The practical replacement of a bonus-malus system

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Abstract

In this paper we will show how to set up a practical bonus-malus system with a finite number of classes. We will use the actual claim amount and claims frequency distributions in order to predict the future observed claims frequency when the new bonus-malus system will be in use. The future observed claims frequency is used to set up an optimal bonus-malus system as well as the transient and stationary distributions of the drivers in the new bonus-malus system. When the number of classes as well as the transition rules of the new bonus-malus system have been adopted, the premium levels are obtained by minimizing a certain distance between the levels of the practical bonus-malus system and the corresponding optimal bonus-malus system. Some iterations are necessary in order to reach stabilization of the future observed claims frequency and the levels of the practical bonus-malus system.

Keywords

Optimal bonus-malus system, practical bonus-malus system, hunger for bonus, observed claims frequency distribution, actual claims frequency distribution, iterative algorithm, Hofmann Distribution, non-parametric mixed Poisson fit.

1. Introduction

A practical bonus-malus system consists of a table with s classes. A premium level is associated with each class. The entry in the bonus-malus system is generally at level 100%. The levels under 100% are bonuses. The levels above 100% are maluses. A rule is chosen in order to let the drivers move within the bonus-malus system according to the number of claims they report each year to the Company. This is the transition rule. A classical reference for bonus-malus systems is Lemaire (1995).

In this paper we are interested in setting up a methodology so that we may change a bonus-malus system. This is a typical problem that Belgian Companies have to face. Indeed, for a long time, they have used a unique bonus-malus system which is now in contravention of the European directives. They may thus be forced to adapt their bonus-malus system.

Throughout the paper, we will use a numerical example in order to explain our methodology. The following characteristics will be used.
Let us assume that the current bonus-malus system is the following

- The number of classes s is 9 , numbered $0, 1, \ldots, 8$. 0 is the minimum class. 8 is the maximum class.
- Entry of the system is in class 4.
- In the case of a claims free year, the policyholder comes down one class
- In the case of claim(s) being reported, the policyholder climbs up 3 classes per claim.
- The bonuses and maluses (in percentage) are given in the following table :

i	0	1	2	3	4	5	6	7	8
C_i	75	80	90	95	100	150	170	185	250

Table 1: Premium levels of the current bonus-malus system

Let us assume that a new bonus-malus system will have the following characteristics :

- $s = 9$: 9 classes numbered $0, 1, \ldots, 8$. 0 is the minimum class. 8 is the maximum class.
- Entry of the system is in class 4.
- In the case of a claims free year, the policyholder comes down one class
- In the case of claim(s) being reported, the policyholder goes to level 8 whatever the number of claims.

Our problem is to find the bonuses and maluses (in percentage) for the new bonus-malus system.

In Walhin and Paris (2000a) it has been shown how to find the actual claim amount and claims frequency distributions of a motor portfolio if a bonus-malus system is used. The reason why the actual claims frequency distribution is not observed is that the bonus-malus system induces the hunger for bonus (see Lemaire (1977)) with some drivers. Lemaire (1977) derives an algorithm giving the optimal retention of a driver in function of his bonus-malus level. See also Walhin and Paris (2000a) for a short description of Lemaire's algorithm.

We will use the following portfolio which is observed within the current bonus-malus system :

Number of accidents	Number of policyholders
0	103704
1	14075
2	1766
3	255
4	45
5	6
6	2

Table 2: Observed claims frequency distribution

Based on a data set of the observed claim amounts, the actual claims frequency distribution of the drivers was deducted in Walhin and Paris (2000a) as well as the proportion of drivers using their optimal retention given by Lemaire's algorithm.
These results are summarized in section 3 after having recalled the conception of an optimal bonus-malus system in section 2. Section 4 shows how to set up a practical bonus-malus system from an optimal bonus-malus system. Section 5 uses Lemaire's algorithm in order to find iteratively the practical bonus-malus system that will be in accordance with the future observed claims frequency distribution within the new bonus-malus system. Section 6 is devoted to the conclusion.

We will use the following notations :

- N is the random variable representing the number of observed claims within the current bonus-malus system
- $N^{'}$ is the random variable representing the number of actual claims
- $N^{''}$ is the random variable representing the number of claims under the optimal policy for the drivers using Lemaire's algorithm within the current bonus-malus system
- $N^{'''}$ is the random variable representing the number of future observed claims within the new bonus-malus system
- $N^{''''}$ is the random variable representing the number of claims under the optimal policy for the drivers using Lemaire's algorithm within the new bonus-malus system

2. An optimal bonus-malus system

This section is mainly based on Walhin and Paris (1999) where further details can be found. It is generally considered that the number of claims in the interval $(0, t]$ $(N(t))$ is a mixed Poisson process :

$$\begin{aligned} \Pi(k, t|\Lambda) &= e^{-\Lambda t}\frac{(\Lambda t)^k}{k!} \\ \Pi(k, t) &= \int_0^\infty e^{-\lambda t}\frac{(\lambda t)^k}{k!}dU(\lambda) \end{aligned}$$

where Λ is a random variable with cdf $U(\lambda)$.
For the t^{th} year, we take into account the information which consists of the number of accidents during the first t years. The a posteriori premium is :

$$\begin{aligned} \mathbb{E}[N(t+1) - N(t)|N(t) = k] &= \mathbb{E}(\Lambda|N(t) = k) \\ &= \frac{k+1}{t}\frac{\Pi(k+1, t)}{\Pi(k, t)} \end{aligned}$$

This is an application of the expected value principle.
By giving the premium levels such a norm that the a priori premium is 100 it is possible to construct a table with two entries : t the number of observed years and k the number of observed claims. The table gives the premium levels in percentage in function of t and k. We will denote these levels by

$$P_{(k,t)} = \frac{100}{\mathbb{E}\Lambda}\frac{k+1}{t}\frac{\Pi(k+1, t)}{\Pi(k, t)} \tag{1}$$

t/k	0	1	2	3	4	$\ldots$
0	100	$-$	$-$	$-$	$-$	$\ldots$
1	$P_{(0,1)}$	$P_{(1,1)}$	$P_{(2,1)}$	$P_{(3,1)}$	$P_{(4,1)}$	$\ldots$
2	$P_{(0,2)}$	$P_{(1,2)}$	$P_{(2,2)}$	$P_{(3,2)}$	$P_{(4,2)}$	$\ldots$
$\vdots$	$\vdots$	$\ldots$				

Table 3: Optimal bonus-malus table : $P_{(k,t)}$

This formula is general and may be applied e.g. to

- a non-parametric mixed Poisson process

$$\Pi(k, t) = \sum_{j=1}^{r} p_j e^{-\lambda_j t}\frac{(\lambda_j t)^k}{k!} \quad , \quad k \geq 0$$

  where Λ takes discrete values λ_j with probabilities p_j , $j = 1, \ldots, r$.

- the Hofmann process

$$\begin{aligned}\Pi(0,t) &= e^{-\theta(t)} \\ (k+1)\Pi(k+1,t) &= \frac{pt}{(1+ct)^a}\sum_{i=0}^{k}\frac{\Gamma(a+i)}{\Gamma(a)i!}\left(\frac{ct}{1+ct}\right)^i \Pi(k-i,t) \quad , \quad k\geq 0\end{aligned}$$

In this case Λ is continuous and such that $\Pi(0,t) = e^{-\theta(t)}$ with $\theta(t)' = \frac{p}{(1+ct)^a}$. It is possible to write the density function of Λ but it is tedious and unnecessary.

In Walhin and Paris (1999) it is shown that the conception of an optimal bonus-malus system is not smooth enough if the non-parametric mixed Poisson fit is used. Therefore we will concentrate on the Hofmann process for the set up of an optimal bonus-malus system.
The problem is that the reported claims frequency distribution will be described using a non-parametric mixed Poisson distribution. We will choose the parameters p, c, and a by matching the first three moments of the non-parametric mixed Poisson distribution and the Hofmann distribution :

$$\begin{aligned}\sum_{j=1}^{r} p_j\lambda_j &= p \\ \sum_{j=1}^{r} p_j\lambda_j(\lambda_j+1) &= p + acp + p^2 \\ \sum_{j=1}^{r} p_j\lambda_j(\lambda_j^2+3\lambda_j+1) &= p(p+acp) + acp(2+p) + p(1+p)^2 + acp(1+p) + (1+a)ac^2p\end{aligned}$$

Another solution might be to equate the mean and the probabilities of 0 observation and 1 observation :

$$\begin{aligned}\sum_{j=1}^{r} p_j\lambda_j &= p \\ \sum_{j=1}^{r} p_j e^{-\lambda_j} &= e^{-\frac{p}{c(1-a)}((1+c)^{1-a}-1)} \\ \sum_{j=1}^{r} p_j\lambda_j e^{-\lambda_j} &= \frac{p}{(1+c)^a} e^{-\frac{p}{c(1-a)}((1+c)^{1-a}-1)}\end{aligned}$$

Note that formula (1) is based on the expected value principle. Other premium principles may be used in order to construct optimal bonus-malus tables. In Walhin and Paris (1999) the principle of zero utility with an exponential utility function is used but numerical examples show that this does not produce significantly different tables from the ones obtained with the expected value principle.
In Walhin and Paris (2000b) optimal bonus-malus tables are constructed with the use of exponential loss distributions. This gives a free parameter which the actuary can play with in order to introduce some solidarity in the bonus-malus table.

3. The hunger for bonus and the actual claims frequency distribution

This section is essentially based on Walhin and Paris (2000a) where details can be found. With the following hypothesis the algorithm of Lemaire (1977) gives the optimal policy (retention and frequency) of the driver as a function of his bonus-malus level.

Let a bonus-malus system be with s classes : $i = 0, \ldots, s-1$
the claims frequency of a policyholder be Poisson distributed with mean λ
the claim amount distribution be X, with cumulative density function (cdf) $F_X(x)$
β be the actualisation rate forecast for the future
P be the commercial premium, i.e. the base premium at level 100%, including security loading, administration expenses, profit and taxes.
$1-t$ with $0 \leq t < 1$ be the time remaining until the next premium payment
m be the number of claims reported to the Company in $[0, t)$

The observed claims frequency distribution (N) given in table 2 can be fitted by a mixed Poisson distribution. An interesting choice for the mixing distribution is the one leading to the non-parametric mixed Poisson fit, i.e. a mixture of various Poisson distributions. This makes the calculations with Lemaire's algorithm easy because it has to be run only for certain values of the parameter λ. Moreover, the stationary or transient distributions of the drivers within the bonus-malus system are easy to calculate for the same reason (see also Walhin and Paris (1999)). For our data set we find

$\lambda_1 = 0.05461$	$p_1 = 0.56189$
$\lambda_2 = 0.24599$	$p_2 = 0.41463$
$\lambda_3 = 0.95618$	$p_3 = 0.02348$

Table 4: Non-parametric maximum likelihood estimation for the observed portfolio

i.e. we find three types of drivers : 56% of the drivers show a claims frequency of 0.0546, 41% of the drivers show a claims frequency of 0.2459 and 2% of the drivers show a claims frequency of 0.9561. In general there are r types of drivers. Note that for classical motor portfolios r will be equal to 3 or 4 because the frequency is low, leading to a maximum number of claims per policyholder equal to about 7. The reason for being only 3 or 4 classes is given in Simar (1976) and discussed in Walhin and Paris (1999).

Taking into account the hunger for bonus and using this non-parametric fit, Walhin and Paris (2000a) have derived the actual claims frequency distribution of the drivers ($N^{'}$) :

$\lambda_1^{'} = 0.0637$	$p_1 = 0.56189$
$\lambda_2^{'} = 0.3628$	$p_2 = 0.41463$
$\lambda_3^{'} = 1.1107$	$p_3 = 0.02348$

Table 5: Parameters for the actual claims frequency distribution

The proportion $(1-p)$ of drivers using the algorithm of Lemaire is also obtained :

$$1 - p = 58.95\%$$

Note that these results were derived by using the following hypothesis :

- the bonus-malus system is the current bonus-malus system
- the claim amount distribution X is exponentially distributed
- a particular data set has been used for the observed claim amounts (see Walhin and Paris (2000a))
- $\beta = 6\%$
- $P = 35$
- $t = \frac{1}{2}$
- the proportions p_j remain the same with the observed and actual claims frequencies

The rest of the paper will be based on the following hypothesis regarding the introduction of a new bonus-malus system :

- the proportions p_j , $j = 1, \ldots, r$ remain the same for predicted future observed claims frequencies
- the proportion of drivers using Lemaire's algorithm remains the same. In fact this is an important information given by the current bonus-malus system. If we have to implement a bonus-malus system for the first time we have no idea of the value of p and a guess is needed. This in fact explains the title of this paper : The practical *replacement* of a bonus-malus system.

When the drivers use Lemaire's algorithm they will report claims with an optimal frequency distribution : $(\lambda_j^{''''}, p_j \, , \, j = 1, \ldots, r)$. The actual claims frequency distribution $(\lambda_j^{'}, p_j \, , \, j = 1, \ldots, r)$ is given in table 5. The future observed claims frequency distribution $(\lambda_j^{'''}, p_j \, , \, j = 1, \ldots, r)$ will be given by

$$\lambda_j^{'''} = p\lambda_j^{'} + (1-p)\lambda_j^{''''} \quad j = 1, \ldots, r \tag{2}$$

where the notation $(\lambda_j^{*}, p_j) \, , \, j = 1, \ldots, r$ gives the non-parametric mixed Poisson fit of the random variable N^{*} defined in the introduction.

4. A practical bonus-malus system

The optimal bonus-malus system described in section 2 seems difficult to apply because of the infinite number of classes in both directions k and t. Such a system will certainly be too complicated for the policyholders. Therefore most European countries use bonus-malus systems with a finite number of classes.

Basically, the bonus-malus systems with finite number of classes (s) have the Markov property (if not, a redefinition of the system may be needed to show the Markov property). They are presented as the examples of the introduction :

s	0	1	2	3	...	$s-1$
C_s	C_0	C_1	C_2	C_3	...	C_{s-1}

Table 6: Premium levels

Transition rules are given indicating how the drivers move inside the bonus-malus system according to their annual number of reported claims.
If we assume that the number of classes has been chosen as well as the transition rules, the premium levels need to be calculated. This is explained in Coene and Doray (1996) where they minimize a certain distance between the C_i and the $P_{(k,t)}$ in order to determine the C_i. We will use the same methodology as Coene and Doray (1996) with some amendments.
Firstly, it is necessary to construct a table of $C_{(k,t)}$ parallel to the table of $P_{(k,t)}$. $C_{(k,t)}$ gives the different possible values for C_i in function of k and t. For example with the new bonus-malus system described in the introduction we have

t/k	0	1	2	...
0	C_4	$-$	$-$	...
1	C_3	C_8	C_8	...
2	C_2	C_8/C_7	C_8/C_7	...
3	C_1	$C_8/C_7/C_6$	$C_8/C_7/C_6$	...
4	C_0	$C_8/C_7/C_6/C_5$	$C_8/C_7/C_6/C_5$	...
5	C_0	$C_8/C_7/C_6/C_5/C_4$	$C_8/C_7/C_6/C_5/C_4$	...
$\vdots$				

Table 7: $C_{(k,t)}$

Coene and Doray (1996) suggest to choose the maximum class for $C_{(k,t)}$ because it is often the most probable class.

However it is not difficult to find the probabilities of the following events : $[N(t)=k, C_{(k,t)}=C_i]$. Let us define

$$w(k,t,i)=\mathbb{P}[N(t)=k, C_{(k,t)}=C_i]$$

The following general formula is valid for a mixed Poisson process :

$$\mathbb{P}[N(n)-N(n-1)=k_n,\ldots,N(1)-N(0)=k_1]=\frac{\Pi(\sum_{i=1}^n k_i, n)}{n^{\sum_{i=1}^n k_i}}\frac{(\sum_{i=1}^n k_i)!}{k_1!\ldots k_n!}$$

Therefore it is always possible to write the probabilities $w(k,t,i)$ as a $\alpha\Pi(k,t)$ where α depends on the different ways to reach C_i at time t with k claims.
For example

$$\begin{aligned}
\mathbb{P}[C_{(1,1)}=C_8|N(1)=1] &= 1\\
\mathbb{P}[C_{(1,1)}=C_i|N(1)=1] &= 0 \quad \forall i<8\\
\mathbb{P}[C_{(1,2)}=C_8|N(2)=1] &= \frac{\mathbb{P}[N(1)-N(0)=0, N(2)-N(1)=1]}{\mathbb{P}[N(2)=1]}=\frac{1}{2}\\
\mathbb{P}[C_{(1,2)}=C_7|N(2)=1] &= \frac{\mathbb{P}[N(1)-N(0)=1, N(2)-N(1)=0]}{\mathbb{P}[N(2)=1]}=\frac{1}{2}\\
\mathbb{P}[C_{(1,2)}=C_i|N(2)=1] &= 0 \quad \forall i<7
\end{aligned}$$

The most obvious way of calculating the C_i is to minimize the following quadratic error with natural weights :

$$\sum_{(k,t,i)} w(k,t,i)[P_{(k,t)}-C_i]^2$$

This minimization procedure is quite similar to the one derived in Coene and Doray (1996) but it is more natural because we use the exact values of $C_{(k,t)}$ and we do not use the stationary distribution of the drivers in the bonus-malus system.
As mentionned in Coene and Doray (1996) we now have a problem of optimization with s variables. Of course some constraints have to be taken into account :

- The entry class in the bonus-malus system is generally at level 100. So in our example we have to constrain $C_4=100$
- Corresponding to the natural properties

$$\begin{aligned}
\frac{\partial}{\partial t}P_{(k,t)} &\leq 0 \quad \forall k\\
P_{(k+1,t)} &\geq P_{(k,t)} \quad \forall t
\end{aligned}$$

  we have the natural constraint : $C_i \leq C_{i+1} \quad \forall i$
- As we are interested in having integer percentages C_i, it may be interesting to constrain the C_i to being integer, i.e. to make optimization with integers.
- As a consequence of the financial equilibrium of the optimal bonus-malus system

$$\sum_{k=0}^{\infty}\Pi(k,t)P_{(k,t)}=100 \quad \forall t$$

the following constraints may be imposed

$$\begin{aligned}
\sum_{i=0}^{s-1} e_\infty(i) C_i &\geq 100 \\
\sum_{i=0}^{s-1} e_t(i, \mathbf{e_0}) C_i &\geq 100 \qquad \text{for some } t \text{ with } \mathbf{e_0} \text{ given}
\end{aligned}$$

where $e_\infty(i)$ denotes the i^{th} component of the stationary distribution of the drivers in the bonus-malus system ($\mathbf{e}_\infty$) and $e_t(i, \mathbf{e_0})$ denotes the i^{th} component of the transient distribution at time t of the drivers in the bonus-malus system ($\mathbf{e_t}(\mathbf{e_0})$) with initial distribution $\mathbf{e_0}$.

5. An iterative algorithm to find the new bonus-malus system

Let us assume that the number of classes of the new bonus-malus system has been chosen. The transition rules are also known.
On the one hand we have to determine the premium levels (C_i , $i = 0, \ldots, s-1$) associated with each class of the bonus-malus system.
On the other hand the conception of the bonus-malus system is based on the future observed claims frequency distribution $((\lambda_j^{'''}, p_j), j = 1, \ldots, r)$ within the bonus-malus system. The problem is that the future observed claims frequency distribution is not known and depends on the premium levels C_i.
The solution is thus to find the bonus-malus levels as well as the future observed frequency distribution iteratively. The following algorithm is proposed.

Algorithm :

Initializing step :

Choose the number of classes of the practical bonus-malus system as well as the transition rules.
Use the actual claims frequency distribution as an initial guess for the future observed claims frequency distribution.

Iterations :

Do

Use the future observed claims frequency distribution obtained as a non-parametric mixture of Poisson distributions in order to find the observed claims frequency distribution under the form of a Hofmann distribution by matching the first three moments.
With the Hofmann distribution obtain an optimal bonus-malus table.
From the optimal bonus-malus table obtain the premium levels of the practical bonus-malus system.

Make an initial choice for the future observed claims frequency distribution
Run Lemaire's algorithm to obtain the optimal claims frequency distribution of the drivers and thus the future observed frequency distribution.
Do

Use the observed claims frequency distribution obtained by Lemaire's algorithm in order to rerun Lemaire's algorithm

Until convergence

Until convergence.

We will now set up the practical bonus-malus system based on the hypothesis made in the introduction.
The optimization program used in order to match as best as possible the $P_{(k,t)}$ and the C_i is the following :

$$Min \sum_{k=0}^{4} \sum_{t=0}^{10} \sum_{i=0}^{8} w(k,t,i)[P_{(k,t)} - C_i]^2 \tag{3}$$

under the following constraints

$$\begin{aligned}
C_i & \quad \text{are integers} \\
C_4 &= 100 \\
C_{i+1} - C_i &\geq 0 \\
\sum_{i=0}^{8} e_\infty(i) C_i &\geq 100 \\
\sum_{i=0}^{8} e_0(i) C_i &\geq 100 \\
\sum_{i=0}^{8} e_2(i, \mathbf{e_0}) C_i &\geq 100
\end{aligned}$$

$$\sum_{i=0}^{8} e_4(i, \mathbf{e_0}) C_i \geq 100$$

where $\mathbf{e_0} = \{0.5729, 0.0562, 0.0661, 0.0784, 0.0421, 0.0441, 0.0457, 0.0429, 0.0516\}$ i.e. the stationary distribution of the drivers in the current bonus-malus system.
We will now fully describe the first iteration of our algorithm.
The hypothesis for the algorithm of Lemaire are the following

- $\beta = 6\%$
- $P = 35$
- $t = \frac{1}{2}$
- X is exponentially distributed with mean 85.32. This is the actual claim amount distribution derived in Walhin and Paris (2000a)

As an initializing step we use the actual claims frequency distribution :

$\lambda_1^{'} = 0.0637$	$p_1 = 0.56189$
$\lambda_2^{'} = 0.3628$	$p_2 = 0.41463$
$\lambda_3^{'} = 1.1107$	$p_3 = 0.02348$

Table 8: Parameters of the actual claims frequency distribution (non-parametric fit)

The corresponding Hofmann distribution is given by matching the first three moments :

$p = 0.2123$
$c = 0.2153$
$a = 0.8915$

Table 9: Parameters of the actual claims frequency distribution (Hofmann parametric form)

An optimal bonus-malus table is immediately obtained with the expected value premium principle :

k/t	0	1	2	3	4
1	84	158	237	317	399
2	73	136	203	271	340
3	64	119	177	236	296
4	57	106	157	210	263
5	52	96	142	189	236
6	48	87	129	171	214
7	44	80	118	157	196
8	41	74	109	145	181
9	38	69	101	135	168
10	36	65	95	126	157

Table 10: Optimal bonus-malus table

The practical bonus-malus table is derived by solving the minimization procedure (3):

s	0	1	2	3	4	5	6	7	8
C_s	89	89	93	99	100	130	134	141	155

Table 11: Practical bonus-malus levels associated with table 10

By iterating Lemaire's algorithm, we find the future observed claims frequency distribution :

Iteration	$\lambda_1^{'''}$	$\lambda_2^{'''}$	$\lambda_3^{'''}$
1	0.0637	0.3628	1.1107
2	0.0455	0.3134	1.0839
3	0.0449	0.3095	1.0832
4	0.0449	0.3091	1.0831
5	0.0449	0.3091	1.0831

Table 12: Iterations in order to find the $\lambda_j^{'''}$

The future observed claims frequency distribution is then

$\lambda_1^{'''} = 0.0449$	$p_1 = 0.56189$
$\lambda_2^{'''} = 0.3091$	$p_2 = 0.41463$
$\lambda_3^{'''} = 1.0831$	$p_3 = 0.02348$

Table 13: Future observed claims frequency distribution

This is the end of the first iteration of our algorithm.

The following iterations are summarized in the next table :

Iteration	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	λ_1'''	λ_2'''	λ_3'''
1	89	89	93	99	100	130	134	141	155	0.0449	0.3091	1.0831
2	82	85	95	100	100	133	143	163	204	0.0380	0.2708	1.0415
3	81	81	93	100	100	137	149	171	215	0.0368	0.2620	1.0332
4	79	84	93	100	100	140	152	174	218	0.0359	0.2581	1.0318
5	79	82	93	100	100	139	152	175	220	0.0359	0.2570	1.0295
6	79	82	93	100	100	139	152	175	220	0.0359	0.2570	1.0295

Table 14: Iterations of the algorithm

Let us note that with this bonus-malus system the future premium income is

t	P
0	100
2	113.51
4	117.57
∞	118.38

Table 15: Future premium income

This means that in such a situation the insurer will be able to propose a reduction of the premium because the financial desequilibrium is in his favour. Note that if the insurer decides to do so, it may have an influence on the behaviour of the drivers using Lemaire's algorithm. Then a new observed frequency distribution should be calculated as well as the new transient and stationary distributions.

Let us have a look at the bonus-malus system obtained if we ease the constrain

$$\sum_{i=0}^{8} e_0(i)C_i \geq 100$$

We find

Iteration	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	$\lambda_1^{'''}$	$\lambda_2^{'''}$	$\lambda_3^{'''}$
1	58	75	79	89	100	112	123	151	185	0.0359	0.2658	1.0463
2	57	74	80	92	100	129	145	173	218	0.0332	0.2430	1.0228
3	58	73	80	92	100	132	149	176	225	0.0329	0.2399	1.0170
4	58	73	80	92	100	133	149	177	226	0.0328	0.2393	1.0162
5	58	73	80	92	100	133	150	177	226	0.0328	0.2392	1.0164
6	58	72	81	92	100	134	150	177	226	0.0328	0.2391	1.0167
7	58	73	81	92	100	133	150	177	226	0.0328	0.2392	1.0165
8	58	72	81	92	100	134	150	177	226	0.0328	0.2391	1.0167

Table 16: Iterations of the algorithm

As we can see, the solution is given by iteration 7 or iteration 8. The reason why we do not converge is that the C_i are restricted to be integers.
In this case the future premium income is

t	P
0	86.07
2	100
4	104.17
∞	105.51

Table 17: Future premium income

If the company adopts this bonus-malus system, it accepts to loose money during the first two years. The profit from the subsequent years will rapidly absorb the losses of the first two years.

6. Future research

The optimal retention of the policyholders probably depends on the type of scale the insurer adopts. However the present paper does not investigate the problem of the optimal choice of the scale of the new bonus-malus system. This is an important point that might be studied

in the processus we propose (see Lemaire (1995) for further details on the choice of the scale). Note also that the bonus-malus system need not be a scale. Indeed the French bonus-malus system works on the basis of bonus-malus coefficients without referring to a scale.
Clearly our model misses the aspect of tariff, or covariates. It is clear that a fraction of the variablity in the claim frequency is explained by some covariates, e.g. sex, age, type of car, ... Future research is then to adapt our frequencies, as well as the variable p such that they depend on some covariates. The problem does not seem to be trivial and the lack of practical data will be crucial. Nevertheless it is clear that the use of a single average commercial premium is a big simplification in our model.
As was already the case in Walhin and Paris (2000a), the problem of proving the convergence of the proposed algorithm remains open.

7. Conclusion

We have derived in this paper an iterative algorithm based on Lemaire's algorithm in order to calculate the premium levels associated with a practical bonus-malus system. The actual claims frequency distribution, obtained from the algorithm of Walhin and Paris (2000a), is used to predict the future observed claims frequency distribution. The latter is a function of the premium levels because of the hunger for bonus. The premium levels are a function of the future observed claims frequency distribution because the optimal bonus-malus table and the transient and stationary distributions of the drivers in the bonus-malus system are constructed with the use of the future observed claims frequency distribution. Therefore the need for an iterative algorithm.
The technique used by Coene and Doray (1996) in order to set up a practical bonus-malus system from an optimal bonus-malus table has been reviewed and extended.
Parametric and non-parametric mixed Poisson distributions are used throughout the paper.
It is important to have an estimate of the proportion of the policyholders using Lemaire's algorithm. This estimate is available from the algorithm of Walhin and Paris (2000a) in case a bonus-malus system was in use before. If not, a guess has to be done.

Acknowledgments

It is our pleasure to thank Professor Michel Denuit for his careful remarks that led to a better presentation of the paper. Also the ideas of section 6 are due to him.

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