A general family of Bivariate Mixed Poisson distributions

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Abstract

In this paper we study a class of bivariate mixed Poisson distributions by extending the Hofmann's distribution from the univariate case to the bivariate case.
We show how to evaluate the bivariate aggregate claims distribution and we adjust some insurance portfolio's given in the literature.
Bonus-malus systems for two kinds of claims and bonus malus systems for yearly changing frequencies are also derived.
This study typically extends the use of bivariate mixed negative binomial, Poisson and Poisson Inverse Gaussian distributions.

Keywords

Bivariate mixed distributions, Hofmann's distribution, maximum likelihood, aggregate claims distribution, recursive algorithm, stable algorithm, bonus-malus system

1. Introduction

In this paper we study a family of bivariate counting distributions. These distributions are of interest in actuarial sciences when one wants to work with frequencies of dependent variables such as material damage and bodily injury claims in third party liability automobile insurance. The class of distributions we will choose is also interesting if one wants to study the joint distribution of the number of claims on several observation years if the frequency is decreasing or increasing geometrically. This is particularly interesting for the construction of bonus-malus systems.

The general family we present in this paper has particular cases Negative Binomial and Poisson Inverse Gaussian that have been discussed in Besson and Partrat (1992) and in Partrat (1994).
By extending the univariate Hofmann's distribution described in Walhin and Paris (2000), we give a general setting for studying bivariate mixed Poisson distributions.
The rest of the paper is organized as follows. Section 2 describes the bivariate mixed Poisson distribution. Section 3 extends the univariate Hofmann's distribution to the bivariate case. Section 4 addresses the problem of estimating the parameters of the distribution. Section 5 shows how to find a trivial recursion giving the aggregate claims distribution that is unfortunately unstable. Section 6 gives a stable recursion for the aggregate claims distribution which is based on a double staged algorithm. Section 7 compares the fits obtained with our distribution against other fits given in the literature for the data used in Besson and Partrat (1992) and in Partrat (1994). Sections 8 and 9 treat the evaluation of bonus-malus systems depending on two types of claims or with yearly changing frequencies. Section 10 gives the conclusion.

2. The model

We are interested in studying a mixed bivariate distribution

$$\mathbb{P}(N=n, M=m) = \int_0^\infty \mathbb{P}(N=n, M=m|\Lambda)dU(\lambda) \tag{1}$$

Furthermore we assume that

- Conditionally on Λ the random variables N and M are independent
- The conditional distributions of N and M are univariate Poisson with parameter respectively λ and $v\lambda$

As $v = \frac{\mathbb{E}M}{\mathbb{E}N}$ we see that this model will be interesting in order to study a counting variable for which the mean frequency is varying geometrically from one year to another. This fact will be used in section 8 for constructing bonus malus systems allowing for a frequency decreasing (or increasing) geometrically each year.
From these assumptions, Partrat (1994) deduces easily that

$$\mathbb{P}(N=n, M=m) = C_{n+m}^n \frac{v^m}{(1+v)^{n+m}} \int_0^\infty e^{-(1+v)\lambda} \frac{((1+v)\lambda)^{n+m}}{(n+m)!} dU(\lambda) \tag{2}$$

As mentionned in Hesselager (1996) and in Partrat (1994), the model can be rewritten as

$$\mathbb{P}(N=n|K=k) = C_k^n \rho_1^n \rho_2^{k-n}$$

with

$$K = N + M$$

$$
\begin{aligned}
\rho_1 &= \frac{1}{1+v} \\
\rho_2 &= \frac{v}{1+v}
\end{aligned}
$$

This result is easily demonstrated :
Proof

$$
\begin{aligned}
\mathbb{P}(N=n|K=k) &= \frac{\mathbb{P}(N=n, M=k-n)}{\mathbb{P}(K=k)} \\
&\Leftrightarrow \\
\mathbb{P}(N=n, M=m) &= C^n_{m+n}\rho_1^n\rho_2^m\mathbb{P}(K=n+m)
\end{aligned}
$$

with

$$
\begin{aligned}
\rho_1 &= \frac{1}{1+v} \\
\rho_2 &= \frac{v}{1+v} \\
\mathbb{P}(K=n+m) &= \mathbb{P}(N+M=n+m) \\
&= \int_0^\infty \mathbb{P}(N+M=n+m|\Lambda)dU(\lambda) \\
&= \int_0^\infty e^{-(1+v)\lambda}\frac{(\lambda(1+v))^{n+m}}{(n+m)!}dU(\lambda)
\end{aligned}
$$

■

3. The bivariate Hofmann's distribution

Walhin and Paris (2000) described the Hofmann's distribution. Let us recall some concepts: Let $N(t)$ be the number of claims in $(0,t]$. We assume $N(0)=0$. We have

$$
\Pi(n,t) = \mathbb{P}[N(t)=n] = \int_0^\infty e^{-\lambda t}\frac{(\lambda t)^n}{n!}dU(\lambda) \tag{3}
$$

By choosing

$$
\begin{aligned}
\Pi(0,t) &= e^{-\theta(t)} \\
\theta(t) &\geq 0 \\
\theta(0) &= 0
\end{aligned}
$$

$$
\frac{d}{dt}\theta(t) \text{ completely monotone}
$$

Walhin and Paris (2000) show that $N(t)$ can also be interpreted as a compound Poisson model:

$$N(t) = \sum_{i=1}^{L(t)} \xi_i$$

where $L(t)$ is Poisson distributed with mean $\theta(t)$ and independent of the ξ_i that are independent and identically distributed. We will use this property in the sequel. A good choice for the function $\theta(t)$ is the choice made by Hofmann and studied in Walhin and Paris (2000) and in Kestemont and Paris (1985) :

$$\theta'(t) = \frac{p}{(1+ct)^a} \quad p > 0, \quad c > 0, \quad a \geq 0$$

By integration, one has

$$\begin{aligned}
\theta(t) &= \frac{p}{c(1-a)}[(1+ct)^{1-a} - 1] \text{ if } a \neq 1 \\
\theta(t) &= \frac{p}{c} \ln(1+ct) \text{ by continuity for } a = 1
\end{aligned}$$

Particular cases of interest are : $a = 0$ (Poisson), $a = 0.5$ (Poisson Inverse Gaussian), $a = 1$ (Negative Binomial), $a = 2$ (Polya-Aeppli) ,$a \to \infty, c \to 0, ac \to b$ (Neymann Type A).
Some properties are

$$\begin{aligned}
\psi_{N(t)}(z) &= \sum_{n=0}^{\infty} \Pi(n,t) z^n = \Pi(0, t - tz) = e^{-\theta(t-tz)} \\
EN(t) &= pt \\
VarN(t) &= pt + pact^2
\end{aligned}$$

where we denote the probability generating function of $N(t)$ by $\psi_{N(t)}(z)$.
In the sequel we will write $\Pi_{p,c,a}(n,t)$ for $\Pi(n,t)$ in order to recall the use of the Hofmann's family of counting distributions.
Let us use the Hofmann's distribution in our bivariate case. Let us assume that $U(\lambda)$ in (1) is the family of cumulative density functions of the mixing distribution giving the univariate Hofmann's distribution. The density of the mixing distribution can be obtained but it is not necessary for our developments.
From (2) and (3) we immediately get

$$\mathbb{P}(N = n, M = m) = C_{n+m}^{n} \frac{v^m}{(1+v)^{n+m}} \Pi(n+m, 1+v) \tag{4}$$

where it is easy to see that

$$\Pi_{p,c,a}(n+m, 1+v) = \Pi_{(1+v)p,(1+v)c,a}(n+m, 1)$$

In fact our model introduces dependency such that :

$$\begin{aligned}
N &\sim Ho(p, c, a) \\
M &\sim Ho(pv, cv, a) \\
N + M &\sim Ho(p(1+v), c(1+v), a)
\end{aligned}$$

This clearly generalizes the reasoning of Partrat (1994) where only Λ Gamma or Inverse Gaussian are considered.

4. Estimation of the parameters

Let us assume that we have observed a sample $(n_i, m_i) \quad 1 \leq i \leq n$ of (N, M).
By maximum likelihood we get

$$\begin{aligned} l(v,p,c,a) &= \ln \prod_{i=1}^{n} \mathbb{P}(N = n_i, M = m_i) \\ &= C + v \sum_{i=1}^{n} m_i - (1+v) \sum_{i=1}^{n} (n_i + m_i) + l_{n+m}(v,p,c,a) \end{aligned}$$

where $l_{n+m}(v,p,c,a)$ is the loglikelihood for the univariate Hofmann distribution $Ho((1+v)p, (1+v)c, a)$ with the sample $(n_i + m_i)\,, 1 \leq i \leq n$. As shown in Besson and Partrat (1992), $\frac{d}{dv} l_{n+m}(v,p,c,a) = 0$ at the maximum likelihood estimate. So we immediately get

$$\hat{v} = \frac{\bar{m}}{\bar{n}}$$

By standard results on the univariate Hofmann's distribution (see Walhin and Paris (2000)), we know that the maximum likelihood estimate of the mean is the observed frequency :

$$\hat{p}(1 + \hat{v}) = \bar{n} + \bar{m}$$

where $\bar{n}$ and $\bar{m}$ are the observed means of N and M.
So the estimates $\hat{p}$ and $\hat{v}$ are derived analytically. The other two estimates $\hat{c}$ and $\hat{a}$ have to be found by standard numerical maximization techniques. Note that one has to be carefull because the likelihood may be very flat and local extrema are not excluded.

5. A trivial recursion for the aggregate claims distribution

In the conclusion of his paper, Partrat (1994) addresses the problem of finding recursions like the Panjer's recursion (Panjer (1981)) in order to give the distribution of the aggregate claims.
In this section we derive a trivial recursion that unfortunately will be seen to be unstable.
Let X be the claim amount distribution of type N and Y the claim amount distribution of type M. X and Y are supposed to be independent. We are interested in

$$(S,T) = (\sum_{i=1}^{N} X_i, \sum_{i=1}^{M} Y_i)$$

Let us use the following notations :

Probability functions

$$\begin{aligned}
\mathbb{P}(N=n, M=m) &= p(n,m) \\
\mathbb{P}(K=k) &= r(k) \quad (K = N+M) \\
\mathbb{P}(X=x) &= f_X(x) \\
\mathbb{P}(Y=y) &= f_Y(y) \\
\mathbb{P}(S=s, T=t) &= f_{(S,T)}(s,t)
\end{aligned}$$

Probability generating functions

$$\begin{aligned}
\psi_X(u) &= \sum_{x=0}^{\infty} f_X(x) u^x \\
\psi_Y(v) &= \sum_{y=0}^{\infty} f_Y(x) v^y \\
\psi_K(u) &= \sum_{r=0}^{\infty} r(k) u^k \\
\phi_{N,M}(u,v) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p(n,m) u^n v^m \\
\phi_{S,T}(u,v) &= \sum_{x=0}^{\infty}\sum_{y=0}^{\infty} f_{(S,T)}(x,y) u^x v^y
\end{aligned}$$

Theorem 1

A simple recursion for $p(n,m)$ is given by

$$p(0,0) = \Pi_{((1+v)p,(1+v)c,a)}(0, 1+v) \tag{5}$$

$$np(n,m) = \frac{1}{v}(m+1)p(n-1,m+1) \text{ if } n > 0 \tag{6}$$

$$mp(n,m) = v(n+1)p(n+1,m-1) \text{ if } m > 0 \tag{7}$$

Proof

(5) follows immediately from (4).
(6) is given by

$$\begin{aligned}
\Pi_{p,c,a}(n+m,1+v) &= p(n,m)\frac{n!m!}{(n+m)!}\frac{(1+v)^{n+m}}{v^m} \\
&= p(n-1,m+1)\frac{(n-1)!(m+1)!}{(n+m)!}\frac{(1+v)^{n+m}}{v^{m+1}} \\
&\Leftrightarrow \\
np(n,m) &= \frac{1}{v}(m+1)p(n-1,m+1)
\end{aligned}$$

and (7) holds similarly. ■

Note that in order to run this recursion, it is necessary to have a full initialization that is $p(n,0)$, $n>0$ or $p(0,m)$, $m>0$

Theorem 2

A simple recursion for $f_{(S,T)}(x,y)$, $x>0$, $y>0$ *is given by*

(8) $$\sum_{j=1}^{y} jf_Y(j)xf_{(S,T)}(x,y-j) = \frac{1}{v}\sum_{i=1}^{x} if_X(i)yf_{(S,T)}(x-i,y)$$

Proof

Multiplying (6) by $\psi_X^{n-1}(u)\frac{d}{du}\psi_X(u)\psi_Y^m(v)\frac{d}{dv}\psi_Y(v)$ on both sides, rearranging and inverting immediately gives (8). ■

The recursion (8) clearly will not be stable due to numbers close to zero that have to be substracted. Moreover the initialization ($f_{(S,T)}(x,0)$ and $f_{(S,T)}(0,y)$) is not given because the structure of $\Pi_{p,c,a}(n+m,1+v)$ is not used.
So we should look for another type of recursions.
In the case of the negative binomial distribution the answer has been given by Hesselager (1996). In his paper, Hesselager(1996) gives a stable algorithm for the evaluation of the density of (S,T) for the particular case of the bivariate negative binomial distribution, i.e. when Λ is Gamma distributed.

6. A stable double-staged recursion for the aggregate claims distribution

In this section we will use the same methodology as in Hesselager in order to derive stable algorithms for the distribution of (S,T). In a first time we will use the fact that the Hofmann's distribution can be seen as a Compond Poisson distribution :

$$(N,M) = (\xi_1 + \cdots + \xi_L, \eta_1 + \cdots + \eta_L)$$

where the ξ_i and η_i are not independent and L is Poisson distributed independently of the (ξ_i, η_i). Let us introduce further notations :

$$\begin{aligned}
\mathbb{P}(\xi = n, \eta = m) &= f_{(\xi,\eta)}(n,m) \\
\phi_{\xi,\eta}(u,v) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f_{(\xi,\eta)}(n,m)u^n v^m
\end{aligned}$$

We use the same methodology as in Walhin and Paris (2000) :

$$\begin{aligned}
(U,V) &= (X_1 + \cdots + X_\xi, Y_1 + \cdots Y_\eta) \\
(S,T) &= (U_1 + \cdots U_L, V_1 + \cdots + V_L)
\end{aligned}$$

The bivariate Panjer's algorithm described in Walhin and Paris (1999a) will be used in order to find the distribution of (S,T) knowing the distribution of (U,V).
In a first time we are interested to derive the distribution of (U,V). Therefore we need first to derive the properties of (ξ,η).
Taking advantage of the fact that,in our particular case, with $K = N+M$, $(N|K)$ is binomial distributed, Hesselager (1996) derives

$$\phi_{N,M}(u,v) = \psi_K(\rho_1 u + \rho_2 v)$$

In the case of the Bivariate Hofmann's distribution we have

$$\begin{aligned}
\phi_{N,M}(u,v) &= e^{-\theta(1+v-(1+v)(\rho_1 u + \rho_2 v))} \\
&= e^{-\theta(1+v)(1-\phi_{\xi,\eta}(u,v))}
\end{aligned}$$

where

$$\begin{aligned}
\phi_{\xi,\eta}(0,0) &= 0 \\
\phi_{\xi,\eta}(u,v) &= 1 - \frac{\theta((1+v)(1-(\rho_1 u + \rho_2 v)))}{\theta(1+v)} \qquad (9)
\end{aligned}$$

The Taylor development around $(1+v)$ of (9) gives after a few calculations

$$f_{(\xi,\eta)}(n,m) = \frac{\theta^{(m+n)}(1+v)}{\theta(1+v)}\frac{(-1)^{m+n-1}}{(m+n)!}v^m C^n_{n+m}$$

As we have

$$\begin{aligned}
\theta(1+v) &= \frac{p}{c(1-a)}\left[(1+c(1+v))^{1-a} - 1\right] \\
\theta^{(n+m)}(1+v) &= (-1)^{n+m-1}pc^{n+m-1}\frac{\Gamma(a+m+n-1)}{\Gamma(a)}(1+c(1+v))^{1-a-n-m}
\end{aligned}$$

we immediately get

$$\begin{aligned} f_{(\xi,\eta)}(n,m) &= \frac{C_{n+m}^n}{(n+m)!}c^{n+m}\frac{1-a}{\Gamma(a)}v^m\Gamma(a+m+n-1)\frac{[1+c(1+v)]^{1-a-n-m}}{(1+c(1+v))^{1-a}-1} \\ &= C_{n+m}^n\rho_1^n\rho_2^m\mathbb{P}(W=n+m) \end{aligned}$$

with

$$\begin{aligned} \rho_1 &= \frac{1}{1+v} \\ \rho_2 &= 1-\rho_1 \\ \mathbb{P}(W=n+m) &= \frac{c^{n+m}}{(n+m)!}\frac{1-a}{\Gamma(a)}\Gamma(a+m+n-1)(1+v)^{n+m}\frac{[1+c(1+v)]^{1-a-n-m}}{(1+c(1+v))^{1-a}-1} \end{aligned}$$

W depends only on n and m through $n+m$.
We have that $W=\xi+\eta$. Indeed we have

$$\begin{aligned} \mathbb{P}(\xi=k|\xi+\eta=n) &= \frac{\mathbb{P}(\xi=k,\eta=n-k)}{\mathbb{P}(\xi+\eta=n)} \\ &= \frac{\frac{n!}{k!(n-k)!}\rho_1^k\rho_2^{n-k}\mathbb{P}(W=n)}{\mathbb{P}(\xi+\eta=n)} \end{aligned}$$

Summing the preceding equation on k gives

$$1=1\frac{\mathbb{P}(W=n)}{\mathbb{P}(\xi+\eta=n)}$$

which is the desired result.

With the particular form of W, we immediately find, with $w(k)=\mathbb{P}(W=k)$

$$\frac{w(k)}{w(k-1)}=\frac{c(1+v)}{1+c(1+v)}+\frac{c(1+v)(a-2)}{1+c(1+v)}\frac{1}{k} \quad k\geq 2$$

which is well in the form of the $(r,s,1)$ class with

$$\begin{aligned} r &= \frac{c(1+v)}{1+c(1+v)} \\ s &= \frac{c(1+v)(a-2)}{1+c(1+v)} \\ w(0) &= 0 \\ w(1) &= c(1+v)(1-a)\frac{[1+c(1+v)]^{-a}}{[1+c(1+v)]^{1-a}-1} \end{aligned}$$

We have that W is a member of the $(r,s,1)$ class of counting distributions.
Now let us study the distribution of

$$(U,V)=(X_1+\cdots+X_\xi,Y_1+\cdots+Y_\eta)$$

If W is a member of the $(r,s,1)$ class we have :

$$\frac{d}{du}\Psi_W(u)[1-ru]=w(1)+(r+s)\Psi_W(u) \tag{10}$$

We are now able to extend Hesselager's methodology to find the aggregate claims distribution.

Theorem 3

We have

$$f_{(\xi,\eta)}(1,0) = \rho_1 w(1) \tag{11}$$

$$f_{(\xi,\eta)}(n,m) = \rho_1\left(r+\frac{s}{n}\right) f_{(\xi,\eta)}(n-1,m) + r\rho_2 f_{(\xi,\eta)}(n,m-1) \quad n \geq 1 \quad \text{unless if } (n,m)=(1,0) \tag{12}$$

$$f_{(\xi,\eta)}(0,1) = \rho_2 w(1) \tag{13}$$

$$f_{(\xi,\eta)}(n,m) = \rho_2\left(r+\frac{s}{n}\right) f_{(\xi,\eta)}(n,m-1) + r\rho_1 f_{(\xi,\eta)}(n-1,m) \quad m \geq 1 \quad \text{unless if } (n,m)=(0,1) \tag{14}$$

Proof

We know that $(\xi|\xi+\eta)$ is binomial distributed. So

$$\phi_{\xi,\eta}(u,v) = \phi_W(\rho_1 u + \rho_2 v) \tag{15}$$

By differentiating (15) with respect to u, and using (10), we get

$$(1 - r\rho_1 u - r\rho_2 v)\frac{\partial}{\partial u}\phi_{\xi,\eta}(u,v) = \rho_1(w(1) + (r+s)\phi_{\xi,\eta}(u,v))$$

Inverting this expression we immediately get (11) and (12).
(13) and (14) are derived similarly. ∎

Theorem 4

We have

$$f_{(U,V)}(0,0) = 1 - \frac{\theta((1+v)(1-(\rho_1 f_X(0) + \rho_2 f_Y(0))))}{\theta(1+v)} \tag{16}$$

$$f_{(U,V)}(x,0) = \rho_1 w(1) f_X(x) + \rho_1 \sum_{i=0}^{x}(r+\frac{si}{x}) f_X(i) f_{(U,V)}(x-i,y) + \rho_2 r \sum_{j=0}^{y} f_Y(j) f_{(U,V)}(x,y-j) \quad , \quad x>0 \tag{17}$$

$$f_{(U,V)}(x,y) = \rho_1 \sum_{i=0}^{x}(r+\frac{si}{x}) f_X(i) f_{(U,V)}(x-i,y) + \rho_2 r \sum_{j=0}^{y} f_Y(j) f_{(U,V)}(x,y-j) \quad , \quad x>0\,, y>0 \tag{18}$$

$$f_{(U,V)}(0,y) = \rho_2 w(1) f_Y(y) + \rho_2 \sum_{i=0}^{y}(r+\frac{si}{y}) f_Y(i) f_{(U,V)}(x,y-i) + \rho_1 r \sum_{j=0}^{x} f_X(j) f_{(U,V)}(x-j,y) \quad , \quad y>0 \tag{19}$$

$$f_{(U,V)}(x,y) = \rho_2 \sum_{i=0}^{y}(r+\frac{si}{y}) f_Y(i) f_{(U,V)}(x,y-i) + \rho_1 r \sum_{j=0}^{x} f_X(j) f_{(U,V)}(x-j,y) \quad , \quad y>0\,, x>0 \tag{20}$$

Proof

As $f_{(U,V)}(0,0) = \phi(f_X(0), f_Y(0))$ we immediately find (16) by using equation (9).
From (12) we get

$$f_{(\xi,\eta)}(n,m) = \rho_1(r+\frac{s}{n})q(n-1,m) + r\rho_2 f_{(\xi,\eta)}(n,m-1) \tag{21}$$

$$\Leftrightarrow$$

$$n f_{(\xi,\eta)}(n,m) = r\rho_1(n-1) f_{(\xi,\eta)}(n-1,m) + \rho_1(r+s) f_{(\xi,\eta)}(n-1,m) + r\rho_2 n f_{(\xi,\eta)}(n,m-1) \tag{22}$$

Multiplying both sides of (22) by $u\psi_X^{n-1}(u)\frac{d}{du}\psi_X(u)\psi_Y^m(v)$ and summing on $n = 1 \to \infty, m = 1 \to \infty$ gives

$$u\frac{\partial}{\partial u}\phi_{(U,V)}(u,v) = r\rho_1 u\frac{\partial}{\partial u}\phi_{(U,V)}(u,v)\psi_X(u) + (r+s)\rho_1 u\frac{d}{du}\psi_X(u)\phi_{(U,V)}(u,v) + r\rho_2\phi_{(U,V)}(u,v)\psi_Y(v)$$

Inverting and rearranging this expression gives (18).
Multiplying both sides of (22) by $u\psi_X^{n-1}(u)\frac{d}{du}\psi_X(u)\psi_Y^m(0)$, summing on $n = 1 \to \infty, m = 0 \to \infty$ and adding $q(1,0)\psi_x(u)$ on both sides and multiplying by u on both sides gives

$$u\frac{\partial}{\partial u}\phi_{(U,V)}(u,v) = \rho_1 w(1)\psi_X(u) + r\rho_1 u\frac{\partial}{\partial u}\phi_{(U,V)}(u,v)\psi_X(u) + (r+s)\rho_1 u\frac{d}{du}\psi_X(u)\phi_{(U,V)}(u,v) + r\rho_2\phi_{(U,V)}(u,v)\psi_Y(v)$$

Inverting and rearranging this expression gives (17).
(19) and (20) are derived similarly. ■

Knowing the distribution of (U,V) it remains to evaluate

$$(S,T) = (U_1 + \cdots + U_L, V_1 + \cdots + V_L)$$

This is easily done with the bivariate Panjer's algorithm described in Walhin and Paris (1999a) for the particular case $L \sim Po(\theta(1+v))$.
We have

$$f_{(S,T)}(0,0) = e^{-\theta(1+v)(1-f_{(U,V)}(0,0))} \tag{23}$$

$$f_{(S,T)}(j,k) = \sum_m^j \sum_n^k [\theta(1+v)\frac{m}{j}] f_{(S,T)}(j-m,k-n) f_{(U,V)}(m,n) \quad j \geq 1 \tag{24}$$

$$f_{(S,T)}(j,k) = \sum_m^j \sum_n^k [\theta(1+v)\frac{n}{k}] f_{(S,T)}(j-m,k-n) f_{(U,V)}(m,n) \quad k \geq 1 \tag{25}$$

where we use the following notation :

$$\sum_{x_1,\ldots,x_k}^{s_1,\ldots,s_k} g(x_1,\ldots,x_k) = \sum_{x_1=0}^{s_1} \ldots \sum_{x_k=0}^{s_k} g(x_1,\ldots,x_k) - g(0,\ldots 0)$$

Numerical example :

Let us work with the portfolio given in table 5 and the Hofmann fit of this portfolio.
Let us assume that the distributions X (material damage) and Y (bodily injury) are given by :

X	$f(x)$	Y	$g(y)$
1	0.2	5	0.2
2	0.2	10	0.36
3	0.2	20	0.22
4	0.1	50	0.11
5	0.1	100	0.11
10	0.1		
20	0.1		

Table 1: Claims distributions

We find the aggregate claims distribution :

	$S_1 = 0$	$S_1 = 1$	$S_1 = 2$	$S_1 = 3$	$\cdots$
$S_2 = 0$	0.9410275	0.0010876	0.0019589	0.0000042	
$S_2 = 5$	0.0100336	0.0000219	0.0000395	0.0000001	
$S_2 = 10$	0.0101349	0.0000223	0.0000403	0.0000001	
$S_2 = 15$	0.0102375	0.0000228	0.0000411	0.0000001	
$\cdots$					

Table 2: Aggregate Claims distribution

Remark :

In the case of the bivariate mixed negative binomial distribution, a single-staged algorithm is given in Hesselager (1996).

7. Fits

In this section we fit two data sets given in Partrat(1994).
Data set 1 gives the yearly frequencies of hurricanes affecting two zones (zone 1 and zone 3) of the United States.
Data set 2 gives the yearly frequencies of an automobile third party liability portfolio, divided in material damage (type 1) and bodily injury (type 2) claims.

		$zone3$			
$zone1$		0	1	2	3
0	obs	27	9	3	2
	$a = 0$	27.66	13.03	3.07	0.48
	$a = 0.0057$	27.59	13.05	3.08	0.48
	BP	28.24	12.71	2.86	0.48
1	obs	24	13	1	0
	$a = 0$	20.44	9.65	2.28	0.36
	$a = 0.0057$	20.47	9.68	2.29	0.36
	BP	20.30	9.79	2.35	0.42
2	obs	8	2	1	0
	$a = 0$	7.57	3.58	0.85	0.13
	$a = 0.0057$	7.59	3.59	0.84	0.13
	BP	7.29	3.75	0.96	0.19
3	obs	1	0	2	0
	$a = 0$	1.87	0.89	0.21	0.03
	$a = 0.0057$	1.87	0.88	0.21	0.03
	BP	2.12	1.16	0.32	0.06

Table 3: Data set 1

For this data set, the maximum likelihood procedure for the bivariate Hofmann distribution gives almost the bivariate mixed Poisson distribution, i.e. our model considers that the random variables are independent. This is not a surprise if we consider the model fitted by Partrat (1994) where he uses a bivariate Poisson distribution obtained by trivariate reduction. We copy his results in our table.

	Poisson	Hofmann	Bivariate Poisson
v	0.1084	0.1084	
p	0.0565	0.0565	
c		1.0220	
a	0	0.005784	
λ_1			0.71876
λ_2			0.44994
μ			0.02317
l	-187.9615	-187.9607	-187.88

Table 4: Parameter estimates and loglikelihood - Data set 1

		Bodily injury		
Material damage		0	1	2
0	*obs*	171345	918	2
	$a = 0$	171086.9	946.0	2.6
	$a = 1$	171348.7	897.1	4.6
	$a = 0.5$	171348.9	897.5	4.6
	$a = 0.2982$	171345.9	898.6	4.5
1	*obs*	8273	73	0
	$a = 0$	8726.4	48.2	0.1
	$a = 1$	8275.6	86.3	0.7
	$a = 0.5$	8279.2	84.9	0.8
	$a = 0.2982$	8289.4	82.7	0.8
2	*obs*	389	5	0
	$a = 0$	222.5	1.2	0.0
	$a = 1$	398.1	6.2	0.1
	$a = 0.5$	391.7	6.9	0.1
	$a = 0.2982$	381.8	7.6	0.1
3	*obs*	31	1	0
	$a = 0$	3.8	0.0	0.0
	$a = 1$	19.1	0.4	0
	$a = 0.5$	21.4	0.6	0
	$a = 0.2982$	23.5	0.8	0
4	*obs*	1	0	0
	$a = 0$	0.0	0.0	0.0
	$a = 1$	0.9	0.0	0.0
	$a = 0.5$	1.3	0.0	0.0
	$a = 0.2982$	1.9	0.1	0.0

Table 5: Data set 2

	Poisson	Negative Binomial	PIG	Hofmann
v	0.1084	0.1084	0.1084	0.1084
p	0.0565	0.0565	0.0565	0.0565
c		$0,0560$	0.1143	0.1940
a	0	1	0.5	0.2982
l	-43251.57	-43143.11	-43141.78	-43141.27

Table 6: Parameter estimates and loglikelihood - Data set 2

8. Bonus-malus systems with changing frequency

In this section we extend the hypothesis of section 2 in order to construct bonus-malus systems allowing for a mean frequency to vary geometrically from a year to another.
Using the same notations than for introducing the Hofmann's distribution, we introduce a new process :

$$\mathbb{P}(N(t) = n) = \Pi(n, \frac{1-v^t}{1-v})$$

This distribution obeys the hypothesis given in Besson and Partrat (1992) :

- $N_1 - N_0|\Lambda, \cdots, N_{t+1} - N_t|\Lambda$ are independent
- $N_i - N_{i-1} \sim Po(\lambda v^{i-1})$

Then by Walhin and Paris (1999b) we get the percentage premium for the expected premium principle (non loaded premium) :

$$\frac{100}{p}\frac{k+1}{t}\frac{\Pi(k+1, \frac{1-v^t}{1-v})}{\Pi(k, \frac{1-v^t}{1-v})}$$

and for the exponential utility principle (loaded premium) :

$$100\frac{\ln\left(\frac{t}{t-w}\right)^k \frac{\Pi(k, \frac{1-v^{t-w}}{1-v})}{\Pi(k, \frac{1-v^t}{1-v})}}{\ln \Pi(0, \frac{1-v^{-w}}{1-v})}$$

with security loading w.
Let us see the effect of a frequency decreasing of 2% each year on the bonus-malus coefficients. We use the Partrat's portfolio for which we aggregate bodily injury and material damage.

The fit is :

	Observed	Fitted
0	171345	171345, 9
1	9191	9187, 9
2	464	469, 0
3	36	32, 0
4	2	2, 8
5	0	0, 3

Table 7: Automobile portfolio

A classical bonus-malus table is given by

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$t=1$	94, 85	180, 60	362, 19	624, 95	914, 69
2	90, 69	164, 45	316, 10	537, 26	784, 81
3	87, 22	151, 94	281, 55	471, 55	687, 11
4	84, 26	141, 91	254, 69	420, 57	611, 02
5	81, 69	133, 67	233, 21	379, 90	550, 13
10	72, 50	107, 33	168, 75	259, 05	367, 99
20	62, 33	83, 31	116, 73	164, 66	224, 56

Table 8: Bonus malus table with constant frequency

A bonus-malus table under the hypothesis that the frequency diminishes of 2% per year is given by

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$t=1$	94, 85	180, 60	362, 19	624, 95	914, 69
2	90, 76	164, 00	314, 20	533, 37	778, 99
3	87, 41	151, 31	278, 73	465, 66	678, 20
4	84, 59	141, 27	251, 46	413, 66	600, 47
5	81, 17	133, 09	229, 86	372, 54	538, 75
10	73, 76	107, 52	166, 22	252, 33	356, 85
20	65, 07	85, 22	116, 56	161, 12	217, 09

Table 9: BM table with yearly decreasing frequency at rate 2%

We remark that the bonus for absence of claims is slower than in the case of a stable frequency. This is coherent with our intuition. Of course a more rapidly changing frequency will have a more dramatic effect. Let us analyze a frequency changing of 5% each year :

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$t=1$	$94,85$	$180,60$	$362,19$	$624,95$	$914,69$
2	$90,88$	$163,31$	$311,34$	$527,51$	$770,20$
3	$87,69$	$150,37$	$274,52$	$456,84$	$664,83$
4	$85,07$	$140,29$	$246,67$	$403,37$	$584,69$
5	$82,87$	$132,21$	$224,89$	$361,59$	$521,77$
10	$75,58$	$107,72$	$162,42$	$242,37$	$340,21$
20	$68,89$	$87,72$	$116,05$	$155,73$	$205,81$

Table 10: BM table with yearly decreasing frequency at rate 5%

9. Bonus-malus systems with two types of claims

In this section we construct a bonus-malus system depending on two types of claims. For example we will use bodily injury and material damage claims.
Let us remind that the pure premium asked to drivers is

$$\mathbb{E}N\mathbb{E}C \tag{26}$$

where N is the number of claims and C is the cost of claims.
In an a posteriori rating we have

$$\mathbb{E}(N(t+1)-N(t)|N(t))\mathbb{E}C \tag{27}$$

Our aim is to introduce the aspect of bodily injury and material damage claims in order to give more liability to drivers with bodily injury claims.
(26) becomes

$$\mathbb{E}N\mathbb{E}MD+\mathbb{E}M\mathbb{E}BI$$

where N and M are the number of material damage and bodily injury claims respectively and MD and BI are the respective costs.
(27) becomes

$$\mathbb{E}(N(t+1)-N(t)|N(t),M(t))\mathbb{E}MD+\mathbb{E}(M(t+1)-M(t)|N(t),M(t))\mathbb{E}BI$$

Unfortunately we have that

$$\begin{aligned}
\mathbb{E}(N(t+1)-N(t)|N(t)=n,M(t)=m) &= \frac{\int_0^\infty \lambda e^{-\lambda t(1+v)}(\lambda t)^{n+m}dU(\lambda)}{\int_0^\infty e^{-\lambda t(1+v)}(\lambda t)^{n+m}dU(\lambda)} \\
\mathbb{E}(M(t+1)-M(t)|N(t)=n,M(t)=m) &= v\frac{\int_0^\infty \lambda e^{-\lambda t(1+v)}(\lambda t)^{n+m}dU(\lambda)}{\int_0^\infty e^{-\lambda t(1+v)}(\lambda t)^{n+m}dU(\lambda)}
\end{aligned}$$

i.e. these quantities depend only of $n+m$. So the model is not able to catch the difference between two kind of claims in an a posteriori rating.
We will use univariate formulae :

$$\mathbb{E}(N(t+1)-N(t)|N(t))\mathbb{E}MD+\mathbb{E}(M(t+1)-M(t)|M(t))\mathbb{E}BI \tag{28}$$

In order to work with percentage premiums, we need to simplify the expression (28). Let us assume that the rapport between $\mathbb{E}BI$ and $\mathbb{E}MD$ is α :

$$\alpha=\frac{\mathbb{E}BI}{\mathbb{E}MD}$$

Then we have

$$\left[\mathbb{E}(N(t+1)-N(t)|N(t)=n)+\alpha\mathbb{E}(M(t+1)-M(t)|M(t)=m)\right]\mathbb{E}MD \tag{29}$$

Percentage premiums in function of $\mathbb{E}MD$ will be given by

$$100\frac{\mathbb{E}(N(t+1)-N(t)|N(t)=n)+\alpha\mathbb{E}(M(t+1)-M(t)|M(t)=m)}{p+\alpha pv}$$

We have applied this formula with the fit of the data set 2 and with an $\alpha=15$. We find

	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$
$n=0$	97	159	295	489	700
$n=1$	130	192	327	522	732
$n=2$	199	261	397	591	802
$n=3$	299	361	497	691	902
$n=4$	409	471	607	801	1012

Table 11: BM table $t=1$

	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$
$n=0$	86	140	253	416	596
$n=1$	101	154	267	430	611
$n=2$	126	179	292	456	636
$n=3$	163	217	330	493	673
$n=4$	208	262	375	538	718

Table 12: BM table $t=9$

Note that the bonus malus tables hereabove have the nice property to be financially balanced.

Using similar arguments than in Walhin and Paris (1999b), it is possible to construct a finite class bonus malus system.
For example we have choosen :
9 classes : 0 to 8
Entry : class 3
No claim : -1
Material damage claim : +1
Bodily injury claim : +3
The bonus-malus table is calculated as :

s	0	1	2	3	4	5	6	7	8
C_s	92	100	100	100	143	161	186	216	285

Table 13: Premium levels of the markovian bonus-malus system

10. Conclusion

In this paper we have extended the use of traditional bivariate mixed Poisson distributions into a general family of counting distributions. This family has nice properties. On the one hand it authorizes a maximum likelihood estimation in an univariate setting. On the other hand it gives stable algorithms for the evaluation of the bivariate aggregate claims distribution.
The fits of some insurance portfolio's are improved thanks to the use of the bivariate Hofmann distribution. However some portfolios may be better fitted with other bivariate distributions.
The construction of bonus-malus systems with a frequency changing geometrically has been put in a formal setting that clearly generalizes Besson and Partrat (1992) reasoning.
The construction of bonus-malus systems based on two types of claims is proposed. However it does not use the dependence between N and M.

Remark

A first version of this paper has been presented at the Royal Statistical Society Conference 1999, Warwick.

REFERENCES

Besson, J.L. and Partrat, C. (1992). Trend et Systèmes de Bonus-Malus. *Astin Bulletin*, 22:11–31.

Hesselager, O. (1996). Recursions for Certain Bivariate Counting Distributions and their Compound Distributions. *Astin Bulletin*, 26:35–52.

Kestemont, R.M. and Paris, J. (1985). Sur l'Ajustement du Nombre de Sinistres. *Bulletin of the Swiss Actuaries*, 85:157–164.

Panjer, H.H. (1981). Recursive Evaluation of a Family of Compound Distributions. *Astin Bulletin*, 12:22–26.

Partrat, C. (1994). Compound Model for Two Dependent Kinds of Claims. *Insurance : Mathematics and Economics*, 15:219–231.

Sundt, B. and Jewell, W.S. (1981). Further Results on Recursive Evaluation of Compound Distributions. *Astin Bulletin*, 12:27–39.

Walhin, J.F. and Paris, J. (1999). Using Mixed Poisson Distributions in connection with Bonus-Malus Systems. *Astin Bulletin*, 29:81–99.

Walhin, J.F. and Paris, J. (2000a). The Effect of Excess of Loss Reinsurance with Reinstatements on the Cedent's Portfolio. Discussion Paper 0003, Institut de Statistique, UCL.

Walhin, J.F. and Paris, J. (2000b). A Large Family of Discrete and Overdispersed Probability Laws. *forthcoming*.

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