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Abstract

This paper aims to derive bounds for the cumulative distribution function of a function of dependent risks. The results presented here complement a recent work by Denuit, Genest and Marceau (1999) where sums of correlated random variables were considered. *Keywords:* Dependent risks, copulae, Fréchet bounds

1 Introduction and Motivation

The present paper aims to provide a method to derive bounds on the cumulative distribution function (cdf, in short) of a function of dependent random variables (claims severities, remaining lifetimes or stochastic interest rates for instance). Whereas standard actuarial mathematics always assume independence, it is clear that this theoretical situation seldom holds in practice. Therefore, we show here how to handle possibly correlated random variables when (almost) no information about their dependence structure is available. This allows the actuary to quantify the impact of a possible correlation among the risks he faces.

Let us point out several examples concerning correlated random variables. In life insurance, policies sold to married couples clearly involve dependent random variables (namely, the spouses' remaining lifetimes). Various methods dealing with such a situation have been proposed, e.g. by Carrière and Chan (1986), Norberg (1989), Wolthuis (1994), Frees, Carrière and Valdez (1996), Dhaene, Vanneste and Wolthuis (2000), Denuit, Dhaene, Le Bailly de Tilleghem and Teghem (2001) and Denuit and Cornet (1999a,b). Another fine example of dependent random variables in an actuarial context has been studied by Klugman and Parsa (1999), where the correlation structure between a loss amount and its ALAE is investigated. Catastrophe insurance (*id est* policies covering the consequences of events like earthquakes, hurricanes or tornados, for instance) of course deals with dependent risks; this aspect has been addressed e.g. by Dhaene and Goovaerts (1996,1997), Bäuerle and Müller (1998) and Cossette, Gaillardetz, Marceau and Rioux (1999). Among the possible tools for taking dependence into account in a non-life context, copula models seem to be of primary interest; they have been used e.g. by Carrière (1994, 1998, 2000), Frees and Valdez (1998), Genest, Ghoudi and Rivest (1998), Wang (1998), Embrechts, McNeil and Strauman (1999, 2000), Klugman and Parsa (1999), Denuit, Genest and Marceau (1999,2001), Cossette, Gaillardetz, Marceau and Rioux (1999) and Genest, Marceau and Mesfioui (2000).

In this paper, we examine the following problematic. Consider two random variables X_1 and X_2, with specified cdf's F_1 and F_2, as well as a measurable function $\Psi : I\!R^2 \to I\!R$. Typically, $\Psi(x_1, x_2)$ is the amount paid by an insurance company if $X_1 = x_1$ and $X_2 = x_2$, or a suitable risk measure. The joint distribution of the random couple (X_1, X_2) is unknown to the actuary, but he feels that the dependence structure between X_1 and X_2 might cause severe problems to the company (in the sense that large values of X_1 tend to occur with large values of X_2, and *vice versa*). Therefore, he would like to determine bounds on the cdf of the random variable $\Psi(X_1, X_2)$, provided Ψ satisfies some reasonable regularity conditions. Taking $\Psi(X_1, X_2) = X_1 + X_2$ leads to the case treated in Denuit *et al.* (1999).

In this context, generalized inverses will play a central role. We thus recall some results that can be found for instance in De Vylder, Dhaene and Goovaerts (1999). Let $\varphi : I\!R \to I\!R$ be a non-decreasing function. The left-continuous inverse φ^{-1} of φ is defined as

$$\varphi^{-1}(x) = \inf\{t \in I\!R | \varphi(t) \geq x\}, \tag{1.1}$$

with the convention that $\inf \emptyset = +\infty$. Similarly, the right-continuous inverse $\varphi^{-1\bullet}$ of φ is defined as

$$\varphi^{-1\bullet}(x) = \sup\{t \in I\!R | \varphi(t) \leq x\}, \tag{1.2}$$

with the convention that $\sup\emptyset = -\infty$. Provided that φ is left-continuous, the equivalence

$$x \leq \varphi^{-1\bullet}(y) \Leftrightarrow \varphi(x) \leq y \tag{1.3}$$

always holds. Now, if φ is non-increasing, its left-continuous φ^{-1} and right-continuous $\varphi^{-1\bullet}$ inverses are respectively given by

$$\varphi^{-1}(x) = \inf\{t \in I\!R | \varphi(t) \leq x\}, \tag{1.4}$$

and

$$\varphi^{-1\bullet}(x) = \sup\{t \in I\!R | \varphi(t) \geq x\}. \tag{1.5}$$

Provided that φ is right-continuous, the equivalence

$$\varphi^{-1}(y) \leq x \Leftrightarrow y \geq \varphi(x) \tag{1.6}$$

always holds.

Similar problems have been previously addressed in the literature. For instance, if Ψ is a non-decreasing function such that $\partial^2\Psi/\partial x_1 \partial x_2 \geq 0$ then Property 4.4(i) in Denuit, Lefèvre and Mesfioui (1999) yields the stop-loss inequalities

$$\begin{aligned} E\max\left\{\Psi(F_1^{-1}(U), F_2^{-1}(1-U)) - t, 0\right\} &\leq E\max\left\{\Psi(X_1, X_2) - t, 0\right\} \\ &\leq E\max\left\{\Psi(F_1^{-1}(U), F_2^{-1}(U)) - t, 0\right\} \end{aligned} \tag{1.7}$$

which are valid for any level t of the deductible, where U is a random variable uniformly distributed over $[0,1]$.

In general, (1.7) does not provide bounds on the cdf of $\Psi(X_1, X_2)$; moreover, it requires second-order regularity condition on Ψ to be valid. Our aim is to determine lower and upper bounds on the cdf of $\Psi(X_1, X_2)$ when all that is known about the risks X_1 and X_2 is their marginal distributions F_1 and F_2. We only require Ψ to be continuous and monotone in each argument. It is worth mentioning that similar problems have been studied by Frank, Nelsen and Schweizer (1987), Williamson and Downs (1990), Williamson (1991) and, in an actuarial context, by Denuit, Genest and Marceau (1999) and Embrechts, McNeil and Strauman (1999, 2000).

In the numerical illustrations, we focus on $\Psi(x_1, x_2) = x_1/x_2$ and $\Psi(x_1, x_2) = x_1 x_2$. The reason is that both functions are non-linear and possess a nice actuarial interpretation. Typically, $\Psi(x_1, x_2) = x_1/x_2$ has the form of a loss ratio: x_1 is the total loss experienced by the company and x_2 represents incomes of the company (premiums paid by policyholders but also amounts due by reinsurers and returns on financial assets). Therefore, the numerator and denominator become dependent, especially if the company bought financial products hedging insurance risks. In the second example, $\Psi(x_1, x_2) = x_1 x_2$, x_1 could be the total loss experienced by the insurer, and x_2 the indicator for some event, or a percentage depending on some index; then $\Psi(x_1, x_2)$ may be considered as the amount of indemnity produced by a reinsurance agreement.

The paper is organized as follows. In Section 2, we first consider the bivariate case. After having briefly recalled the definition of the copulae, we derive bounds for the cdf of $\Psi(X_1, X_2)$.

We also show how it is possible to improve these bounds when additional information about the correlation structure of the random couple (X_1, X_2) is available. To close this section, we address the numerical aspects of the method. Then, in Section 3, we extend the results to the multivariate case and we propose stochastic bounds on $\Psi(X_1, \ldots, X_n)$ for continuous monotone $\Psi : \mathbb{R}^n \to \mathbb{R}$, when all it is known about the X_i's is their marginal cdf's $F_1, \ldots, F_n$.

2 Bounds on a function of two dependent risks

2.1 Bivariate Copula models

Copulae are functions that join (or "couple") multivariate distribution functions to their one-dimensional marginals. Technically speaking, copulae are distribution functions whose marginals are uniform. Since Fréchet and Höffding, it is well known that all copulae satisfy

$$C_L(t_1, t_2) \leq C(t_1, t_2) \leq C_U(t_1, t_2) \text{ for all } (t_1, t_2) \in [0, 1]^2, \tag{2.1}$$

where the copulae C_L and C_U are given by

$$C_L(t_1, t_2) = \max\{t_1 + t_2 - 1, 0\} \text{ and } C_U(t_1, t_2) = \min\{t_1, t_2\}, \;\; (t_1, t_2) \in [0, 1]^2;$$

see e.g. Nelsen (1999). C_L and C_U are known as the Fréchet lower and upper bounds copulae, respectively. Coming back to the stop-loss bounds in (1.7), it is easily seen that

$$P\left[F_1^{-1}(U) \leq x_1, F_2^{-1}(1 - U) \leq x_2\right] = C_L(F_1(x_1), F_2(x_2))$$

and

$$P\left[F_1^{-1}(U) \leq x_1, F_2^{-1}(U) \leq x_2\right] = C_U(F_1(x_1), F_2(x_2)).$$

If the random couple (X_1, X_2) has the copula C_U then (X_1, X_2) is said to be comonotonic. Comonotonicity thus represents perfect positive dependence (since both X_1 and X_2 are non-decreasing functions of the same underlying random variable U).

Now, let $F_{(X_1,X_2)}$ be a two-dimensional distribution function with univariate marginals F_1 and F_2. There exists a copula C such that

$$F_{(X_1,X_2)}(t_1, t_2) = C(F_1(t_1), F_2(t_2)) \text{ for all } (t_1, t_2) \in \mathbb{R}^2. \tag{2.2}$$

The copula C contains all the dependency informations about the random couple (X_1, X_2). Copulae theory therefore provides a natural setting for the study of questions dealing with properties of distribution functions with fixed marginals. In particular, when $C(t_1, t_2) = C_I(t_1, t_2) \equiv t_1 t_2$, the components of the random couple (X_1, X_2) are mutually independent. Henceforth, we assume that the marginal cdf's F_1 and F_2 are continuous; this ensures that the copula C achieving the representation (2.2) is unique, whereas some technical difficulties arise when the F_i's exhibit jump points (for more details, see e.g. Nelsen (1999)).

The dual of a copula C is the function $C^d(.,.)$ defined by

$$C^d(t_1, t_2) = t_1 + t_2 - C(t_1, t_2), \;\; (t_1, t_2) \in [0, 1]^2. \tag{2.3}$$

It is easily seen from (2.2) that

$$C^d(F_1(t_1), F_2(t_2)) = P[X_1 \le t_1 \cup X_2 \le t_2],\ \ (t_1,t_2) \in [0,1]^2.$$

For instance, the dual of C_L is given by

$$C_L^d(t_1,t_2) = \min\{t_1+t_2, 1\},\ \ (t_1,t_2) \in [0,1]^2;$$

the latter will play a central role in the next section.

2.2 Distributional bounds on $\Psi(X_1, X_2)$

Assume one wants to get bounds on the distribution function of $\Psi(X_1, X_2)$ in terms of the marginals F_1 and F_2. Frank, Nelsen and Schweizer (1987, Theorem 5.1) were the first who solved this problem. We recall their result here, and we provide an elementary proof of it, which appears to be new.

Proposition 2.1. *Let (X_1, X_2) be a bivariate risk with marginals F_1 and F_2. Given a non-decreasing and continuous function $\Psi : \mathbb{R}^2 \to \mathbb{R}$, let us define for arbitrary s and x in $\mathbb{R}$ the continuous function $\varphi_x : \mathbb{R} \to \mathbb{R}$ by $t \mapsto \varphi_x(t) = \Psi(x,t)$. Then, the inequalities*

$$F_{\min}(s|\Psi) \le P[\Psi(X_1,X_2) \le s] \le F_{\max}(s|\Psi) \tag{2.4}$$

hold for all $s \in \mathbb{R}$, with

$$F_{\min}(s|\Psi) = \sup_{t_1\in\mathbb{R}} C_L\big(F_1(t_1), F_2(\varphi_{t_1}^{-1\bullet}(s))\big)$$

and

$$F_{\max}(s|\Psi) = \inf_{t_1\in\mathbb{R}} C_L^d\big(F_1(t_1), F_2(\varphi_{t_1}^{-1\bullet}(s))\big).$$

Proof. Let C be the copula such that (2.2) holds for the random couple (X_1, X_2). Then it is clear from (1.3) that $X_1 > x$ and $X_2 > \varphi_x^{-1\bullet}(s)$ together imply $\Psi(X_1, X_2) > s$. We then have for any $s \in \mathbb{R}$ that

$$\begin{aligned} P[\Psi(X_1,X_2) \le s] &\le P[X_1 \le x \cup X_2 \le \varphi_x^{-1\bullet}(s)] \\ &= C^d(F_1(x), F_2(\varphi_x^{-1\bullet}(s))) \\ &\le C_L^d(F_1(x), F_2(\varphi_x^{-1\bullet}(s))). \end{aligned}$$

Therefore,

$$P[\Psi(X_1,X_2) \le s] \le \inf_{x\in\mathbb{R}} C_L^d(F_1(x), F_2(\varphi_x^{-1\bullet}(s))),$$

which is the right inequality of (2.4). In order to get the left one, it suffices to note that for any $s \in \mathbb{R}$

$$\begin{aligned} P[\Psi(X_1,X_2) \le s] &\ge C(F_1(x), F_2(\varphi_x^{-1\bullet}(s))) \\ &\ge C_L(F_1(x), F_2(\varphi_x^{-1\bullet}(s))), \end{aligned}$$

and the best lower bound is finally obtained by taking the "sup". □

Remark 2.2. It is worth mentioning that $F_{\min}(\cdot|\Psi)$ can be cast into

$$F_{\min}(s|\Psi) = \sup_{(t_1,t_2)\in I\!R^2|\Psi(t_1,t_2)=s} C_L\big(F_1(t_1), F_2(t_2)\big). \tag{2.5}$$

On the contrary, we do not have in general that

$$F_{\max}(s|\Psi) = \inf_{(t_1,t_2)\in I\!R^2|\Psi(t_1,t_2)=s} C_L^d\big(F_1(t_1), F_2(t_2)\big). \tag{2.6}$$

The explanation behind this fact is the following. The condition $\Psi(t_1, t_2) = s$ is equivalent with $\varphi_{t_1}(t_2) = s$ which is the same as

$$\begin{cases} \varphi_{t_1}(t_2) \leq s \\ \varphi_{t_1}(t_2) > s - \epsilon \text{ for all } \epsilon > 0 \end{cases} \Leftrightarrow \begin{cases} t_2 \leq \varphi_{t_1}^{-1\bullet}(s) \\ t_2 > \varphi_{t_1}^{-1\bullet}(s-\epsilon) \text{ for all } \epsilon > 0 \end{cases}$$

according to (1.3), i.e.

$$\varphi_{t_1}^{-1\bullet}(s-) \leq t_2 \leq \varphi_{t_1}^{-1\bullet}(s).$$

Now, since $C_L(F_1(t_1), F_2(t_2))$ is non-decreasing in (t_1, t_2), it follows that the supremum in the right hand-side of (2.5) is taken at the right endpoint of the interval $[\varphi_{t_1}^{-1\bullet}(s-), \varphi_{t_1}^{-1\bullet}(s)]$ implying the equality in (2.5). However, this reasoning cannot be repeated for the non-decreasing function $C_L^d(F_1(t_1), F_2(t_2))$ for which we have to take the infimum.

Williamson and Downs (1990, Theorem 3) proved the pointwise best possible nature of the bounds (2.4). More precisely, they showed that there always exists a copula such that the distribution function of $\Psi(X_1, X_2)$ meets the bounds (2.4) at some given point s. In other words, one cannot construct tighter bounds. In the particular case $\Psi(x_1, x_2) = x_1 + x_2$, Proposition 2.1 reduces to Proposition 1 in Denuit, Genest and Marceau (1999).

It is worth mentioning that Proposition 2.1 is easily adapted to monotone functions Ψ. This is formally stated in the next corollary.

Corollary 2.3. *Under the assumptions of Proposition 2.1 with Ψ non-decreasing in the first argument and non-increasing in the second argument, the bounds in (2.4) become*

$$F_{\min}(s|\Psi) = \sup_{t_1\in I\!R} \left\{F_1(t_1) - C_U\big(F_1(t_1), F_2(\varphi_{t_1}^{-1}(s))\big)\right\},$$

and

$$F_{\max}(s|\Psi) = \inf_{t_1\in I\!R} \left\{1 - F_2\big(\varphi_{t_1}^{-1}(s)\big) + C_U^d\Big(F_1(t_1), F_2\big(\varphi_{t_1}^{-1}(s)\big)\Big)\right\}.$$

Proof. The reasoning of Proposition 2.1 is easily adapted to deal with the present situation. Indeed, we now have from (1.6) that

$$\begin{aligned} P[\Psi(X_1, X_2) \leq s] &\leq P[X_1 \leq x \cup X_2 \geq \varphi_x^{-1}(s)] \\ &= P[X_2 \geq \varphi_x^{-1}(s)] + P[X_1 \leq x, X_2 \leq \varphi_x^{-1}(s)] \\ &= 1 - F_2(\varphi_x^{-1}(s)) + C(F_1(x), F_2(\varphi_x^{-1}(s))) \\ &\leq 1 - F_2(\varphi_x^{-1}(s)) + C_U(F_1(x), F_2(\varphi_x^{-1}(s))). \end{aligned}$$

On the other hand,

$$\begin{aligned}
P[\Psi(X_1, X_2) \leq s] &\geq P[X_1 \leq x, X_2 \geq \varphi_x^{-1}(s)] \\
&= F_1(x) - C(F_1(x), F_2(\varphi_x^{-1}(s))) \\
&\geq F_1(x) - C_U(F_1(x), F_2(\varphi_x^{-1}(s))),
\end{aligned}$$

and this ends the proof. □

To end with, if Ψ is decreasing, (2.4) becomes

$$1 - F_{\max}(-s| - \Psi) \leq P[\Psi(X_1, X_2) \leq s] \leq 1 - F_{\min}(-s| - \Psi).$$

This is easily obtained from Proposition 2.1 since Ψ is non-decreasing if, and only if, $-\Psi$ is non-increasing.

2.3 Improvements of the bounds

If we know moreover that the copula C describing the structure of dependence of the random couple (X_1, X_2) is such that

$$C(t_1, t_2) \geq C_0(t_1, t_2) \geq C_L(t_1, t_2) \text{ for all } (t_1, t_2) \in [0, 1]^2, \tag{2.7}$$

for some given copula C_0, then we get better bounds by substituting C_0 for C_L in (2.4). This is formalized in the next result; the proof follows the same lines as for Proposition 2.1 and is therefore omitted.

Proposition 2.4. *Let (X_1, X_2) be a bivariate risk with marginals F_1 and F_2, and satisfying (2.2) with the copula C that fulfills (2.7). Given a non-decreasing and continuous function $\Psi : \mathbb{R}^2 \to \mathbb{R}$, the inequalities*

$$F_{\min}(s|\Psi, C_0) \leq P[\Psi(X_1, X_2) \leq s] \leq F_{\max}(s|\Psi, C_0) \tag{2.8}$$

hold for all $s \in \mathbb{R}$, with

$$F_{\min}(s|\Psi, C_0) = \sup_{t_1 \in \mathbb{R}} C_0\big(F_1(t_1), F_2(\varphi_{t_1}^{-1\bullet}(s))\big),$$

and

$$F_{\max}(s|\Psi, C_0) = \inf_{t_1 \in \mathbb{R}} C_0^d\big(F_1(t_1), F_2(\varphi_{t_1}^{-1\bullet}(s))\big).$$

Obviously, Remark 2.2 also applies in this case (simply substituting C_0 for C_L or C_0^d for C_L^d). Again, Williamson and Downs (1990, Theorem 3) proved the pointwise best possible nature of these bounds.

Let us briefly expand on the meaning of a majoration like in (2.7). Since $C(t_1, t_2) \geq C_0(t_1, t_2)$ actually means that C_0 precedes C in the correlation order of Dhaene and Goovaerts (1996, Definition 2 and Theorem 1), we get that

$$\mathrm{Cov}_{C_0}[\phi_1(X_1), \phi_2(X_2)] \leq \mathrm{Cov}_C[\phi_1(X_1), \phi_2(X_2)]$$

for any non-decreasing functions ϕ_1 and ϕ_2 for which the covariances exist, where $\mathrm{Cov}_C[\phi_1(X_1), \phi_2(X_2)]$ (resp. $\mathrm{Cov}_{C_0}[\phi_1(X_1), \phi_2(X_2)]$) is the covariance of $\phi_1(X_1)$ and $\phi_2(X_2)$ given that the joint distribution function of (X_1, X_2) is $C(F_1, F_2)$ (resp. $C_0(F_1, F_2)$). Therefore, $\mathrm{Cov}_{C_0}[X_1, X_2]$ is a lower bound on the covariance between X_1 and X_2. This highlights the dependency induced by $C_0 \neq C_L$. Note that the interpretation of $C_0 \equiv C_I$ in (2.7) is particularly simple. It says that X_1 and X_2 are positively quadrant dependent (PQD, in short). This is to say that the independent version of (X_1, X_2) precedes (X_1, X_2) in the correlation order, *id est*

$$P[X_1 > t_1, X_2 > t_2] \geq P[X_1 > t_1]P[X_2 > t_1] \text{ for all } (t_1, t_2) \in I\!R^2.$$

Roughly speaking, this means that the probability that X_1 and X_2 both realize large (resp. small) values is larger than the corresponding probability in the case of independence.

2.4 Numerical illustrations

Let us first consider that we only know the F_i's, and that we do not have any additional information about the dependence structure of the X_i's. In general, it is not possible to get closed form expressions for $F_{\min}(.|\Psi)$ and $F_{\max}(.|\Psi)$, and we have to resort to numerical methods. To be specific, we approximate $F_{\min}(.|\Psi)$ and $F_{\max}(.|\Psi)$ by $\hat{F}^N_{\min}(.|\Psi)$ and $\hat{F}^N_{\max}(.|\Psi)$, by means of a slight adaptation of the method given in Denuit, Genest and Marceau (1999). First, we choose some large integer N and for each $r = 1, 2, \ldots, N-1$, we compute the (r/N)th quantile of F_i, $i = 1, 2$, $q_i(r/N)$ say, given by

$$q_i(r/N) = F_i^{-1}(r/N), \;\; i = 1, 2.$$

Let us also define

$$q_i(0) = c_i \text{ and } q_i(1) = d_i$$

for some reals such that

$$-\infty < c_i < q_i(1/N) \text{ and } q_i(1 - 1/N) < d_i < +\infty.$$

In typical actuarial applications, c_i would be 0 and d_i could be taken as the expected maximal loss. Then, the approximated bounds $\hat{F}^N_{\min}(.|\Psi)$ and $\hat{F}^N_{\max}(.|\Psi)$ are given by

$$\hat{F}^N_{\min}(s|\Psi) = \frac{1}{N} \sum_{r=1}^{N} \mathcal{I}_{q_{\min}(r/N|\Psi)}(s) \tag{2.9}$$

and

$$\hat{F}^N_{\max}(s|\Psi) = \frac{1}{N} \sum_{r=0}^{N-1} \mathcal{I}_{q_{\max}(r/N|\Psi)}(s), \tag{2.10}$$

where $\mathcal{I}_c$ denotes the indicator function of the interval $[c, +\infty[$, and $q_{\min}(.|\Psi)$ and $q_{\max}(.|\Psi)$ are defined as follows:

1. if Ψ is non-decreasing, then

$$q_{\min}(r/N|\Psi) = \min_{r\leq\ell\leq N}\{\Psi(q_1(\ell/N), q_2(1-(\ell-r)/N))\} \tag{2.11}$$

and

$$q_{\max}(r/N|\Psi) = \max_{0\leq\ell\leq r}\{\Psi(q_1(\ell/N), q_2((r-\ell)/N))\}\,; \tag{2.12}$$

2. if Ψ is non-decreasing in the first argument and non-increasing in the second, then

$$q_{\min}(r/N|\Psi) = \min_{r\leq\ell\leq N}\{\Psi(q_1(\ell/N), q_2((\ell-r)/N))\} \tag{2.13}$$

and

$$q_{\max}(r/N|\Psi) = \max_{0\leq\ell\leq r}\{\Psi(q_1(\ell/N), q_2(1-(r-\ell)/N))\}\,. \tag{2.14}$$

According to Williamson and Downs (1990, pp. 118-123), it follows that $\hat{F}^N_{\min}(s|\Psi)$ and $\hat{F}^N_{\max}(s|\Psi)$ tend to $F_{\min}(s|\Psi)$ and $F_{\max}(s|\Psi)$ as N tends to infinity, respectively. Also, from Williamson and Downs (1990, Theorem 4), we have

$$\hat{F}^N_{\min}(s|\Psi) < F_{\min}(s|\Psi) < F_{\max}(s|\Psi) < \hat{F}^N_{\max}(s|\Psi).$$

Let us now turn to the improved bounds, *id est* those obtained when (2.7) is known to hold. The approximations of the bounds $F_{\min}(.|\Psi, C_0)$ and $F_{\max}(.|\Psi, C_0)$, denoted as $\hat{F}^N_{\min}(.|\Psi, C_0)$ and $\hat{F}^N_{\max}(.|\Psi, C_0)$, are given by

$$\hat{F}^N_{\min}(s|\Psi, C_0) = \frac{1}{N}\sum_{r=1}^{N}\mathcal{I}_{\tilde{q}_{\min}(r/N|\Psi,C_0)}(s)$$

and

$$\hat{F}^N_{\max}(s|\Psi, C_0) = \frac{1}{N}\sum_{r=0}^{N-1}\mathcal{I}_{\tilde{q}_{\max}(r/N|\Psi,C_0)}(s),$$

where $\tilde{q}_{\min}(.|\Psi, C_0)$ and $\tilde{q}_{\max}(.|\Psi, C_0)$ are defined as follows:

1. if Ψ is non-decreasing, then

$$\tilde{q}_{\min}(r/N|\Psi, C_0) = \min_{r\leq\ell\leq N}\{\Psi(q_1(\ell/N), q_2(\nu_{r,\ell}))\}$$

and

$$\tilde{q}_{\max}(r/N|\Psi, C_0) = \max_{0\leq\ell\leq r}\left\{\Psi(q_1(\ell/N), q_2(\nu^*_{r,\ell}))\right\},$$

with for $\nu_{r,\ell}$ the solution of the equation

$$C_0(\ell/N, \nu_{r,\ell}) = r/N$$

and for $\nu^*_{r,\ell}$ the one of

$$C^d_0(\ell/N, \nu^*_{r,\ell}) = r/N.$$

2. if Ψ is non-decreasing in the first argument and non-increasing in the second, then

$$\tilde{q}_{\min}(r/N|\Psi, C_0) = \min_{r\leq\ell\leq N} \{\Psi(q_1(\ell/N), q_2(\nu_{r,\ell})\}$$

and

$$\tilde{q}_{\max}(r/N|\Psi, C_0) = \max_{0\leq\ell\leq r} \left\{\Psi(q_1(\ell/N), q_2(\nu^*_{r,\ell}))\right\}.$$

Similarly, from Williamson and Downs (1990), $\hat{F}^N_{\min}(s|\Psi, C_0)$ and $\hat{F}N_{\max}(s|\Psi, C_0)$ approach asymptotically $F_{\min}(s|\Psi, C_0)$ and $F_{\max}(s|\Psi, C_0)$ respectively.

In order to illustrate the accuracy of the bounds on the cdf of $\Psi(X_1, X_2)$, we considered two specific examples for Ψ. In the first case, we chose $\Psi(x_1, x_2) = x_1/x_2$ and plotted the graph of Figure 1 with X_1 exponentially distributed with unit mean and X_2 Pareto distributed with mean 1 and variance 2, *id est*

$$P[X_2 \leq x] = 1 - \left(\frac{3}{3+x}\right)^4, \quad x \in I\!R^+.$$

Since $\Psi(x_1, x_2) = x_1/x_2$ is non-decreasing in the first argument and non-increasing in the second one, (2.13) and (2.14) become

$$q_{\min}(r/N|\Psi) = \min_{r\leq\ell\leq N} \{q_1(\ell/N)/q_2((\ell-r)/N)\}$$

and

$$q_{\max}(r/N|\Psi) = \max_{0\leq\ell\leq r} \{q_1(\ell/N)/q_2(1-(r-\ell)/N)\}.$$

The dashed line corresponds to the cdf of $\Psi(X_1, X_2)$ when X_1 and X_2 are mutually independent.

As another example, the graph of Figure 2 corresponds to the bounds on the cdf of $\Psi(X_1, X_2)$ with $\Psi(x_1, x_2) = x_1 x_2$, X_1 and X_2 as described in the preceding illustration. In this case, Ψ is non-decreasing so that (2.11) and (2.12) become

$$q_{\min}(r/N|\Psi) = \min_{r\leq\ell\leq N} \{q_1(\ell/N)q_2(1-(\ell-r)/N)\}$$

and

$$q_{\max}(r/N|\Psi) = \max_{0\leq\ell\leq r} \{q_1(\ell/N)q_2((r-\ell)/N)\}.$$

Now, let us consider the exponentially distributed risks X_1 and X_2, with respective means 20 and 1. If X_1 and X_2 are comonotonic then

$$\frac{X_1}{X_2} =_d \frac{-20\ln(1-U)}{-\ln(1-U)} = 20 \text{ almost surely,}$$

must hold, where U denotes a random variable uniformly distributed over $[0, 1]$. This explains why in Figure 3 $F_{\min} = 0$ on $[0, 20[$, and $F_{\max} = 1$ on $[20, +\infty[$. Let us now turn to Figure 4, where the product $X_1 X_2$ is considered, with X_1 and X_1 as described before. If X_1 and X_2 are comonotonic, we have that

$$X_1 X_2 =_d 20 X_2^2,$$

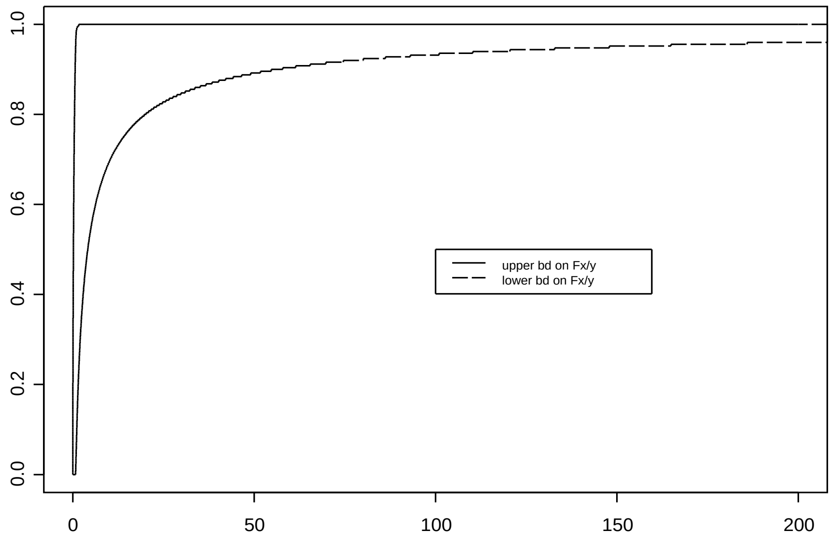

Figure 1: *Bounds on the cumulative distribution function of X_1/X_2, X_1 exponentially distributed with unit mean, X_2 Pareto distributed with mean 1 and variance 2.*

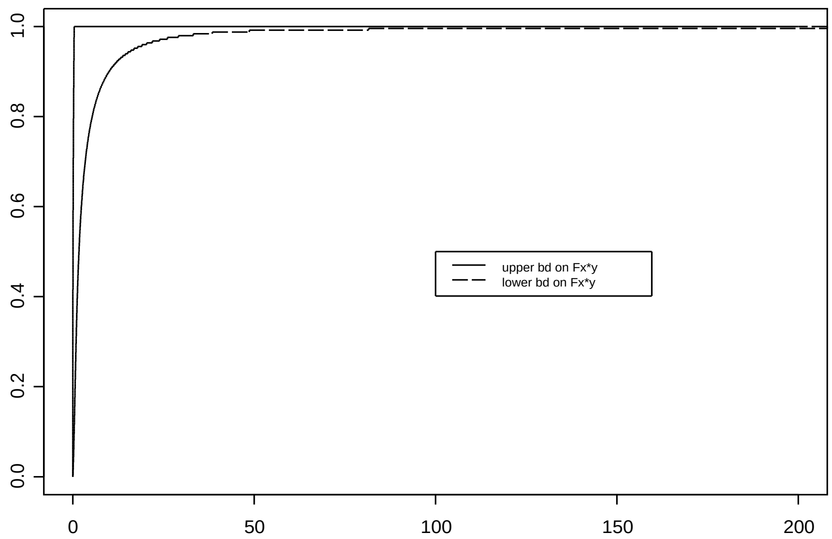

Figure 2: *Bounds on the cumulative distribution function of X_1X_2, X_1 exponentially distributed with unit mean, X_2 Pareto distributed with mean 1 and variance 2.*

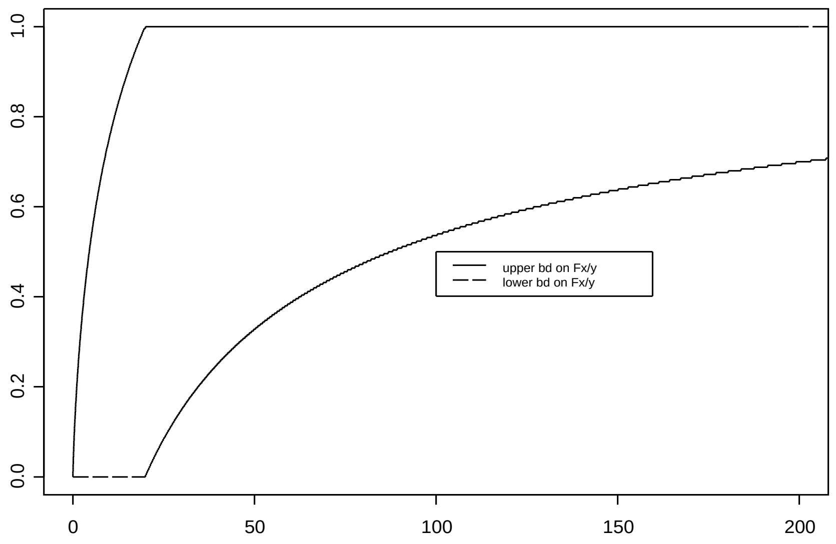

Figure 3: *Bounds on the cumulative distribution function of X_1/X_2, X_1 and X_2 exponentially distributed with respective means 20 and 1.*

so that

$$P[X_1X_2 \leq z] = P[X_2 \leq \sqrt{z/20}] = 1 - \exp(-\sqrt{z/20}).$$

Of course, the curve $z \mapsto 1-\exp(-\sqrt{z/20})$ lies between the bounds $F_{\min}$ and $F_{\max}$ depicted in Figure 4. It is worth mentioning that once the marginals are fixed, the best upper bound on $E[\max\{X_1X_2 - t, 0\}]$ is obtained when X_1 and X_2 are comonotonic, see (1.7). In other words, if the marginals are exponential with means 20 and 1, as above,

$$E[\max\{X_1X_2 - t, 0\}] \leq \int_{z=t}^{+\infty} \exp(-\sqrt{z/20})dz.$$

3 Bounds on a function of n dependent risks

3.1 Multivariate copula models

As for the bivariate case, any multivariate distribution function can be represented in a way that emphasizes the separate role of the marginals and the dependence structure. To be specific, let $F_{(X_1,\dots,X_n)}$ be the cumulative distribution function of a n-dimensional random vector $(X_1,\dots,X_n)$ with marginal cdf's $F_1,\dots,F_n$, respectively. Then, there exists a multivariate distribution function C with uniform marginals such that the representation

$$F_{(X_1,\dots,X_n)}(x_1,\dots,x_n) = C(F_1(x_1),\dots,F_n(x_n)) \tag{3.1}$$

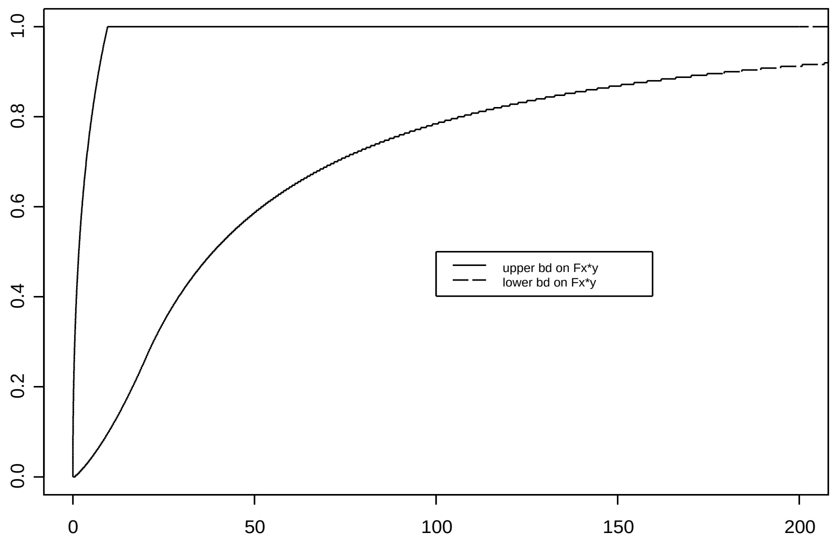

Figure 4: *Bounds on the cumulative distribution function of* X_1X_2, X_1 *and* X_2 *exponentially distributed with respective means 20 and 1.*

holds.

As in (2.1), all n-dimensional copulae C satisfy

$$C_L(t_1, \dots, t_n) \leq C(t_1, \dots, t_n) \leq C_U(t_1, \dots, t_n) \tag{3.2}$$

for all $(t_1, \dots, t_n) \in [0,1]^n$, where C_L and C_U, given by

$$C_L(t_1, \dots, t_n) = \max\left\{\sum_{i=1}^n t_i - (n-1), 0\right\}, \ \ (t_1, \dots, t_n) \in [0,1]^n,$$

and

$$C_U(t_1, \dots, t_n) = \min\{t_1, \dots, t_n\}, \ \ (t_1, \dots, t_n) \in [0,1]^n,$$

are known as the Fréchet lower and upper bounds, respectively. Let us point out that C_U is a *bona fide* copula whereas this is not necessarily true for C_L when $n \geq 3$. For more details, see e.g. Nelsen (1999).

As in (2.3), for each copula C, we define its dual C^d as

$$C^d(F_1(x_1), \dots, F_n(x_n)) = P\left[\cup_{i=1}^n \{X_i \leq x_i\}\right].$$

Observe that no simple relation as (2.3) exists between C and C^d when $n \geq 3$.

3.2 Distributional bounds on $\Psi(X_1, \dots, X_n)$

We provide a multivariate extension of Proposition 2.1.

Proposition 3.1. *Let $(X_1, \dots, X_n)$ be a n-dimensional random vector with marginals $F_1, \dots, F_n$. Given a non-decreasing and continuous function $\Psi : I\!R^n \to I\!R$, let us define for arbitrary s, x_1, x_2, $\dots$, and x_{n-1} in $I\!R$ the continuous function $\varphi_{x_1,\dots,x_{n-1}} : I\!R \to I\!R$ by*

$$t \mapsto \varphi_{x_1,\dots,x_{n-1}}(t) = \Psi(x_1, \dots, x_{n-1}, t).$$

Then, the inequalities

$$F_{\min}(s|\Psi) \leq P[\Psi(X_1, \dots, X_n) \leq s] \leq F_{\max}(s|\Psi) \tag{3.3}$$

hold for all $s \in I\!R$, with

$$F_{\min}(s|\Psi) = \sup_{(t_1,\dots,t_{n-1})\in I\!R^{n-1}} C_L\big(F_1(t_1), \dots, F_{n-1}(t_{n-1}), F_n(\varphi^{-1\bullet}_{t_1,\dots,t_{n-1}}(s))\big),$$

and

$$F_{\max}(s|\Psi) = \inf_{(t_1,\dots,t_{n-1})\in I\!R^{n-1}} \min\left\{\sum_{i=1}^{n-1} F_i(t_i) + F_n(\varphi^{-1\bullet}_{t_1,\dots,t_{n-1}}(s)), 1\right\}.$$

Proof. Let C be the copula such that (3.1) holds for the random vector $(X_1, \dots, X_n)$. Then it is clear that $X_1 > x_1$, $X_2 > x_2$, $\dots$, $X_{n-1} > x_{n-1}$ and $X_n > \varphi^{-1\bullet}_{x_1,\dots,x_{n-1}}(s)$ imply $\Psi(X_1, \dots, X_n) > s$, so that

$$\begin{aligned} P[\Psi(X_1, \dots, X_n) \leq s] &\leq P[X_1 \leq x_1 \cup \dots \cup X_{n-1} \leq x_{n-1} \cup X_n \leq \varphi^{-1\bullet}_{x_1,\dots,x_{n-1}}(s)] \\ &\leq \sum_{i=1}^{n-1} F_i(x_i) + F_n(\varphi^{-1\bullet}_{x_1,\dots,x_{n-1}}(s)), \end{aligned}$$

whence the right hand side inequality follows in (3.3). For the other inequality, it suffices to note that

$$P[\Psi(X_1, \dots, X_n) \leq s] \geq C_L(F_1(x_1), \dots, F_{n-1}(x_{n-1}), F_n(\varphi^{-1\bullet}_{x_1,\dots,x_{n-1}}(s))),$$

and this completes the proof. $\square$

Remark 3.2. We adapt here the comments in Remark 2.2 to the multivariate case. The condition $\Psi(t_1, \dots, t_n) = s$ is equivalent with

$$\left\{\begin{array}{l} \varphi_{t_1,\dots,t_{n-1}}(t_n) \leq s \\ \varphi_{t_1,\dots,t_{n-1}}(t_n) > s - \epsilon \text{ for all } \epsilon > 0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} t_n \leq \varphi^{-1\bullet}_{t_1,\dots,t_{n-1}}(s) \\ t_n > \varphi^{-1\bullet}_{t_1,\dots,t_{n-1}}(s-\epsilon) \text{ for all } \epsilon > 0 \end{array}\right.$$

i.e.

$$\varphi^{-1\bullet}_{t_1,\dots,t_{n-1}}(s-) \leq t_n \leq \varphi^{-1\bullet}_{t_1,\dots,t_{n-1}}(s).$$

The increasingness of $C_L(F_1(t_1), F_2(t_2))$ ensures that the supremum over $\{(t_1, \dots, t_n) \in I\!R^n | \Psi(t_1, \dots, t_n) = s\}$ is taken at the right endpoint of the interval $[\varphi^{-1\bullet}_{t_1,\dots,t_{n-1}}(s-), \varphi^{-1\bullet}_{t_1,\dots,t_{n-1}}(s)]$ implying that (2.5) readily extends to dimension ≥ 3 as

$$F_{\min}(s|\Psi) = \sup_{(t_1,\dots,t_n)\in I\!R^n | \Psi(t_1,\dots,t_n)=s} C_L\big(F_1(t_1), \dots, F_n(t_n)\big).$$

Exactly as in the bivariate case, a representation like (2.6) is in general not valid.

Note that the computation of $F_{\min}(.|\Psi)$ and $F_{\max}(.|\Psi)$ in (3.3) involves an optimum over a surface. In practice, $F_{\min}(.|\Psi)$ and $F_{\max}(.|\Psi)$ can be accurately evaluated iteratively provided that Ψ can be written as

$$\Psi(x_1,\ldots,x_n)=\varrho\,(\ldots\varrho(\varrho(\varrho(x_1,x_2),x_3),x_4)\ldots,x_n)$$

for some function $\varrho:\mathbb{R}^2\to\mathbb{R}$; see Denuit, Genest and Marceau (1999, Section 3).

It is worth mentioning that the results of Proposition 3.1 can be adapted to functions Ψ non-increasing in some of their arguments and non-decreasing in the others.

3.3 Improvement of the bounds

Now, assume as in Subsection 2.3 that we have at our disposal some partial knowledge of the dependence structure existing between the X_i's, namely that we have a copula C_0 providing a lower bound for C, *id est*

$$C(t_1,\cdots,t_n)\geq C_0(t_1,\ldots,t_n)\text{ for all }(t_1,\ldots,t_n)\in[0,1]^n, \tag{3.4}$$

and a dual copula C_1^d providing an upper bound for C^d, *id est*

$$C^d(t_1,\ldots,t_n)\leq C_1^d(t_1,\ldots,t_n)\text{ for all }(t_1,\ldots,t_n)\in[0,1]^n. \tag{3.5}$$

In such a case, we are in a position to prove the following result, which provides better bounds on the cdf of $\Psi(X_1,\ldots,X_n)$ than $F_{\min}(.|\Psi)$ and $F_{\max}(.|\Psi)$ in (3.3).

Proposition 3.3. *Let $(X_1,\ldots,X_n)$ be a n-dimensional random vector with marginals $F_1,\ldots,F_n$ satisfying (3.4) and (3.5). Given a non-decreasing and continuous function $\Psi:\mathbb{R}^n\to\mathbb{R}$, the inequalities*

$$F_{\min}(s|\Psi,C_0)\leq P[\Psi(X_1,\ldots,X_n)\leq s]\leq F_{\max}(s|\Psi,C_1) \tag{3.6}$$

hold for all $s\in\mathbb{R}$, with

$$F_{\min}(s|\Psi,C_0)=\sup_{(t_1,\ldots,t_{n-1})\in\mathbb{R}^{n-1}}C_0\big(F_1(t_1),\ldots,F_{n-1}(t_{n-1}),F_n(\varphi^{-1\bullet}_{t_1,\ldots,t_{n-1}}(s))\big),$$

and

$$F_{\max}(s|\Psi,C_1)=\inf_{(t_1,\ldots,t_{n-1})\in\mathbb{R}^{n-1}}C_1^d\big(F_1(t_1),\ldots,F_{n-1}(t_{n-1}),F_n(\varphi^{-1\bullet}_{t_1,\ldots,t_{n-1}}(s))\big).$$

Obviously, Remark 3.2 also applies in this case (simply substituting C_0 for C_L or C_1^d for C_L^d).

For instance, when (3.4) is satisfied with

$$C_0(t_1,\ldots,t_n)=\prod_{i=1}^n t_i,$$

the vector $(X_1, \ldots, X_n)$ is said to be positively lower orthant dependent; when (3.5) is fulfilled with

$$C_1^d(t_1, \ldots, t_n) = 1 - \prod_{i=1}^{n}(1 - t_i),$$

then $(X_1, \ldots, X_n)$ is said to be positively upper orthant dependent, and finally, when (3.4) and (3.5) are simultaneously verified, $(X_1, \ldots, X_n)$ is said to be positively orthant dependent. For more details, the interested reader is referred to Szekli (1995, pp. 144-145) and Denuit, Genest and Marceau (2001), for instance.

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