

$$20.519 \leq 21.131.$$

Summarising the results we have

Fréchet lower bound	19.469
exact result	20.519
independent case	21.131
Fréchet upper bound	21.279

Table 5.4: Example 1 : pure premium

It is also interesting to analyse other moments of the cover, or premiums obtained by the PH-transform premium principle. We have

	exact	independent
$\mathbb{E}Cover$	20.519	21.131
$\mathbb{E}Cover^2$	265.04	261.32
$\mathbb{E}Cover^3$	4124.3	3915.8
$\mathbb{E}Cover^4$	70331	64760

Table 5.5: Example 1 : other moments

ρ	exact	independent
1	20.519	21.131
0.75	34.898	35.420
0.50	60.786	61.034
0.25	108.71	108.55

Table 5.6: Example 1 : PH-transform premium principle

We observe that, unfortunately, the other moments, as well as the premiums obtained with the PH-transform premium principle, are not ordered anymore.

5.2 Numerical example 2

Some preliminary statistics are :

	$X^{Re-drop}$	X^{Re-top}	$Y^{Re-drop}$
$\mathbb{E}$	200	16.14	1.83
$\mathbb{V}ar$	0	1817.63	206.31

Table 5.7: Example 2 : summands

	S^{Re-top}	$T^{Re-drop}$	$U^{Re-drop}$
$\mathbb{E}$	600	4.84	4.57
$\mathbb{V}ar$	12000	623.4	524.15

Table 5.8: Example 2 : aggregate claims

The correlation between S^{Re-top} and $T^{Re-drop}$ is 0.35.

Let us calculate the pure premium for our cover :

$$\mathbb{E}Cover = \sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{u=0}^{\infty} f_{S,T}(s,t) f_U(u) \max(0, s+t+u-400) = 2.252.$$

Now we are able to obtain easy upper and lower bounds. Indeed, let

$$\begin{aligned} F_{S^{min},T^{min}} &= \max[F_S(x_1) + F_T(x_2) - 1; 0], \\ F_{S^{max},T^{max}} &= \min[F_S(x_1), F_T(x_2)]. \end{aligned}$$

Using Theorem 4.13 in connection with Theorem 4.3 we have that

$$(S^{min}, T^{min}) \leq_c (S, T) \leq_c (S^{max}, T^{max}).$$

Using Lemma 4.4, we have that

$$(S^{min}, T^{min} + U) \leq_c (S, T + U) \leq_c (S^{max}, T^{max} + U).$$

Using Theorem 4.10, we have that

$$\mathbb{E}[\max(0; S^{min} + T^{min} + U - 400)]$$

$$\leq$$

$$\mathbb{E}[\max(0; S + T + U - 400)]$$

$$\leq$$

$$\mathbb{E}[\max(0; S^{max} + T^{max} + U - 400)].$$

These numerical bounds are

$$0.952 \leq 2.252 \leq 5.471.$$

We are now able to improve the lower bound, which is unfortunately less interesting than in the previous example. Let $X^{Re-top,\perp}$ and $X^{Re-drop,\perp}$ be the independent versions of X^{Re-top} and $X^{Re-drop}$, that is $X^{Re-top,\perp}$ and $X^{Re-drop,\perp}$ are independent and have the same distributions as X^{Re-top} and $X^{Re-drop}$, respectively.
Let us define

$$\begin{aligned} S^{\perp} &= X_1^{Re-top,\perp} + \cdots + X_N^{Re-top,\perp}, \\ T^{\perp} &= X_1^{Re-drop,\perp} + \cdots + X_N^{Re-drop,\perp}. \end{aligned}$$

It is clear, from their construction, that X^{Re-top} and $X^{Re-drop}$ are comonotonic random variables. We then have that

$$\begin{aligned} F_{X^{Re-top,\perp},X^{Re-drop,\perp}}(x_1,x_2) &= F_{X^{Re-top,\perp}}F_{X^{Re-drop,\perp}}(x_1,x_2) \\ &\leq \min(F_{X^{Re-top}}(x_1), F_{X^{Re-drop}}(x_2)) \\ &= F_{X^{Re-top},X^{Re-drop}}(x_1,x_2) \quad , \quad \forall x_1, x_2 \geq 0. \end{aligned}$$

Therefore

$$(X^{Re-top,\perp}, X^{Re-drop,\perp}) \leq_c (X^{Re-top}, X^{Re-drop}).$$

Using theorem 4.6 we have that

$$(S^{\perp}, T^{\perp}) \leq_c (S, T).$$

However $S^{\perp}$ and $T^{\perp}$ are not independent and there is no interest to work with this random vector. Then, let us define the independent versions of $S^{\perp}$ and $T^{\perp}$: $S^{\perp\perp}$ and $T^{\perp\perp}$. Using Theorem 4.8 we have that

$$(S^{\perp\perp}, T^{\perp\perp}) \leq_c (S^{\perp}, T^{\perp}).$$

By transitivity we then obtain

$$(S^{\perp\perp}, T^{\perp\perp}) \leq_c (S, T).$$

Using Lemma 4.4 we have that

$$(S^{\perp\perp}, T^{\perp\perp} + U) \leq_c (S, T + U).$$

Using Theorem 4.11 we have that

$$\mathbb{E}[\max(0; S^{\perp\perp} + T^{\perp\perp} + U - 400)] \leq \mathbb{E}[\max(0; S + \max(0; T + U - 400)].$$

Numerically we have that

$$1.153 \leq 2.252.$$

Summarising the results we have

Fréchet lower bound	0.952
independent case	1.153
exact result	2.252
Fréchet upper bound	5.471

Table 5.9: Example 2 : pure premium

Contrary to example 1, we are now able to compare other moments and premiums obtained by the PH-transform premium principle. Numerical results give

	exact	independent
$\mathbb{E}X$	2.252	1.152
$\mathbb{E}X^2$	486.9	242.3
$\mathbb{E}X^3$	140198	61874
$\mathbb{E}X^4$	51084848	19062223

Table 5.10: Example 2 : other moments

ρ	exact	independent
1	2.252	1.152
0.75	7.815	4.611
0.50	30.22	20.60
0.25	170.04	143.41

Table 5.11: Example 2 : PH-transform premium principle

In this case the characterization of the stop-loss order and theorem 8 are directly applicable because we have simply

$$Cover^{\perp} \leq_{sl} Cover.$$

Typically, a reinsurer will not offer an unlimited cover, at least for property business. Therefore the cover of example 2 should be limited in practice and could read

$$Cover = \min(400, \max(0, S + T + U - 400)),$$

which we denote example 2bis.

Pricing this realistic cover thus requires exact computations because we cannot show that the derived bounds remain valid. We obtain the following result

Fréchet lower bound	0.952
independent case	1.153
exact result	2.252
Fréchet upper bound	5.471

Table 5.12: Example 2 with limited cover

On this example, the bounds remain valid. This is probably the case because the probabiloity of exhausting the cover is really small.

6 Generalization

6.1 Extension of the cover to n layers

Let us now reexamine the problem in a very general setting. Let us assume that every large claim X_i is splitted in

$$\begin{cases} X_i^{Re-1} = \psi_1(X_i) \\ \vdots \\ X_i^{Re-n} = \psi_n(X_i) \end{cases}$$

where the functions $\psi_1, \ldots, \psi_n$ are non-decreasing. The cover offered by the reinsurer is then built as Cover=$\Psi(S_1, \ldots, S_n, U_1, \ldots, U_k)$ for some non-decreasing function Ψ, where

$$S_j = \sum_{i=1}^{N} X_i^{Re-j} = \sum_{i=1}^{N} \psi_j(X_i)$$

and the U_j's are independent of the S_j's (typically, the U_j's depend on the small claims). Henceforth, we will impose a further condition on Ψ, namely supermodularity.

Definition 6.1. A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is said to be supermodular if for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ it satisfies

$$\phi(\boldsymbol{x}) + \phi(\boldsymbol{y}) \leq \phi(\boldsymbol{x} \wedge \boldsymbol{y}) + \phi(\boldsymbol{x} \vee \boldsymbol{y}),$$

where the operators $\wedge$ and $\vee$ denote coordinatewise minimum and maximum, respectively.

This is equivalent to requiring that

$$\phi(x_1, \ldots, x_i + \epsilon, \ldots, x_j + \delta, \ldots, x_n) - \phi(x_1, \ldots, x_i + \epsilon, \ldots, x_j, \ldots, x_n)$$

$$\geq \phi(x_1, \ldots, x_i, \ldots, x_j + \delta, \ldots, x_n) - \phi(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n)$$

holds for all $\boldsymbol{x} \in \mathbb{R}^n$, $1 \leq i < j \leq n$ and all $\epsilon, \delta > 0$. Considering the latter inequality, an intuitive explanation of the notion of supermodularity can be given as follows. Let $x_1, x_2, \ldots, x_n$ be claim amounts and let $\phi(x_1, x_2, \ldots, x_n)$ be the corresponding amount paid by the reinsurer to the insurer. Then supermodularity of ϕ means that the consequences of an increase of a single claim are worse for the reinsurer the higher the other claims are. This

translates the need of security of the direct insurer and makes supermodularity a plausible assumption for Ψ.
Note that the function

$$(s_1, s_2, u) \mapsto \Psi(s_1, s_2, u) = \max\{s_1 + s_2 + u - \alpha, 0\}$$

is supermodular (this comes from condition (1c) on page 150 of Marshall and Olkin (1979) together with Property D2 page 151 (ibidem).
On the contrary, the function

$$(s_1, s_2, u) \mapsto \Psi(s_1, s_2, u) = \min\Big\{\alpha, s_1 + \max\{s_2 + u - \beta, 0\}\Big\}$$

is not supermodular. To see it, let us choose ϵ, δ, s_1, s_2 and u such that

$$\begin{aligned}
\rho &= \max\{0, s_2 + \delta + u - \beta\} > 0 \\
\max\{0, s_2 + u - \beta\} &= 0 \\
\min\{\alpha, s_1 + \epsilon, +\rho\} &= \min\{\alpha, s_1 + \rho\} = \alpha \\
\min\{\alpha, s_1 + \epsilon\} &= s_1 + \epsilon \\
\min\{\alpha, s_1\} &= s_1.
\end{aligned}$$

It is easy to see that the preceding relations respect the conditions on ϵ and δ : $\epsilon, \delta > 0$. For $\Psi(s_1, s_2, u)$ to be supermodular, the following relations has to hold true :

$$\Psi(s_1 + \epsilon, s_2 + \delta, u) - \Psi(s_1 + \epsilon, s_2, u) \geq \Psi(s_1, s_2 + \delta, u) - \Psi(s_1, s_2, u)$$

for all values of $s_1, s_2, u, \epsilon, \delta$. With the particular case described hereabove we find

$$\alpha - s_1 - \epsilon \geq \alpha - s_1,$$

or equivalently $\epsilon \leq 0$ which violates the hypothesis on ϵ. This counter example shows that the function $\Psi(s_1, s_2, u)$ is not supermodular.
So, the results derived in this section do not apply in this particular case. This means that example 1 cannot be treated within the present generalization.Therefore the analysis proposed in the preceding sections is relevant. Regarding example 2, it may be treated within the present generalization.
The function

$$(s_1, s_2, u) \mapsto \Psi(s_1, s_2, u) = \min\Big\{\alpha, \max\{s_1 + s_2 + u - \beta, 0\}\Big\},$$

coresponding to example 2bis may also be shown not to be supermodular. So within this section devoted to the generalization of the techniques in higher dimensions, we will not be able to treat example 2bis.

6.2 Comonotonicity

Let us recall the definition of comonotonicity in the multivariate case.

Definition 6.2. The random vector $\boldsymbol{X}$ is said to be comonotonic if it is distributed as $(\phi_1(Z), \phi_2(Z), \ldots, \phi_n(Z))$, where Z is a real-valued random variable and the ϕ_i's are non-decreasing real-valued functions.

From this definition, we see that comonotonic risks can be considered as "common monotonic" in the sense that such risks are not able to compensate each other. Note that $\boldsymbol{X}$ is comonotonic if, and only if, its distribution function $F_{\boldsymbol{X}}$ coincides with the Fréchet upper bound, i.e.

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = \min\{F_1(x_1), \ldots, F_n(x_n)\} \text{ for all } x_1, x_2, \ldots, x_n \in \mathbb{R}^n.$$

6.3 Supermodular order

Definition 4.2 of correlation order does not carry directly to higher dimensions. An appropriate multivariate analog to $\preceq_c$ is the supermodular order. The supermodular order has been cursorily studied in the applied probability literature; standard references include Shaked and Shanthikumar (1997) and Bäuerle & Müller (1998). It is defined by means of supermodular functions.

Definition 6.3. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two n-dimensional random vectors such that $\mathbb{E}[\phi(\boldsymbol{X})] \leq \mathbb{E}[\phi(\boldsymbol{Y})]$ for all supermodular functions $\phi : \mathbb{R}^n \to \mathbb{R}$, provided the expectations exist. Then $\boldsymbol{X}$ is said to be smaller than $\boldsymbol{Y}$ in the supermodular order (denoted by $\boldsymbol{X} \preceq_{sm} \boldsymbol{Y}$).

Since the functions $\boldsymbol{y} \mapsto \mathbb{I}[\boldsymbol{y} > \boldsymbol{x}]$ and $\boldsymbol{y} \mapsto \mathbb{I}[\boldsymbol{y} \leq \boldsymbol{x}]$ are supermodular for each fixed $\boldsymbol{x}$, it is immediate that if $\boldsymbol{X} \preceq_{sm} \boldsymbol{Y}$ then $X_i =_d Y_i$ for $i = 1, 2, \ldots, n$. Therefore, if $\boldsymbol{X} \preceq_{sm} \boldsymbol{Y}$ then $\boldsymbol{X}$ and $\boldsymbol{Y}$ have necessarily to belong to the same Fréchet space.
It is worth emphasizing that in the special case $n = 2$, $\boldsymbol{X} \preceq_{sm} \boldsymbol{Y} \Leftrightarrow \boldsymbol{X} \preceq_c \boldsymbol{Y}$.
Supermodular ordering is a useful tool for comparing dependence structures of random vectors. As pointed out above, only distributions with the same marginals can be compared in the supermodular sense. Moreover, all functions $\phi(\boldsymbol{x}) = x_i x_j$ for $i \neq j$ are supermodular. Hence,

$$\boldsymbol{X} \preceq_{sm} \boldsymbol{Y} \Rightarrow \mathrm{Cov}(X_i, X_j) \leq \mathrm{Cov}(Y_i, Y_j) \text{ for any } i \neq j.$$

Roughly speaking, the supermodular stochastic order compares the strength of the positive dependence of the underlying random vectors.
Let us now recall the following result, given in Shaked and Shanthikumar (1997).

Theorem 6.4. *Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_m$ be a set of independent random vectors where the dimension of $\boldsymbol{X}_i$ is k_i, $i = 1, 2, \ldots, m$. Let $\boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots, \boldsymbol{Y}_m$ be another set of independent random vectors where the dimension of $\boldsymbol{Y}_i$ is k_i, $i = 1, 2, \ldots, m$. If $\boldsymbol{X}_i \preceq_{sm} \boldsymbol{Y}_i$ for $i = 1, 2, \ldots, m$, then*

$$(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_m) \preceq_{sm} (\boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots, \boldsymbol{Y}_m).$$

That is, the supermodular order is closed under conjunctions.

We will also need the following result, that can be found in Shaked and Shantikumar (1997).

Theorem 6.5. *The upper Fréchet bound provides the strongest dependence structure in the $\preceq_{sm}$-sense, that is $\boldsymbol{X} \preceq_{sm} \boldsymbol{X}^{\max}$, where $\boldsymbol{X}^{\max}$ is a comonotonic random vector with the same univariate marginal distribution as $\boldsymbol{X}$.*

6.4 Ordering the approximations to $(S_1,\ldots,S_n)$

Each random vector $(X_i^{Re-1},\ldots,X_i^{Re-n})$ is obviously comonotonic (since every component is a non-decreasing function of the same underlying random variable X_i), so that

$$(X_i^{Re-1,\perp},\ldots,X_i^{Re-n,\perp}) \preceq_{sm} (X_i^{Re-1},\ldots,X_i^{Re-1}).$$

From the latter stochastic inequality, we get from the independence of the X_i's that

$$\left(S_1^{\perp},\ldots,S_n^{\perp}\right) \preceq_{sm} (S_1,\ldots,S_n) \text{ where } S_j^{\perp}=\sum_{i=1}^{N} X_i^{Re-j,\perp},$$

because $\preceq_{sm}$ is closed under compouding (the proof follows the reasoning hold in Lemma 4.4 and Theorems 4.5-4.6 ; see e.g. Shaked and Shantikumar (1997) for more details). Moreover, we also have that

$$(S_1,\ldots,S_n) \preceq_{sm} (S_1^{\max},\ldots,S_n^{\max}).$$

where $(S_1^{max},\ldots,S_n^{max})$ is a comonotonic version of $(S_1,\ldots,S_n)$.
Let us recall the following result, obtained by Denuit, Genest and Marceau (2001).

Theorem 6.6. *Let $(M_1,\ldots,M_n)$ be a vector of integer-valued random variables, and given sequences $(X_{1k}),\ldots,(X_{nk})$ of positive random variables, consider the random sums*

$$S_{M_i}=\sum_{k=1}^{M_i} X_{ik}, \quad 1\le i\le n. \tag{6.1}$$

Let $\mathbf{M}=(M_1,\ldots,M_n)$, $\mathbf{M}'=(M_1',\ldots,M_n')$ be two vectors of integer-valued random variables, and let $\mathbf{S_M}=(S_{M_1},\ldots,S_{M_n})$ and $\mathbf{S_{M'}}=(S_{M_1'},\ldots,S_{M_n'})$ be defined according to (6.1). Then, $\mathbf{M}\prec_{sm}\mathbf{M}'\Rightarrow\mathbf{S_M}\prec_{sm}\mathbf{S_{M'}}$.

Let us define

$$S_j^{\perp\perp}=\sum_{i=1}^{N_j} X_i^{Re-j,\perp}$$

where the N_j's are independent copies of N. Theorem 6.5 ensures that $(N_1,\ldots,N_n)\preceq_{sm}$ $(N,\ldots,N)$. We then get from Theorem 6.6 that

$$\left(S_1^{\perp\perp},\ldots,S_n^{\perp\perp}\right) \preceq_{sm} \left(S_1^{\perp},\ldots,S_n^{\perp}\right).$$

Since $\preceq_{sm}$ is closed under concatenation (see Theorem 6.4), we have

$$\begin{aligned}(S_1^{\perp\perp},\ldots,S_n^{\perp\perp},U_1,\ldots,U_k) &\preceq_{sm} (S_1^{\perp},\ldots,S_n^{\perp},U_1,\ldots,U_k)\\ &\preceq_{sm} (S_1,\ldots,S_n,U_1,\ldots,U_k)\\ &\preceq_{sm} (S_1^{\max},\ldots,S_n^{\max},U_1,\ldots,U_k).\end{aligned}$$

Hence, we can obtain bounds on the pure premium since Ψ has been assumed supermodular. However we are not able to compare other moments of the over or premiums obtained with the PH-transform premium principle.

7 Conclusion

We have derived in this paper some additional results on the correlation order and the stop-loss order. These results helped to analyse bounds for the pure premium of Top & Drop Excess of Loss covers. With the hypothesis that the reinsurance cover is a function of a comonotonic random vector, we have observed that it is possible to obtain bounds at least for the pure premium. In particular we observed that for a particular treaty, the incorrectly assumed hypothesis of independence provides an upper bound for the pure premium of the treaty. This happens when the cover of the treaty is limited and when the comonotonic random variables expressed as an excess of the same underlying random variable. Unfortunately we have seen on an example that the other moments of the cover are no longer ordered and we do not have a theoretical result on these orders.

On a second example, the incorrectly assumed hypothesis of independence was not conservative, which shows the danger of not working with the exact model when it is known. Furthermore the upper and lower Fréchet bounds may be quite far from the exact result, as shown in example 2.

The theoretical results used in dimension 2 have been extended in higher dimensions by using the supermodular ordering and the notion of association. However we have observed that our first numerical example could not be treated within the multivariate generalization. A particular treatment in dimension 2 was necessary to handle it correctly.

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