Smooth generators of integral stochastic orders

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November 20, 2000

Abstract

The purpose of this paper is to show that many integral stochastic orders have a generator consisting of infinitely differentiable functions. This especially holds for all stochastic orders with characterizations via difference operators. The usefulness of this result is demonstrated in two applications relating to stochastic ordering of multivariate normal distributions and ordering of random vectors in a random environment.
Keywords: integral stochastic orders; infinitely differentiable generator; difference operator; multivariate normal distribution; random environment.
AMS 1991 subject classifications: 60E15, 62H05.

1 Introduction

Stochastic orders are binary relations defined on sets of probability distributions, which formally describe intuitive ideas like "being larger", "being more variable" or "being more dependent". Such stochastic order relations are now used in many areas of probability and statistics. For a comprehensive treatment see e.g. Shaked and Shanthikumar (1994), Stoyan (1983) or Szekli (1995).

Many of these orders have characterizations as so called *integral stochastic orders*, i.e. there is a class $\mathfrak{F}$ of measurable functions such that $P \leq_{\mathfrak{F}} Q$ holds if, and only if, $\int f \mathrm{d}P \leq \int f \mathrm{d}Q$ for all $f \in \mathfrak{F}$. A general study of this type of orders has been given in Müller (1997). Typically there are several different classes of functions which characterize a given order. There is always a largest one, called *maximal generator*. This has been characterized in Theorem 3.7 of Müller (1997). Such large generators are interesting for applications of the orders $\leq_{\mathfrak{F}}$. However, for verifying that an integral stochastic order holds it is desirable to have small generators. In some cases (like the usual stochastic order $\leq_{st}$ on the real line) there are nice small generators, which make it easy to check the order. For some other orders, however, no small generators are available. This especially applies to the multivariate case.

For technical reasons, it is often helpful to know that it is sufficient to check $\int f \mathrm{d}P \leq \int f \mathrm{d}Q$ for all $f \in \mathfrak{F}$ which are sufficiently smooth. Indeed, differentiability of test functions is assumed here and there in proofs without justifying that this is allowed. Smoothness makes us able e.g. to use integration by parts.

It is the aim of this paper to show that for all well known integral stochastic orders it is allowed to assume that the generator consists only of infinitely differentiable functions. We derive our main theoretical results in Section 2, where it is shown that provided $\leq_{\mathfrak{F}}$ is closed under convolution and weak convergence, its generator can be restricted to smooth functions. In Section 3 this will be applied to a variety of integral stochastic orderings, especially to all orders with a generator that can be characterized via difference operators. In Section 4, we show how these results can be used for the comparison of multivariate normal distributions and for the ordering of random vectors in a random environment.

2 Main result

Let us first fix the notation. We use the setting of Müller (1997). Specifically, let $(S, \mathcal{A})$ be an arbitrary measure space and let $b : S \to [1, \infty)$ be a measurable function, called *weight* *function.* We consider the set $\mathcal{B}_b$ of b-bounded measurable functions $f : S \to \mathbb{R}$, for which

$$\|f\|_b := \sup_{s\in S} \frac{|f(s)|}{b(s)} < \infty.$$

Then any set of functions $\mathcal{F} \subset \mathcal{B}_b$ defines an order relation $\leq_{\mathcal{F}}$ on $\mathbb{P}_b$, the set of all probability measures on $(S, \mathcal{A})$ with $\int b\mathrm{d}P < \infty$. $\mathcal{R}_{\mathcal{F}}$ shall be the corresponding maximal generator, which according to Theorem 3.7 of Müller (1997) is the closed convex cone (in the corresponding weak topology) spanned by $\mathcal{F}$ and the constant functions. In the following S will always be some Euclidean space $S = \mathbb{R}^d$ endowed with the Borel-σ-algebra, and as usual $\mathcal{C}^\infty$ shall denote the set of all functions on S that are infinitely often differentiable.

We will show that in all relevant cases $\mathcal{R}_{\mathcal{F}} \cap \mathcal{C}^\infty$ is an alternative generator of $\mathcal{F}$. Typically this can be achieved by a two step procedure as follows. First show that the generator can be restricted to continuous functions. Then find for an arbitrary continuous $f \in \mathcal{R}_{\mathcal{F}}$ a sequence $\{f_n,\ n \in \mathbb{N}\}$ of infinitely differentiable functions $f_n \in \mathcal{R}_{\mathcal{F}}$ such that

$$\lim_{n\to\infty} \int f_n \mathrm{d}P = \int f \mathrm{d}P \tag{1}$$

for all $P \in \mathbb{P}_b$. In order to obtain the f_n's, we resort on the following well known construction. Let g be an infinitely differentiable probability density with a compact support, and let $g_n(\mathbf{x}) = n^d \cdot g(n \cdot \mathbf{x})$. Then it is well known that the sequence

$$f_n(\mathbf{x}) = f * g_n(\mathbf{x}) := \int f(\mathbf{x} - \mathbf{y}) g_n(\mathbf{y}) \mathrm{d}\mathbf{y} \tag{2}$$

satisfies (1), see e.g. Rudin (1987), Theorem 9.10. Thus it only remains to be shown that $f_n \in \mathcal{R}_{\mathcal{F}}$. This question, however, is related to the question whether or not $\leq_{\mathcal{F}}$ is closed under convolution, which means that $P_1 \leq_{\mathcal{F}} P_2$ implies $P_1 * Q \leq_{\mathcal{F}} P_2 * Q$ for all $Q \in \mathbb{P}_b$. In fact, we have the following general result.

Theorem 2.1. *Let $\leq_{\mathcal{F}}$ be an integral stochastic order generated by a class $\mathcal{F}$ of continuous functions. If $\leq_{\mathcal{F}}$ is closed under convolution then there is a generator $\mathcal{G}$ of this order relation which only consists of infinitely differentiable functions. Especially, $\mathcal{G} = \mathcal{R}_{\mathcal{F}} \cap \mathcal{C}^\infty$ is such a generator.*

Proof. We can assume without loss of generality that $\mathcal{F}$ consists of all continuous functions in $\mathcal{R}_{\mathcal{F}}$. Since $\leq_{\mathcal{F}}$ is closed under convolution, it follows from Theorem 4.2 e) in Müller (1997) that $\mathcal{F}$ is invariant under translations, and therefore $f(\cdot - \mathbf{y}) \in \mathcal{F}$ for all $\mathbf{y}$. Thus Theorem 3.6 in Müller (1997) ensures that the function f_n defined in (2) is in $\mathcal{R}_{\mathcal{F}}$. Now define

$\mathcal{G} = \mathcal{R}_{\mathcal{F}} \cap \mathcal{C}^\infty$. Since $\mathcal{G} \subset \mathcal{R}_{\mathcal{F}}$ it is obvious that $P \leq_{\mathcal{F}} Q$ implies $P \leq_{\mathcal{G}} Q$. On the other hand, we have just shown that $f_n \in \mathcal{G}$ for all $f \in \mathcal{F}$. Hence $P \leq_{\mathcal{G}} Q$ implies $\int f_n \mathrm{d}P \leq \int f_n \mathrm{d}Q$ and by equation (1) $\int f \mathrm{d}P \leq \int f \mathrm{d}Q$ for all $f \in \mathcal{F}$. Therefore we also have that $P \leq_{\mathcal{G}} Q$ implies $P \leq_{\mathcal{F}} Q$, i.e. $\leq_{\mathcal{F}}$ and $\leq_{\mathcal{G}}$ are identical. □

Recall that a stochastic order $\leq_{\mathcal{F}}$ is said to be closed under weak convergence, if for weak convergent sequences $\{P_n,\ n \in \mathbb{N}\}$ and $\{Q_n,\ n \in \mathbb{N}\}$ converging to P and Q respectively,

$$P_n \leq_{\mathcal{F}} Q_n \quad \text{for all } n \in \mathbb{N} \text{ implies } P \leq_{\mathcal{F}} Q.$$

In Theorem 4.2.f) of Müller (1997) it has been shown that an integral stochastic order is closed under weak convergence if, and only if, it has a generator consisting of bounded continuous functions. Hence we can derive the following corollary from Theorem 2.1.

Corollary 2.2. *If $\leq_{\mathcal{F}}$ is closed under weak convergence and under convolution then it has a generator $\mathcal{G} \subset \mathcal{C}^\infty$.*

3 Stochastic orders defined by difference operators

There is a large number of stochastic orders with a generator that can be characterized in terms of difference operators. Define for a function $f : \mathbb{R}^d \to \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^d$, $i \in \{1, ..., d\}$ and $\varepsilon > 0$ the difference operator Δ_i^ε as

$$\Delta_i^\varepsilon f(\mathbf{x}) = f(\mathbf{x} + \varepsilon \mathbf{e}_i) - f(\mathbf{x}),$$

where $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ denotes the i-th unit vector. In case $d = 1$ we simply write

$$\Delta^\varepsilon f(x) = f(x + \varepsilon) - f(x).$$

For a tuple $\boldsymbol{\imath} = (i_1, \ldots, i_k)$ of numbers $i_j \in \{1, \ldots, d\}$ we define the generator $\mathcal{F}_{\boldsymbol{\imath}^+}$ which shall consist of all functions such that

$$\Delta_{i_1}^{\varepsilon_1} ... \Delta_{i_k}^{\varepsilon_k} f(\mathbf{x}) \geq 0 \quad \text{for all } \mathbf{x} \in \mathbb{R}^d,\ \varepsilon_1, ..., \varepsilon_k > 0.$$

and let $\mathcal{F}_{\boldsymbol{\imath}^-}$ be the set of functions with

$$\Delta_{i_1}^{\varepsilon_1} ... \Delta_{i_k}^{\varepsilon_k} f(\mathbf{x}) \leq 0 \quad \text{for all } \mathbf{x} \in \mathbb{R}^d,\ \varepsilon_1, ..., \varepsilon_k > 0.$$

Given a set $\Im$ of such 'signed indices' let

$$\mathcal{F}_{\Im} = \bigcap_{\boldsymbol{\imath} \in \Im} \mathcal{F}_{\boldsymbol{\imath}}.$$

Many integral stochastic orders that have been considered in the literature possess a generator of the form $\mathcal{F}_{\Im}$, as it can be seen from the following examples.

Let us start with univariate stochastic orders, i.e. the case $d = 1$. Denote by $\boldsymbol{\imath}_k = (1, ..., 1)$ the tuple consisting of k repetitions of the number 1. Notice that

$$\Delta^{\varepsilon} f(x) \geq 0 \quad \text{for all } x \in \mathbb{R} \text{ and all } \varepsilon > 0$$

if and only if f is increasing. Hence $\mathcal{F}_{\boldsymbol{\imath}_1^+}$ is the set of all increasing function and thus the generator of the (usual) stochastic order $\leq_{st}$. Similarly,

$$\Delta^{\varepsilon_1} \Delta^{\varepsilon_2} f(x) \geq 0 \quad \text{for all } x \in \mathbb{R} \text{ and all } \varepsilon_1, \varepsilon_2 > 0$$

if and only if f is convex. Hence $\mathcal{F}_{\boldsymbol{\imath}_2^+}$ generates the convex order $\leq_{cx}$, and taking the intersection yields that $\mathcal{F}_{\Im_2}$ with $\Im_2 = \{\boldsymbol{\imath}_1^+, \boldsymbol{\imath}_2^+\}$ generates the increasing convex order $\leq_{icx}$. Similarly $\mathcal{F}_{\boldsymbol{\imath}_2^-}$ generates the concave order $\leq_{cv}$, and the increasing concave order $\leq_{icv}$ is obtained from $\Im_2^- = \{\boldsymbol{\imath}_1^+, \boldsymbol{\imath}_2^-\}$. Considering higher differences yields the so called s-convex orderings considered by Denuit et al. (1998). There the ordering generated by $\mathcal{F}_{\boldsymbol{\imath}_s^+}$ is called $\leq_{s-cx}$ and the ordering $\leq_{s-icx}$ has the generator $\mathcal{F}_{\Im_s} = \{\boldsymbol{\imath}_1^+, ..., \boldsymbol{\imath}_s^+\}$. Another ordering with such a characterization is the Laplace order $\leq_{Lt}$ which is defined as follows. $X \leq_{Lt} Y$ holds if $Ee^{-tX} \geq Ee^{-tY}$ for all $t > 0$. At first sight there seems not be a characterization of the kind described above. Reuter and Riedrich (1981), however, have shown that the maximal generator of Laplace order consists of all functions with

$$(-1)^{n+1} \Delta^{\varepsilon_1} ... \Delta^{\varepsilon_n} f(x) \geq 0 \quad \text{for all } x \in \mathbb{R},\ \varepsilon_1, ..., \varepsilon_n > 0 \text{ and all } n = 1, 2, ...$$

Thus it is generated by $\mathcal{F}_{\Im}$ with $\Im = \{\boldsymbol{\imath}_1^+, \boldsymbol{\imath}_2^-, \boldsymbol{\imath}_3^+, ...\}$

There are also numerous multivariate stochastic orders that can be characterized in this way. Notice that a function $f : \mathbb{R}^d \to \mathbb{R}$ is increasing if and only if

$$\Delta_i^{\varepsilon} f(\mathbf{x}) \geq 0 \quad \text{for all } \mathbf{x} \in \mathbb{R}^d,\ \varepsilon > 0 \text{ and } i = 1, ..., d.$$

Hence the multivariate version of $\leq_{st}$ is generated by $\mathcal{F}_{\Im_{st}}$ for $\Im_{st} = \{i^+ : i = 1, ..., d\}$.

A function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be supermodular if

$$\Delta_i^{\varepsilon} \Delta_j^{\delta} f(\mathbf{x}) \geq 0 \quad \text{for all } \mathbf{x} \in \mathbb{R}^d,\ \varepsilon, \delta > 0 \text{ and } 1 \leq i < j \leq d. \tag{3}$$

Properties and applications of the stochastic order generated by this class of functions have been considered by Bäuerle (1997), Shaked and Shanthikumar (1997) and Müller and Scarsini (2000), among others. It is easy to see that this supermodular order (denoted by $\leq_{sm}$) is generated by $\mathfrak{F}_{\Im_{sm}}$ where $\Im_{sm} = \{(i,j)^+ : 1 \leq i < j \leq d\}$. A similar concept is that of the directionally convex order $\leq_{dcx}$ which has shown to be useful e.g. in Bäuerle and Rolski (1998), Meester and Shanthikumar (1999) and Müller and Scarsini (1999). A function is called directionally convex if it is supermodular and, in addition, convex in each argument, when the others are held fixed. Hence it is generated by $\mathfrak{F}_{\Im_{dcx}}$, where $\Im_{dcx} = \{(i,j)^+ : i,j = 1,...,d\}$. Combining supermodularity respectively directional convexity with monotonicity yields the increasing supermodular order $\leq_{ism}$ which is generated by $\mathfrak{F}_{\Im_{sm}} \cap \mathfrak{F}_{\Im_{st}}$ and the increasing directional convex order $\leq_{idcx}$ generated by $\mathfrak{F}_{\Im_{dcx}} \cap \mathfrak{F}_{\Im_{st}}$.

Finally we want to mention the lower and upper orthant orders, defined as

$$\mathbf{X} \leq_{uo} \mathbf{Y} \quad \text{if } P(X_1 > x_1, ..., X_d > x_d) \leq P(Y_1 > x_1, ..., Y_d > x_d) \quad \text{for all } x_1, ..., x_d$$

and

$$\mathbf{X} \leq_{lo} \mathbf{Y} \quad \text{if } P(X_1 \leq x_1, ..., X_d \leq x_d) \geq P(Y_1 \leq x_1, ..., Y_d \leq x_d) \quad \text{for all } x_1, ..., x_d.$$

These orders both have generators defined by difference operators, too. In fact the maximal generator of $\leq_{uo}$ is the set of all functions with

$$\Delta_{i_1}^{\varepsilon_1} ... \Delta_{i_k}^{\varepsilon_k} f(\mathbf{x}) \geq 0 \quad \text{for all } \mathbf{x} \in \mathbb{R}^d,\ \varepsilon_1, ..., \varepsilon_k > 0$$

and all distinct $i_1, ..., i_k$, $1 \leq k \leq d$. This result can be traced back to Rüschendorf (1980). Hence $\leq_{uo}$ is generated by $\mathfrak{F}_{\Im_{uo}}$, where $\Im_{uo} = \{(i_1, ..., i_k)^+ : i_1, ..., i_k \text{ distinct}, 1 \leq k \leq d\}$. A similar characterization is possible for $\leq_{lo}$ as it is generated by the class of all functions f such that $f(-\mathbf{x})$ is in $\mathfrak{F}_{\Im_{uo}}$.

Now we will show that for all these orderings Theorem 2.1 applies. First observe the following result.

Theorem 3.1. *Let $\Im$ be an arbitrary set of signed indices and let $\leq_{\mathfrak{F}_{\Im}}$ be the stochastic order generated by the corresponding class $\mathfrak{F}_{\Im}$ of functions. Then $\leq_{\mathfrak{F}_{\Im}}$ is closed under convolution.*

Proof. It is obvious from the definition that for a fixed index $\boldsymbol{\imath}$ the sets $\mathfrak{F}_{\boldsymbol{\imath}^+}$ and $\mathfrak{F}_{\boldsymbol{\imath}^-}$ are invariant under translations. Hence $\mathfrak{F}_{\Im}$ as an intersection of such sets is again invariant under translations. Thus the assertion follows from Theorem 4.2. e) in Müller (1997). $\square$

Since all of the orders mentioned above are known to have generators consisting only of continuous functions, we can apply Theorem 2.1 to show that all these orders have a generator consisting of infinitely differentiable functions. The nice feature of the next theorem is that it means that we can just translate the conditions given in terms of differences into conditions given in terms of derivatives. Therefore let us define for a given $\boldsymbol{\imath} = (i_1, ..., i_k)$ the set $\mathfrak{F}^{\infty}_{\boldsymbol{\imath}^+}$ ($\mathfrak{F}^{\infty}_{\boldsymbol{\imath}^-}$) as the set of all infinitely differentiable functions with the property that

$$\frac{\partial^k}{\partial x_{i_1} ... \partial x_{i_k}} f(\mathbf{x}) \geq (\leq)\ 0 \qquad \text{for all } \mathbf{x} \in \mathbb{R}^d,$$

and for a given a set $\Im$ of such signed indices let

$$\mathfrak{F}^{\infty}_{\Im} = \bigcap_{\boldsymbol{\imath} \in \Im} \mathfrak{F}^{\infty}_{\boldsymbol{\imath}}.$$

We get the following result.

Theorem 3.2. *Let $\mathfrak{F}_{\Im}$ be the generator of one of the stochastic orders $\leq_{st}, \leq_{s-cx}, \leq_{s-icx}$ $, \leq_{s-cv}, \leq_{s-icv}, \leq_{Lt}, \leq_{sm}, \leq_{ism}, \leq_{dcx}, \leq_{idcx}, \leq_{uo}$ or $\leq_{lo}$. Then $\leq_{\mathfrak{F}_{\Im}}$ is also generated by $\mathfrak{F}^{\infty}_{\Im}$ which consists only of infinitely differentiable functions.*

Proof. In view of Theorem 2.1 we have to show that the mentioned orders are closed under convolution and that there is a generator consisting of continuous functions. The closure under convolution has been shown in general in Theorem 3.1. Hence it remains to be shown that there is always a generator consisting of continuous functions. For $\leq_{st}$ this is known since Kamae et al. (1977). For the supermodular and increasing supermodular order this has been shown in Müller and Scarsini (2000). A similar proof is possible for the orthant order. In all other cases it is obvious that even the maximal generator contains only continuous functions. □

We want to mention explicitly a few important examples that are contained in Theorem 3.2. Given two d-dimensional random vectors $\mathbf{X}$ and $\mathbf{Y}$,

(i) $\mathbf{X} \leq_{sm} \mathbf{Y}$ if, and only if, $Ef(\mathbf{X}) \leq Ef(\mathbf{Y})$ holds for all twice differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$ satisfying $\partial^2/\partial x_i \partial x_j f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^d$ and all $1 \leq i < j \leq d$.

(ii) $\mathbf{X} \leq_{ism} \mathbf{Y}$ if, and only if, $Ef(\mathbf{X}) \leq Ef(\mathbf{Y})$ holds for all twice differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$ satisfying $\partial/\partial x_i f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^d$ and all $1 \leq i \leq d$ and $\partial^2/\partial x_i \partial x_j f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^d$ and all $1 \leq i < j \leq d$.

(iii) $\mathbf{X} \leq_{dcx} \mathbf{Y}$ if, and only if, $Ef(\mathbf{X}) \leq Ef(\mathbf{Y})$ holds for all twice differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$ satisfying $\partial^2/\partial x_i \partial x_j f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^d$ and all $1 \leq i, j \leq d$.

(iv) $\mathbf{X} \leq_{idcx} \mathbf{Y}$ if, and only if, $Ef(\mathbf{X}) \leq Ef(\mathbf{Y})$ holds for all twice differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$ satisfying $\partial/\partial x_i f(\mathbf{x}) \geq 0$ for $1 \leq i \leq d$ and $\partial^2/\partial x_i \partial x_j f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^d$ and all $1 \leq i, j \leq d$.

At the end of this section we want to mention that there are also stochastic orders to which Theorem 2.1 applies though they can not be characterized via difference operators. The most important examples are the multivariate versions of the convex and increasing convex orders. Specifically, $\mathbf{X}$ is said to precede $\mathbf{Y}$ in the multivariate convex order, written as $\mathbf{X} \leq_{cx} \mathbf{Y}$, if $Ef(\mathbf{X}) \leq Ef(\mathbf{Y})$ holds for any convex function $f : \mathbb{R}^d \to \mathbb{R}$, provided the expectations exist. In this case, Theorem 2.1 applies, since a translation of a convex function again is convex, and any convex function must be continuous. Hence $\mathbf{X} \leq_{cx} \mathbf{Y}$ holds if and only if $Ef(\mathbf{X}) \leq Ef(\mathbf{Y})$ holds for all twice differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$ with a positive semi-definite Hesse matrix. The same applies to the multivariate increasing convex order, written as $\mathbf{X} \leq_{icx} \mathbf{Y}$, which holds if $Ef(\mathbf{X}) \leq Ef(\mathbf{Y})$ holds for any non-decreasing and convex function $f : \mathbb{R}^d \to \mathbb{R}$, provided the expectations exist; in this case, Theorem 2.1 applies, too.

Remark 3.3. A number of notions of positive dependence have been introduced in the literature in an effort to mathematically describe the property that "large values of the components of a random vector go together with large values of the other components." Many of these notions are defined by means of some comparison of the random vector $\mathbf{X}$ with its independent version $\mathbf{X}^{\perp}$ (which shall have the same marginals as $\mathbf{X}$, but independent components); we refer the interested reader e.g. to Szekli (1995) or Scarsini and Shaked (1996). For instance, the upper orthant dependence is defined as $\mathbf{X}^{\perp} \leq_{uo} \mathbf{X}$, the lower orthant dependence as $\mathbf{X} \leq_{lo} \mathbf{X}^{\perp}$. All the results derived in this paper thus obviously apply to dependence notions defined with the help of integral stochastic orderings.

4 Applications.

4.1 Ordering Gaussian vectors

As a first application the results of Section 3 can be used to characterize stochastic ordering of multivariate normal distributions; such characterizations can be found in Müller (2000) and are based on the following identity. Let $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $\mathbf{Y} \sim \mathcal{N}(\boldsymbol{\mu}', \boldsymbol{\Sigma}')$, and let ϕ_λ be the density of $\mathcal{N}(\lambda\boldsymbol{\mu}' + (1-\lambda)\boldsymbol{\mu}, \lambda\boldsymbol{\Sigma}' + (1-\lambda)\boldsymbol{\Sigma})$, $0 \le \lambda \le 1$. Moreover, assume that $f : \mathbb{R}^d \to \mathbb{R}$ is twice continuously differentiable and satisfies some polynomial growth conditions. Then

$$Ef(\mathbf{Y}) - Ef(\mathbf{X}) = \int_0^1 \int \left((\boldsymbol{\mu}' - \boldsymbol{\mu})^T \nabla f(\mathbf{x}) + \frac{1}{2} tr((\boldsymbol{\Sigma}' - \boldsymbol{\Sigma})\mathbf{H}_f(\mathbf{x})) \right) \cdot \phi_\lambda(\mathbf{x}) \, \mathrm{d}\mathbf{x} \, \mathrm{d}\lambda, \tag{4}$$

where ∇f and H_f are the gradient and the Hessian of f, respectively, and $tr(\mathbf{A})$ denotes the trace of the matrix $\mathbf{A}$.

This identity is then used to derive characterizations of the most important integral stochastic orders. In the proofs it is tacitly assumed that it is allowed to restrict the generators to twice differentiable functions. Thus Theorem 2.1 together with Section 3 fills a little gap in the proofs there.

4.2 Ordering random vectors in a random environment

Consider the following model of a d-dimensional random vector $\mathbf{X} = (X_1, \ldots, X_d)$ depending on a random environment $\boldsymbol{\Theta} = (\Theta_1, \ldots, \Theta_d)$ in the following fashion.

1. Component X_i depends only on Θ_i, i.e. given $\boldsymbol{\Theta} = \boldsymbol{\theta}$, the distribution of X_i depends only on θ_i and is independent of θ_k for $k \neq i$.

2. The components of $\mathbf{X}$ are conditionally independent, i.e. given $\boldsymbol{\Theta} = \boldsymbol{\theta}$ the random variables $X_1, \ldots, X_d$ are independent.

Unconditionally, however, there may be dependence in the vector $\mathbf{X} = (X_1, \ldots, X_d)$ induced by the dependence structure of $\boldsymbol{\Theta} = (\Theta_1, \ldots, \Theta_d)$. It is an interesting question how the distribution of $\boldsymbol{\Theta}$ affects the distribution of $\mathbf{X}$, especially how the dependence structure of $\mathbf{X}$ depends on the one of $\boldsymbol{\Theta}$. In the language of stochastic orderings this question can be stated as follows: Let $\boldsymbol{\Theta}$ and $\boldsymbol{\Theta}'$ be two random environments, $\mathbf{X}$ and $\mathbf{X}'$ the corresponding random observations with the same conditional marginal distribution functions

$$F_i(x|\theta) = P(X_i \le x | \Theta_i = \theta) = P(X_i' \le x | \Theta_i' = \theta)$$

and let $\le_{\mathcal{F}}$ be a stochastic order of interest. Under what conditions does $\boldsymbol{\Theta} \le_{\mathcal{F}} \boldsymbol{\Theta}'$ imply $\mathbf{X} \le_{\mathcal{F}} \mathbf{X}'$?

Various problems of this type can be found in the recent literature. Bäuerle (1997, Lemma 3.4) shows such a result for the supermodular order in the context of queueing systems in a random environment. Her proof is based on a coupling argument. In the context of synchronized stochastic systems Li and Xu (2000) study so called *separable Markov chains* which have transition kernels of this type. In their Theorem 2.7 they prove results of the above mentioned nature for supermodular order and orthant orders. In actuarial sciences similar results have recently been given by Hu and Pan (1999), Denuit et al. (2000) and Lillo et al. (2000).

It is not surprising that the answer to the problem mentioned above is related to stochastic monotonicity and stochastic convexity conditions on $F_i(x|\theta)$. Recall that X_i is said to be stochastically increasing (resp. convex) in Θ_i if the function

$$\theta \mapsto E[f(X_i)|\Theta_i = \theta]$$

is increasing (resp. convex), whenever f is an increasing (resp. convex) function. In the proof below we will use the well known facts that X_i is stochastically increasing in Θ_i if, and only if,

$$\theta \mapsto P(X_i > x|\Theta_i = \theta) = 1 - F_i(x|\theta)$$

is increasing for all x, and that it is stochastically convex if and only if $\theta \mapsto E[X_i|\Theta_i = \theta]$ is linear and

$$\theta \mapsto \pi_i(x|\theta) = E[(X_i - x)_+|\Theta_i = \theta] = \int_x^\infty [1 - F_i(y|\theta)]\mathrm{d}y$$

is convex; see e.g. Denuit and Lefèvre (1999, Proposition 1) for a proof of the latter claim.

Theorem 4.1. *a) Assume that X_i is stochastically increasing in Θ_i for all $i = 1, 2, \ldots, d$ and let $\preceq$ be one of the stochastic orders $\leq_{st}$, $\leq_{sm}$, $\leq_{ism}$, $\leq_{uo}$ and $\leq_{lo}$. Then the following implication holds:*

$$\boldsymbol{\Theta} \preceq \boldsymbol{\Lambda} \Rightarrow \mathbf{X} \preceq \mathbf{Y}.$$

b) If X_i is stochastically convex in Θ_i for all $i = 1, 2, \ldots, d$ then

$$\boldsymbol{\Theta} \leq_{dcx} \boldsymbol{\Lambda} \Rightarrow \mathbf{X} \leq_{dcx} \mathbf{Y}.$$

c) If X_i is stochastically increasing and stochastically convex in Θ_i for all $i = 1, 2, \ldots, d$ then

$$\boldsymbol{\Theta} \leq_{idcx} \boldsymbol{\Lambda} \Rightarrow \mathbf{X} \leq_{idcx} \mathbf{Y}.$$

Proof. Let us write

$$Ef(\mathbf{X}) = \int E[f(\mathbf{X})|\mathbf{\Theta} = \boldsymbol{\theta}]F_{\mathbf{\Theta}}(\mathrm{d}\boldsymbol{\theta}) = Ef^*(\mathbf{\Theta})$$

where

$$f^*(\boldsymbol{\theta}) = \int f(\mathbf{x}) \bigotimes_{k=1}^{d} F_k(\mathrm{d}x_k|\theta_k). \tag{5}$$

Here $\otimes$ denotes the product measure. According to the results derived in Section 3 it is sufficient to consider infinitely differentiable functions f. Moreover, we assume without loss of generality that f has bounded derivatives of all orders. For the supermodular order it has been shown in Müller and Scarsini (2000) that this is sufficient and for the directional convex order in Müller and Scarsini (1999). For all other orders in the theorem this is obvious.

Now the differentiability of f enables us to perform partial integrations to get the following identities:

$$\Delta_i^\varepsilon f^*(\boldsymbol{\theta}) \;=\; \int \frac{\partial}{\partial x_i} f(\mathbf{x}) \big\{F_i(x_i|\theta_i) - F_i(x_i|\theta_i+\varepsilon)\big\} \bigotimes_{k\neq i} F_k(\mathrm{d}x_k|\theta_k)\mathrm{d}x_i \tag{6}$$

for all $i = 1, ..., d$,

$$\begin{aligned}\Delta_i^\varepsilon \Delta_j^\delta f^*(\boldsymbol{\theta}) \;=\; & \int \frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{x}) \big\{F_i(x_i|\theta_i) - F_i(x_i|\theta_i+\varepsilon)\big\} \\ & \cdot\big\{F_j(x_j|\theta_j) - F_j(x_j|\theta_j+\delta)\big\} \bigotimes_{k\neq i,j} F_k(\mathrm{d}x_k|\theta_k)\mathrm{d}x_i\mathrm{d}x_j\end{aligned} \tag{7}$$

for all $i, j = 1, ..., d$ with $i \neq j$, and

$$\Delta_1^{\varepsilon_1}...\Delta_d^{\varepsilon_d} f^*(\boldsymbol{\theta}) \;=\; \int \frac{\partial^d}{\partial x_1 ... \partial x_d} f(\mathbf{x}) \prod_{i=1}^{d} (F_i(x_i|\theta_i) - F_i(x_i|\theta_i+\varepsilon))\mathrm{d}x_1...\mathrm{d}x_d. \tag{8}$$

¿From these equations part a) immediately follows.

If X_i is stochastically convex in Θ_i then $E[X_i|\Theta_i = \theta]$ is linear in θ, i.e. $E[X_i|\Theta_i = \theta] = a_i + b_i\theta$ for all $i = 1, ..., d$. Hence

$$\begin{aligned}&\lim_{x_i\to-\infty} \big\{\pi_i(x_i|\theta_i+2\varepsilon) - 2\pi_i(x_i|\theta_i+\varepsilon) + \pi_i(x_i|\theta_i)\big\} \\ &\quad = E[X_i|\Theta_i = \theta_i+2\varepsilon] - 2E[X_i|\Theta_i = \theta_i+\varepsilon] + E[X_i|\Theta_i = \theta_i] = 0.\end{aligned}$$

Therefore applying another integration by parts to (6) yields

$$\Delta_i^\varepsilon \Delta_i^\varepsilon f^*(\boldsymbol{\theta}) = \int \frac{\partial^2}{\partial x_i^2} f(\mathbf{x}) \big\{\pi_i(x_i|\theta_i+2\varepsilon) - 2\pi_i(x_i|\theta_i+\varepsilon) + \pi_i(x_i|\theta_i)\big\} \bigotimes_{k\neq i} F_k(\mathrm{d}x_k|\theta_k)\mathrm{d}x_i. \tag{9}$$

¿From this identity part b) and c) follow immediately, too. □

Many parametric families of distributions fulfill the assumptions of Theorem 4.1, see e.g. Chapter 6 of Shaked and Shanthikumar (1994).

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