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ON S-CONVEXITY AND RISK AVERSION

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Abstract

The present note first discusses the concept of s-convex pain functions in decision theory. Then, the economic behavior of an agent with such a pain function is represented through the comparison of some recursive lotteries.

Key words: Expected utility theory, Actuarial studies, s-convex pain functions, Stochastic s-convex orders, Recursive lotteries.

1 Introduction

In the expected utility theory (Von Neumann and Morgenstern (1947)), each decision-maker is assumed to possess a utility function that expresses his preferences. To be specific, a utility function u assigns a value $u(x)$ to a monetary amount of \$ x; u describes "how much the decision-maker appreciates a fortune of \$ x". The decision-maker is then said to base his preferences on the "expected utility hypothesis" if he acts in order to maximize his expected utility. This means that for all uncertain future fortunes X and Y, the decision-maker with utility u will prefer Y over X (denoted as $X \preceq_u Y$ in the remainder of the paper) if, and only if, $Eu(X) \leq Eu(Y)$, provided that the expectations exist. Since two random fortunes X and Y with the same distribution function have the same expected utility, the decision-maker will be indifferent between them. This means that the individual's preferences structure is a weak order relation (reflexive, transitive, and total) on the set of the distribution functions of random fortunes; $\preceq_u$ is not antisymmetric but it defines equivalence classes on which a total order exists. For more details about expected utility theory, the interested reader is referred e.g. to Fishburn (1982a) or Schmidt (1998).

All the reasonable utility functions share certain common properties. For instance, it seems natural that an individual always prefers more wealth to less wealth. Therefore, the utility function u is always assumed to be non-decreasing. If u is differentiable, the lure of gain is expressed by the condition $u^{(1)} \geq 0$ (where $u^{(s)}$ denotes the s-th derivative of the function u). Another important concept is the notion of risk aversion: a decision-maker is said to be risk averse when his utility function is concave on its domain. If u is assumed to be twice differentiable, this reduces to $u^{(2)} \leq 0$. One way to justify the concavity assumption is to remark that, given any concave function u, $E[u(X)] \leq u(E[X])$ holds by Jensen inequality for all random fortunes X. Therefore, a risk averse decision-maker always prefers a certain fortune to a random income with the same expected value. A detailed discussion of risk aversion can be found e.g. in Scarsini (1994).

Expected utility theory has greatly contributed to understanding the economics of risk and uncertainty for the past several decades. As Trowbridge (1989) pointed out, utility theory forms also the philosophical basis of actuarial sciences, and yet this has received moderate attention. We mention e.g. Borch (1974) and Bowers *et al.* (1996).

The framework of the present note is decision theory, especially in an actuarial context. A rather natural concept directly linked with a utility function is the so-called pain function that represents some kind of pain associated with a debt by a decision-maker. Our purpose hereafter will be twofold. Firstly, we will discuss the property of s-convexity for such pain functions, and the related s-convex stochastic orderings for comparing two risks. Then, to enlighten the economic meaning of these notions, we will construct recursively a lottery that generates two risks which can be ordered in the s-convex sense by a decision-maker with a s-convex pain function.

2 *S*-Convex Pain Functions

The central notion in actuarial mathematics is the notion of risk that can be described as an event solely due to the whims of fate that brings about some financial loss. The actuary models these risks with the aid of non-negative random variables. The latter represent the random amounts of money the insurance company will have to pay out in order to indemnify the policyholder and/or the third party for the consequences of the occurrence of the insured risk.

Let u be a non-decreasing utility function for a decision-maker. Obviously, the function v defined by

$$v(x) = -u(-x), \quad x \in \mathbb{R}, \tag{2.1}$$

is also a non-decreasing utility function. According to Denuit, Dhaene and Van Wouwe (1999), v will be said to be the pain function of the decision-maker, because $v(x)$ may be viewed as the pain associated with a debt of amount \$ x. Moreover, it is easy to verify that u is convex if and only if v is concave. Hence, saying that a decision-maker with utility function u is risk averse is equivalent to saying that his pain function v is convex.

Let us now consider a risk X. We have that $Eu(-X) = -Ev(X)$. Thus, we get the following equivalence for risks X and Y:

$$E\left[u(-X)\right] \geq E\left[u(-Y)\right] \Leftrightarrow E\left[v(X)\right] \leq E\left[v(Y)\right]. \tag{2.2}$$

In terms of the pain functions, the expected utility hypothesis states that a risk X is preferred over a risk Y (denoted as $X \preceq_v Y$) if, and only if, $Ev(X) \leq Ev(Y)$ provided that the expectations exist, i.e.

$$-Y \preceq_u -X \Leftrightarrow X \preceq_v Y. \tag{2.3}$$

A main criticism addressed to expected utility theory is that ordering of risks depends on a subjective utility function, unknown to an objective observer. In most practical situations, it is indeed extremely difficult to find an explicit expression for a decision-maker's utility function u. Therefore, several authors suggested to focus on the common preferences shared by all the members of classes of reasonable decision-makers, and this gave rise to the theory of the integral stochastic orderings (see e.g. Mosler and Scarsini (1991), as well as the references therein). Note that whereas each individual of the class totally orders the risk, albeit differently, their common preferences only generate a partial ordering. Such an order is useful because it gives information about the risk-preferences of an actuary based on the distribution of the risk alone, not on the actual utility function, except that it satisfies some general conditions. Henceforth, we say that a risk X is dominated by another risk Y in the stochastic ordering $\preceq_{\mathcal{F}}$ if

$$X \preceq_v Y \text{ for all the pain functions } v \text{ in a class } \mathcal{F}. \tag{2.4}$$

Let us point out that we implicitly assume that the amount of premium charged for X and Y are equal. In practical situations, this often comes down to suppose that X and Y have the same first few moments (since most premium calculation principles are based on such moments of the risk to be covered).

The reader interested in actuarial applications of stochastic orderings is referred e.g. to the books by Goovaerts *et al.* (1990), Kaas *et al.* (1994), De Vylder (1996) and Hürlimann (1998). For a general overview of this topic in probability and statistics, see Shaked and Shanthikumar (1994).

Now, for the definition of $\mathcal{F}$ above, a first possibility consists in taking the set of all the pain functions v such that $v^{(1)} \geq 0$. The associated stochastic ordering is the well-known stochastic dominance ($\preceq_{st}$). Note that $X \preceq_{st} Y \Leftrightarrow -Y \preceq_{st} -X$ so that $X \preceq_{st} Y$ means that the loss X is preferred over the loss Y by all the profit-seeking decision-makers. A second possibility consists in choosing for $\mathcal{F}$ the set of all the pain functions v such that $v^{(1)}$ and $v^{(2)} \geq 0$. This leads to the stochastic ordering which is usually named the stop-loss order ($\preceq_{s\ell}$) in actuarial sciences. Obviously, $X \preceq_{s\ell} Y \Leftrightarrow -Y \preceq_{u} -X$ for all the utility functions u such that $u^{(1)} \geq 0$, $u^{(2)} \leq 0$, so that, given two risks X and Y, $X \preceq_{s\ell} Y$ means that the loss X is preferred over the loss Y by all the risk averse profit-seeking decision-makers.

In an attempt to generalize $\preceq_{st}$ and $\preceq_{s\ell}$, Goovaerts *et al.* (1990) have introduced stop-loss orders of higher degrees. For $s \in \mathbb{N}_0$, the s-th degree stop-loss order ($\preceq_{s-s\ell}$) is defined by means of (2.4) with for $\mathcal{F}$ the class of all the pain functions v such that $v^{(1)}, v^{(2)}, \ldots, v^{(s)} \geq 0$. Note that this order is sometimes referred to as the $(s+1)$-th degree (e.g. by Kaas and Hesselager (1995)). We also have that $X \preceq_{s-s\ell} Y \Leftrightarrow -Y \preceq_{u} -X$ for all the utility functions u such that $u^{(1)} \geq 0$, $u^{(2)} \leq 0$, ..., $(-1)^{s+1}u^{(s)} \geq 0$, so that $X \preceq_{s-s\ell} Y$ means that the loss X is preferred over the loss Y by all the decision-makers with non-decreasing utility functions with first s derivatives of alternating signs.

Recently, Denuit *et al.* (1998) have reinforced the $\preceq_{s-s\ell}$ orderings by requiring that, in addition, X and Y have the same first $s-1$ moments. These orders, called s-convex ($\preceq_{s-cx}$), are thus defined by

$$X \preceq_{s-cx} Y \Leftrightarrow \begin{cases} X \preceq_{s-s\ell} Y, \\ EX^k = EY^k \text{ for } k = 1, 2, \ldots, s-1. \end{cases}$$

It has been proved that $X \preceq_{s-cx} Y$ if, and only if, $X \preceq_{v} Y$ for all the pain functions v such that $v^{(s)} \geq 0$. Moreover, an equivalent characterization is as follows: let $\mathcal{U}_{s-cx}$ be the family of functions

$$\mathcal{U}_{s-cx} = \{v : \mathbb{R}^{+} \rightarrow \mathbb{R} | v^{(s)} \geq 0\},$$

and let $\overline{\mathcal{U}}_{s-cx}$ be the closure of $\mathcal{U}_{s-cx}$ in the topology of the pointwise convergence; then

$$X \preceq_{s-cx} Y \Leftrightarrow X \preceq_{v} Y \text{ for all the pain functions } v \in \overline{\mathcal{U}}_{s-cx}.$$

We recall that $\overline{\mathcal{U}}_{s-cx}$ is the set of the convex functions of degree s such as defined e.g. in Pečarić *et al.* (1992), and Roberts and Varberg (1973).

Now, for our purposes, it is useful to derive an equivalent representation of $\overline{\mathcal{U}}_{s-cx}$. Specifically, given any function $v : I\!R^+ \to I\!R$ and for $\epsilon > 0$, define recursively the usual forward differences as

$$\Delta_\epsilon^k u(x) = \Delta_\epsilon^{k-1} u(x+\epsilon) - \Delta_\epsilon^{k-1} u(x), \;\; k \geq 0,$$

with $\Delta_\epsilon^0 u \equiv u$. According to Definition 1.45 in Pečarić *et al.* (1992), the function v is said to be Jensen-convex of order s if

$$\Delta_\epsilon^s u(x) \geq 0 \text{ for all } x \in I\!R^+, \epsilon > 0.$$

Characterization 2.1 $\overline{\mathcal{U}}_{s-cx}$ *is the set of the functions that are Jensen-convex of order* s.

Proof. Consider $v \in \overline{\mathcal{U}}_{s-cx}$. By definition of $\overline{\mathcal{U}}_{s-cx}$, there exists a sequence $\{v_n, \;\; n \in I\!N\}$ of functions in $\mathcal{U}_{s-cx}$ such that $v_n(x) \to v(x)$ as $n \to +\infty$, for all $x \in I\!R^+$. Therefore, we also have that $\Delta_\epsilon^s v_n(x) \to \Delta_\epsilon^s v(x)$. Now, it is easily seen that

$$\Delta_\epsilon^s v_n(x) = \int_{t_1=0}^{\epsilon} \int_{t_2=0}^{\epsilon} \cdots \int_{t_n=0}^{\epsilon} v_n^{(s)}(x + t_1 + t_2 + \ldots + t_n) dt_n \ldots dt_2 dt_1,$$

so that $\Delta_\epsilon^s v_n(x) \geq 0$. Consequently, we deduce that $\Delta_\epsilon^s v(x) \geq 0$ as announced.

Conversely, consider a function u which is Jensen-convex of order s, and let us prove that v then belongs to $\overline{\mathcal{U}}_{s-cx}$. The proof of this point is rather technical and is therefore omitted. The interested reader is referred to Lemma 1 of Fishburn (1982b), as well as to the references mentioned there. $\diamond$

3 Economic comparison through recursive lotteries

As we said above, the meaning of a stochastic comparison in the sense of $\preceq_{st}$ or $\preceq_{s\ell}$ is quite speaking to the actuary. However, it is not clear which characteristics of rational behavior are shared by all the decision-makers with a s-convex pain function. To tackle the question, we are going to classify the behavior of economic agents by comparing recursive lotteries in the way discussed by Mosler and Scarsini (1991). The comparison will be performed recursively, each step refining the classification obtained at the previous step.

Specifically, let us consider a given pain function v. For any $\epsilon \in I\!R_0^+$, we put $v_0(.|\epsilon) \equiv v$ and we then define the sequence $v_k(.|\epsilon)$, $k \in I\!N_0$ as the successive forward differences of v, i.e.

$$v_k(x|\epsilon) = \Delta_\epsilon^k v(x), \;\; k \geq 1. \tag{3.1}$$

Obviously, we have

$$v_k(x|\epsilon) = \sum_{j=0}^{k} \left(\begin{array}{c} k \\ j \end{array} \right) (-1)^j v[x + (k-j)\epsilon], \;\; k \in I\!N_0. \tag{3.2}$$

We begin by observing that $v_1(x|\epsilon) \geq 0$ means that $v(x) \leq v(x+\epsilon)$, that is, the risk $X_1(x,\epsilon) \equiv x$ (almost surely) is preferred to the risk $Y_1(x,\epsilon) \equiv x+\epsilon$ (almost surely) by a decision-maker with pain function v. Furthermore, by Characterization 3.1, we know that $v_1(x|\epsilon) \geq 0$ for all $x \in I\!R^+$ and $\epsilon > 0 \Leftrightarrow v$ non-decreasing. Similarly, $v_2(x|\epsilon) \geq 0$ means that $v(x+\epsilon) \leq [v(x)+v(x+2\epsilon)]/2$, that is, the risk $X_2(x,\epsilon) \equiv x+\epsilon$ (almost surely) is preferred over the risk

$$Y_2(x,\epsilon) \equiv \begin{cases} x \text{ with the probability } 1/2, \\ x+2\epsilon \text{ with the probability } 1/2, \end{cases}$$

by a decision-maker with pain function v. Moreover, by Characterization 3.1, $v_2(x|\epsilon) \geq 0$ for all $x \in I\!R^+$ and $\epsilon > 0 \Leftrightarrow v$ convex.

Now, let us extend this construction by providing a sequence of risks $[X_s(x,\epsilon), Y_s(x,\epsilon)]$, $s \in I\!N_0$, such that for all $x \in I\!R^+$ and $\epsilon > 0$, the risk $X_s(x,\epsilon)$ is preferred to the risk $Y_s(x,\epsilon)$ by a decision-maker with pain function v that is s-convex.

For that, starting from $X_1(x,\epsilon)$ and $Y_1(x,\epsilon)$ given above, we define $X_s(x,\epsilon)$ and $Y_s(x,\epsilon)$, $s \geq 2$, by

$$X_s(x,\epsilon) \equiv \begin{cases} X_{s-1}(x,\epsilon)+\epsilon \text{ with probability } 1/2, \\ Y_{s-1}(x,\epsilon) \text{ with probability } 1/2, \end{cases} \tag{3.3}$$

and

$$Y_s(x,\epsilon) \equiv \begin{cases} X_{s-1}(x,\epsilon) \text{ with probability } 1/2, \\ Y_{s-1}(x,\epsilon)+\epsilon \text{ with probability } 1/2. \end{cases} \tag{3.4}$$

For $s=2$, for instance, (3.3) and (3.4) yield precisely the risks $X_2(x,\epsilon)$ and $Y_2(x,\epsilon)$ built before.

Lemma 3.1 *For $s \in I\!N_0$ and all $x \in I\!R^+$, $\epsilon > 0$,*

$$E[X_s(x,\epsilon)]^k = E[Y_s(x,\epsilon)]^k \text{ for } k = 0,1,\ldots,s-1. \tag{3.5}$$

Proof. We proceed by induction on s. For $s=1$, (3.5) is obviously true since

$$E[X_1(x,\epsilon)]^0 = 1 = E[Y_1(x,\epsilon)]^0.$$

Let us assume that (3.5) holds for $s-1 \in I\!N_0$. ¿From (3.3), we get, for all $k \in I\!N$,

$$\begin{aligned} E[X_s(x,\epsilon)]^k &= \frac{1}{2}\left(E[X_{s-1}(x,\epsilon)+\epsilon]^k + E[Y_{s-1}(x,\epsilon)]^k\right) \\ &= \frac{1}{2}\left(\sum_{j=0}^{k}\binom{k}{j}\epsilon^j E[X_{s-1}(x,\epsilon)]^{k-j} + E[Y_{s-1}(x,\epsilon)]^k\right), \end{aligned}$$

and from (3.4),

$$\begin{aligned} E[Y_s(x,\epsilon)]^k &= \frac{1}{2}\left(E[X_{s-1}(x,\epsilon)]^k + E[Y_{s-1}(x,\epsilon)+\epsilon]^k\right) \\ &= \frac{1}{2}\left(E[X_{s-1}(x,\epsilon)]^k + \sum_{j=0}^{k}\binom{k}{j}\epsilon^j E[Y_{s-1}(x,\epsilon)]^{k-j}\right). \end{aligned}$$

Since, by the recurrence hypothesis,

$$E[X_{s-1}(x,\epsilon)]^j = E[Y_{s-1}(x,\epsilon)]^j \text{ for } j = 0, 1, \ldots, s-2,$$

we then deduce that (3.5) is also valid for s as announced. $\diamond$

We notice that a similar induction argument yields, for all $x \in I\!R^+$, $\epsilon, x > 0$,

$$E[X_s(x,\epsilon)]^k \leq E[Y_s(x,\epsilon)]^k \text{ for } k \geq s.$$

Proposition 3.2 *For $s \in I\!N_0$ and all $x \in I\!R^+$, $\epsilon > 0$,*

$$X_s(x,\epsilon) \preceq_{s-cx} Y_s(x,\epsilon). \tag{3.6}$$

Proof. By construction, $X_s(x,\epsilon)$ and $Y_s(x,\epsilon)$ are discrete random variables valued in $[x, x+s\epsilon]$. Lemma 3.1 states that they have the same first $s-1$ moments, μ_1, μ_2, ..., μ_{s-1}, say. Thus, both $X_s(x,\epsilon)$ and $Y_s(x,\epsilon)$ belong to a common moment space, denoted by $\mathcal{B}_s([x, x+s\epsilon]|\mu_1, \mu_2, \ldots, \mu_{s-1})$.

Now, a careful examination of the support of these random variables shows that 1) $X_s(x,\epsilon)$ has, when s is even, $s/2$ support points inside $[x, x+s\epsilon]$ and, when s is odd, $(s-1)/2$ support points inside $[x, x+s\epsilon]$ plus the initial point x;
2) $Y_s(x,\epsilon)$ has, when s is even, $s/2-1$ support points inside $[x, x+s\epsilon]$ plus the extremal points x and $x+s\epsilon$, and when s is odd, $(s-1)/2$ support points inside $[x, x+s\epsilon]$ plus the final point $x+s\epsilon$.

In the terminology of De Vylder (1996), this means that $X_s(x,\epsilon)$ has a narrow (resp. initial) distribution when s is even (resp. odd), while $Y_s(x,\epsilon)$ has a broad (resp. final) distribution when s is even (resp. odd). By Proposition 4.1 in Denuit, De Vylder and Lefèvre (1999), we then directly deduce that $X_s(x,\epsilon)$ and $Y_s(x,\epsilon)$ correspond to the minimal and maximal distributions on $[x, x+s\epsilon]$ with respect to the s-convex stochastic ordering, respectively. In particular, this implies (3.6). $\diamond$

It is worth underlying that, as shown in the proof, the s-convexity of the pain function means that the decision-maker prefers the s-convex minimum $X_s(x,\epsilon)$ to the s-convex maximum $Y_s(x,\epsilon)$ among all the risks valued in $[x, x+s\epsilon]$, for all $x \in I\!R^+$, $\epsilon > 0$.

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