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A SIMPLE GEOMETRIC PROOF THAT COMONOTONIC RISKS HAVE THE CONVEX-LARGEST SUM

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A simple geometric proof that comonotonic risks have the convex-largest sum

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Abstract

In the recent actuarial literature, several proofs have been given for the fact that if a random vector $(X_1, X_2, \ldots, X_n)$ with given marginals has a comonotonic joint distribution, the sum $X_1 + X_2 + \cdots + X_n$ is the largest possible in convex order. In this note we give a lucid proof of this fact, based on a geometric interpretation of the support of the comonotonic distribution.

1 Introduction

Up to a few years ago, when dealing with stochastic orderings, actuarial risk theory has focused on single risks or sums of independent risks. Here risks denote non-negative random variables such as they occur in the individual and the collective model, see, e.g., Kaas et al. (1994, 2001). Lately, with an eye on financial actuarial applications, the attention has shifted to sums $X_1 + X_2 + \cdots + X_n$ of random variables that may also have negative values. Moreover, their independence is no longer required. Only the marginal distributions are assumed to be fixed. A central result is that in this situation, the sum of the components $X_1 + X_2 + \cdots + X_n$ is the riskiest if the joint distribution of the random variables X_i is *comonotonic*. This means that the support of $(X_1, X_2, \ldots, X_n)$ has the property that each two points in this support have ordered components.

Perhaps due to the fact that so far no standard actuarial textbook has dealt with this topic, independent proofs of this result have appeared in several papers lately. Since Schmeidler (1986), Yaari (1987) and Roëll (1987), the concept of comonotonicity has played an important role in the economic theories of decision

under risk and uncertainty. Hoeffding (1940) and Fréchet (1951) studied the comonotonic distribution without actually proving the sum of the components to be riskier. As far as we know, this result was first mentioned in the actuarial literature in Heilmann (1986), who attributes it to Meilijson and Nadas (1979). Proofs of it involving the more general concept of supermodularity are contained in Müller (1997) (see also Bäuerle and Müller, 1998), Goovaerts and Dhaene (1999) and Goovaerts and Redant (1999). In Goovaerts and Kaas (2001) a proof is given involving limits for random variables with finitely many values. Proofs for the special case $n = 2$ and for the individual life model (two-point distributions) are to be found in Dhaene & Goovaerts (1996) and Dhaene & Goovaerts (1997) respectively. Goovaerts et al. (2000) considers only continuous random variables. In Kaas et al. (2000), see also De Vylder and Dhaene (2001), a general proof is given using an extension of the notion inverse distribution function. Dhaene et al. (2000) prove a slightly more general result. See also Dhaene and Denuit (1999), Wang and Dhaene (1998) and Wang and Young (1998).

A drawback of all these proofs is that none of them contributes a lot to the intuitive understanding of the concepts involved. In this short note, we give a transparent proof which is based on the geometric properties of the comonotonic support, and which we think is more suitable for classroom use. Also, it might inspire others to work on applications of comonotonic risks.

In Section 2, we define the order concepts used, convex order and stop-loss order. In Section 3, we describe the support and the joint cdf of the comonotonic distribution. Section 4 provides simple continuous and discrete examples. Section 5 finally gives the proof that comonotonicity implies a convex-largest sum of the components of a random vector.

2 Convex order and stop-loss order

The natural ordering concept in actuarial science is the *stop-loss order*. A random variable X is less than Y in stop-loss order, written $X \leq_{SL} Y$, if their net stop-loss premiums satisfy

$$E[(X-d)_+] \leq E[(Y-d)_+] \text{ for all real } d.$$

This order has many useful invariance properties. For instance, stop-loss order survives the operations of convolution and compounding on non-negative random variables (risks), and stop-loss larger claims lead to increased ruin probability and higher zero-utility premiums for risk averse decision makers. Risk X is preferred over Y either because it represents a smaller loss, or because it is less spread. See for instance Kaas et al. (1994, 2001) in the framework of actuarial sciences.

The quantity $E[(X-d)_+]$ represents the expected loss over d. With stability in mind, not only excessively large positive values of a random variable are unattractive, but also negative ones. Hence $E[(-X-t)_+]$ should be small too. So in that case, random variable X is preferred over Y if for all real $d=-t$, both

$$\begin{aligned} E[(X-d)_+] &\leq E[(Y-d)_+], \text{ and} \\ E[(d-X)_+] &\leq E[(d-Y)_+]. \end{aligned}$$

If this holds, we say that X is less than Y in *convex order*, written as $X \leq_{cx} Y$. Note that adding d to the first set of inequalities and letting $d \to -\infty$ leads to $E[X] \leq E[Y]$. Subtracting d in the second set of inequalities and letting $d \to +\infty$ produces $E[X] \geq E[Y]$, hence $E[X] = E[Y]$ must hold for two random variables to be comparable in convex order. On the other hand, the first set of inequalities together with equal means can be shown to imply the second set.

Stop-loss order can be shown to be the same as having ordered expected values $E[f(X)]$ for all *non-decreasing* convex functions $f(\cdot)$, see e.g. Goovaerts et al. (1990). Hence it represents the common preferences of all risk-averse decision makers. On the other hand, convex order is the same as ordered expectations for *all* convex functions, see e.g. the standard work on stochastic orders in a more general framework, Shaked and Shanthikumar (1994). This is of course where the name convex order comes from. In a utility context, it represents the common preferences of all risk-averse decision makers between random variables with equal mean. The proof that convex order implies ordered expectations of convex functions generally relies on the classical argument that any convex function can be obtained as the uniform limit of a sequence of piecewise linear functions, each of them being expressible as a linear combination of functions $(x-t)_+$ and $(t-x)_+$. A somewhat simpler proof, relying on partial integration, is given below, not because it is new, but just to keep the paper self-contained.

Lemma 1 *If $X \leq_{cx} Y$ and $f(\cdot)$ is convex, then $E[f(X)] \leq E[f(Y)]$.*

Proof. Consider the function $g(x) = f(x) - f(0) - xf'(0)$. Since $E[X] = E[Y]$, the inequality $E[f(X)] \leq E[f(Y)]$ is equivalent to $E[g(X)] \leq E[g(Y)]$. Write $\overline{F}(x) = 1 - F(x)$. Since $g(0) = g'(0) = 0$, the integrated terms below vanish, so by two partial integrations we get

$$\begin{aligned} E[g(X)] &= \int_{-\infty}^{0} g(x)dF(x) - \int_{0}^{+\infty} g(x)d\overline{F}(x) \\ &= \int_{-\infty}^{0} g'(x)F(x)dx + \int_{0}^{+\infty} g'(x)\overline{F}(x)dx \\ &= \int_{-\infty}^{0} E[(x-X)_+]dg'(x) + \int_{0}^{+\infty} E[(X-x)_+]dg'(x) \end{aligned}$$

from which the result immediately follows, because by the convexity of $f(\cdot)$ and also $g(\cdot)$, we have $dg'(x) \geq 0$ for all x. ■

3 The support of the comonotonic distribution

We start by defining comonotonicity of a set of real n-vectors in R^n. When the support of a random vector is a comonotonic set, also the random vector itself as well as its distribution are called comonotonic.

Definition 1 *The set S in R^n is said to be comonotonic, if, for all $(y_1, y_2, \ldots, y_n)$ and $(z_1, z_2, \ldots, z_n)$ in this set, $y_i \leq z_i$ for some i implies $y_j \leq z_j$ for all j.*

Notice that a comonotonic set is a 'thin' set. Since the upper left and lower right corners of a rectangle may not both be in it, it must be (a subset of) a curve that is monotonically non-decreasing in each component. It cannot contain any subset of dimension larger than 1.

Proposition 2 *The connected closure $\overline{S}$ of the comonotonic set S is a continuous curve which is comonotonic.*

Proof. First form the closure of the set S, by adding to S all its accumulation points. We then have a series of connected closed curves, which still form a comonotonic set. Next, connect the endpoints of consecutive curves by straight lines. Note that this has to be done only countably many times. We are left with $\overline{S}$, which is a continuously increasing curve in R^n. ■

The set $\overline{S}$, having no more 'holes 'and missing pieces, might be parametrized by non-decreasing continuous functions such that

$$\overline{S} = \{(f_1(z), \ldots, f_n(z)) \mid -\infty < z < \infty\}.$$

Hence, it may be traversed in an upward direction (with increasing z).

Proposition 3 *Let the support of random vector $(Y_1, Y_2, \ldots, Y_n)$ be contained in the connected closed curve $\overline{S}$. Then the joint cdf of $(Y_1, Y_2, \ldots, Y_n)$ must have the following form:*

$$\Pr[Y_1 \leq y_1, \ldots, Y_n \leq y_n] = \min_{j=1,\ldots,n} \Pr[Y_j \leq y_j].$$

Proof. We are looking for the total probability of the region $R = R_1 \cap R_2 \cap \cdots \cap R_n$, where $R_j, j = 1, 2, \ldots, n$ is defined as the region $\{t \in R^n | t_j \leq y_j\}$. As vector s traverses $\overline{S}$ in the upward direction, it must reach one of the boundary planes $\{t \in R^n | t_j = y_j\}, j = 1, 2, \ldots, n$ of this region first. Let k be the index corresponding to this boundary plane. Then, $\Pr[Y_k \leq y_k] = \min_j \Pr[Y_j \leq y_j]$ and $\overline{S} \cap R = \overline{S} \cap R_k$ are obvious from the geometric properties of $\overline{S}$. Hence the event $Y_1 \leq y_1, \ldots, Y_n \leq y_n$ has the same probability as $Y_k \leq y_k$, and the proof is completed. ■

For instance in Figure 2 below, $\overline{S}$ is the dotted line. To compute the joint cdf at $(\frac{1}{2}, \frac{3}{2})$, observe that going along $\overline{S}$ upwards, border $x = \frac{1}{2}$ is crossed at $(\frac{1}{2}, 1)$, while the other border $y = \frac{3}{2}$ is crossed only at $(\frac{3}{2}, \frac{3}{2})$. The set of points with positive probability to the left of and below $(\frac{1}{2}, \frac{3}{2})$ coincides with the corresponding set to the left of $x = \frac{1}{2}$.

Consider some cumulative univariate distribution function F. It is well-known that if $U \sim$ Uniform(0,1), the random variable $F^{-1}(U)$ is distributed according to F (probability integral transform). Note that it is irrelevant how we define $y = F^{-1}(u)$ for arguments u where there is an ambiguity, i.e., where $F(y) = u$ holds for an interval of y-values. For the same reason that the cdf of a random variable can have only countably many jumps, it can be shown that there can only be countably many such horizontal segments. [To see this, observe that, in the interval $[-2^n, 2^n]$, there are only finitely many intervals with a length over 2^{-n} where $F(y)$ is constant, and let $n \to \infty$.] Hence, if $g(\cdot)$ and $h(\cdot)$ are two different choices for the inverse cdf, $g(U)$ and $h(U)$ will be equal with probability one. The customary choice is taking $F^{-1}(u)$ to be the left-hand endpoint of the interval of y-values (generally containing one point only) with $F(y) = u$. Then, $F^{-1}(\cdot)$ is non-decreasing and continuous from the left.

Now consider any random vector $(X_1, X_2, \ldots, X_n)$. We have:

Proposition 4 *The following random vector has a comonotonic support, and moreover, it has the same marginal distributions as* $(X_1, X_2, \ldots, X_n)$*:*

$$(Y_1, Y_2, \ldots, Y_n) = (F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_n}^{-1}(U)).$$

Proof. If $(y_1, y_2, \ldots, y_n)$ and $(z_1, z_2, \ldots, z_n)$ are in the support of $(Y_1, Y_2, \ldots, Y_n)$ with $F_{X_i}^{-1}(u) = y_i \leq z_i = F_{X_i}^{-1}(v)$, then $u \leq v$ must hold, and hence $y_j \leq z_j$ for all $j = 1, 2, \ldots, n$.

For the marginals, we have $\Pr[F_{X_i}^{-1}(U) \leq x] = \Pr[U \leq F_{X_i}(x)] = F_{X_i}(x)$ for all x. ■

The support of $(Y_1, Y_2, \ldots, Y_n)$ as in Proposition 4 is the set

$$\left\{ \left(F_{X_1}^{-1}(u), F_{X_2}^{-1}(u), \ldots, F_{X_n}^{-1}(u) \right) | 0 < u < 1 \right\}.$$

Note that, by Proposition 3, any other comonotonic random vector with the same marginals as $(Y_1, Y_2, \ldots, Y_n)$ has the same cdf. In this sense, $(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_n}^{-1}(U))$ is the unique comonotonic random vector with the same marginals as $(X_1, X_2, \ldots, X_n)$.

The following result can be found in Hoeffding (1940) and Fréchet (1951). See also Bäuerle and Müller (1998). In our setup, it can be deduced directly from Propositions 3 and 4.

Corollary 5 *The joint cdf of the comonotonic random vector* $(Y_1, Y_2, \ldots, Y_n)$ *with the same marginal distributions as* $(X_1, X_2, \ldots, X_n)$ *satisfies:*

$$\Pr[Y_1 \leq y_1, \ldots, Y_n \leq y_n] = \min_{j=1,\ldots,n} \Pr[X_j \leq y_j].$$

Note that this proposition implies that the comonotonic cdf is as large as it can possibly be while still having the required marginal distributions. It is equal to an upper bound for it. By the same token, the joint probability of all Y_j having large values is also maximized. The probability of having some Y_j large, some small, is minimized, thus eliminating hedging possibilities. In this light, it is easy to see why the sum $Y_1 + Y_2 + \cdots + Y_n$ is as variable as possible when the Y_j are comonotonic, see Section 5.

4 Two simple examples

First, we give a continuous example with $n = 3$. Let $X \sim$ Uniform on the set $(0, \frac{1}{2}) \cup (1, \frac{3}{2})$, $Y \sim$ Beta(2,2), hence $F_Y(y) = 3y^2 - 2y^3$ on $(0, 1)$, and $Z \sim$ Normal$(0, 1)$. The support of the comonotonic distribution is the set

$$\{(F_X^{-1}(u), F_Y^{-1}(u), F_Z^{-1}(u)) | 0 < u < 1\}.$$

See Figure 1. Actually, not all of the support is depicted. The part left out corresponds to $u \notin (\Phi(-2), \Phi(2))$ and extends along the asymptotes, the vertical lines $(0, 0, z)$ and $(\frac{3}{2}, 1, z)$. The thick continuous line is S, while the dotted line is the straight line needed to make S into the connected curve $\overline{S}$. Note that F_X has a horizontal segment between $\frac{1}{2}$ and 1. The projection of $\overline{S}$ along the z-axis can also be seen to constitute an increasing curve, as do projections along the other axes.

For a discrete example, take $X \sim$ Uniform$\{0, 1, 2, 3\}$ and $Y \sim$ Binomial$(3, \frac{1}{2})$. It is easy to verify that

$$(F_X^{-1}(u), F_Y^{-1}(u)) \quad = \quad (0, 0) \text{ for } 0 < u < \frac{1}{8},$$

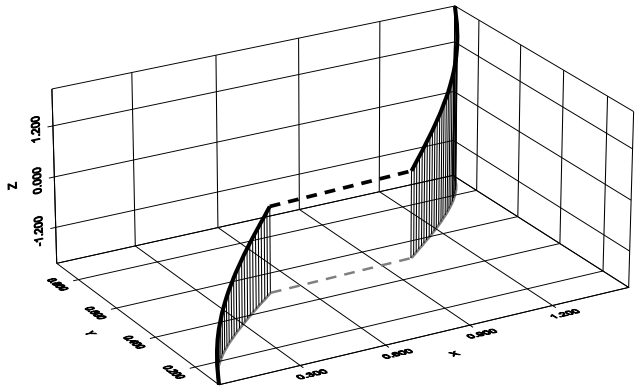

Figure 1: A continuous example with $n = 3$.

$$
\begin{aligned}
&= (0,1) \text{ for } \frac{1}{8} < u < \frac{2}{8}, \\
&= (1,1) \text{ for } \frac{2}{8} < u < \frac{4}{8}, \\
&= (2,2) \text{ for } \frac{4}{8} < u < \frac{6}{8}, \\
&= (3,2) \text{ for } \frac{6}{8} < u < \frac{7}{8}, \\
&= (3,3) \text{ for } \frac{7}{8} < u < 1.
\end{aligned}
$$

The support S of the comonotonic distribution is just these six points, and the curve $\overline{S}$ arises by simply connecting them consecutively with straight lines, the dotted lines in Figure 2. At the boundaries of the intervals for u, one may take the limit from either the left or the right. The straight line connecting $(1,1)$ and $(2,2)$ is not along one of the axes. This happens because at level $u = \frac{1}{2}$, both $F_X(y)$ and $F_Y(y)$ have horizontal segments. Note that any non-decreasing curve connecting $(1,1)$ and $(2,2)$ would have led to a feasible $\overline{S}$. These two points have probability $\frac{2}{8}$, the other points $\frac{1}{8}$.

5 Comonotonicity and convex ordered sums

After all this work, harvesting our main result is quite easy.

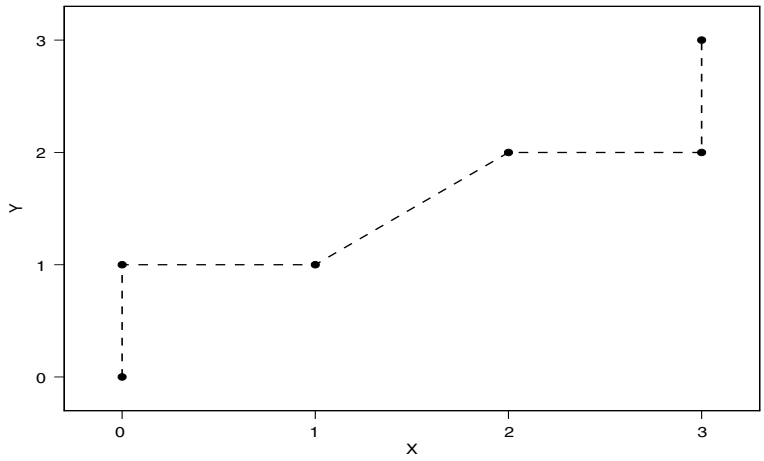

Figure 2: A discrete example.

Theorem 6 *If the random vector* $(Y_1, Y_2, \ldots, Y_n)$ *is comonotonic and has the same marginals as* $(X_1, X_2, \ldots, X_n)$*, then*

$$X_1 + X_2 + \cdots + X_n \leq_{cx} Y_1 + Y_2 + \cdots + Y_n.$$

Proof. It suffices to prove $\leq_{SL}$, since it is obvious that the means of these two random variables are equal. The following holds for all $(x_1, x_2, \ldots, x_n)$ when $d_1 + d_2 + \cdots + d_n = d$:

$$\begin{aligned} & (x_1 + x_2 + \cdots + x_n - d)_+ \\ = \; & \{(x_1 - d_1) + (x_2 - d_2) + \cdots + (x_n - d_n)\}_+ \\ \leq \; & \{(x_1 - d_1)_+ + (x_2 - d_2)_+ + \cdots + (x_n - d_n)_+\}_+ \\ = \; & (x_1 - d_1)_+ + (x_2 - d_2)_+ + \cdots + (x_n - d_n)_+. \end{aligned}$$

Assume that d is such that $0 < P[Y_1 + Y_2 + \cdots + Y_n \leq d] < 1$ holds; if not, the stop-loss premiums of $Y_1 + Y_2 + \cdots + Y_n$ and $X_1 + X_2 + \cdots + X_n$ can be seen to be equal. The connected curve $\overline{S}$ containing the support S of the comonotonic random vector $(Y_1, Y_2, \ldots, Y_n)$ points upwards in all directions, so it is obvious that $\overline{S}$ has exactly one point of intersection with the hyperplane $\{(x_1, \ldots, x_n) | x_1 + \cdots + x_n = d\}$. Let's assume this point of intersection is $(d_1, d_2, \ldots, d_n)$. But then for all points $(y_1, y_2, \ldots, y_n)$ in the support S of $(Y_1, Y_2, \ldots, Y_n)$, we have the following *equality:*

$$(y_1 + y_2 + \cdots + y_n - d)_+ \equiv (y_1 - d_1)_+ + (y_2 - d_2)_+ + \cdots + (y_n - d_n)_+.$$

This is because whenever $y_j > d_j$ for any j, also have $y_k \geq d_k$ for all k; when all $y_j \leq d_j$, obviously the left hand side is 0 as well.

Now replacing constants by the corresponding random variables in the two relations above and taking expectations, we get, since X_j and Y_j have the same marginal distribution for all j,

$$\begin{aligned} & E[(Y_1 + Y_2 + \cdots + Y_n - d)_+] \\ = \; & E[(Y_1 - d_1)_+] + E[(Y_2 - d_2)_+] + \cdots + E[(Y_n - d_n)_+] \\ = \; & E[(X_1 - d_1)_+] + E[(X_2 - d_2)_+] + \cdots + E[(X_n - d_n)_+] \\ \geq \; & E[(X_1 + X_2 + \cdots + X_n - d)_+]. \end{aligned}$$

■

Note that having fixed $\overline{S}$ in a particular instance, in principle we can determine $d_1, d_2, \ldots, d_n$ as in the proof of Theorem 4 for every d, and using these we can express the stop-loss premiums of the comonotonic sum in the marginal stop-loss premiums $E[(X_i - d_i)_+]$, see the last equality in the proof just above. Also, for the inverse cdf of the comonotonic sum in terms of the (inverse) cdfs of the random variables X_i, we have the following expression:

$$F^{-1}_{Y_1+\ldots+Y_n}(u) = F^{-1}_{X_1}(u) + \ldots + F^{-1}_{X_n}(u).$$

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