Some comments on two approximations used for the pricing of reinstatements

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Abstract

The aim of this paper is to give some comments on two approximations used to price reinstatements related to excess of loss reinsurance. For the pro rata capita clause, we will study the rate on line method. For the pro rata temporis clause, we will study the use of a trivial approximation. The effect of an aggregate deductible is also looked at.

Keywords

Excess of loss reinsurance, reinstatements, pro rata capita, pro rata temporis, rate on line, aggregate deductible, order statistic, Panjer's algorithm, convolutions.

1. Introduction

Excess of loss reinsurance is widely used when it is the concern of the ceding company to reduce the cost of individual major losses. If D is the retention of the insurance company, then the reinsurer is liable for

$$R_i = \max(0, X_i - D) \tag{1}$$

where X_i denotes the amount of the ith claim.
Often the reinsurer limits its liability to a level L. $[D, D+L]$ is called the layer. (1) becomes

$$R_i = \min(L, \max(X_i - D, 0)) \tag{2}$$

If there are N claims during the year, the aggregate claims distribution of the reinsurer is

$$S_R = R_1 + \cdots + R_N \tag{3}$$

Unfortunately for the ceding company, the reinsurer does not always offer unlimited capacity. So he will try to limit his aggregate liability. A common way for doing this is to use reinstatements.

It is said that the reinsurer offers k reinstatements if he limits his yearly liability to $k+1$ times the layer L. Sometimes the reinstatements are free. Often they are paid. This means that when a claim affects the layer, the ceding company is obliged to pay a reinstatement premium in order to reinstate the layer.

This reinstatement premium is calculated pro rata capita, i.e. it considers the fraction of the layer used by the potential claims leading to the jth reinstatement premium :

$$\frac{1}{L}\min(L, \max(0, S_R - (j-1)L))$$

This fraction is multiplied by a certain price, or percentage (c_j) of the initial premium P :

$$\frac{Pc_j}{L}\min(L, \max(0, S_R - (j-1)L))$$

The latter formula gives the reinstatement premium to be collected in order to reinstate the layer for the jth time. Note that a single claim R_i may lead to a payment in regard of reinstatement j and $j+1$. However we do not need to write the formulae giving the reinstatement premium regarding claim i. This is not the case when there are pro rata temporis reinstatements. In fact, apart from the pro rata capita calculations, the reinstatement premium is sometimes calculated pro rata temporis as well. This means that we take into account the time remaining until the next reinsurance renewal $(1 - T_i)$ where T_i is the time elapsed until the ith claim.

In this case we have to write the formulae giving the reinstatement premium in regard of claim i. We shall come back to this point in section 4.

It is clear that pricing such specific clauses is not trivial. The case of pro rata capita only is treated in Sundt (1991) and in Walhin and Paris (1999). In section 2 we give the exact formula for the pricing of a pro rata capita only clause. In section 3 we study an approximation called the rate on line method. To the knowledge of the author, the case of pro rata temporis reinstatements is not treated in the literature. In section 4 we give the exact formula when there is only one reinstatement, which represents the most common case. We make some comments on a trivial approximate formula in section 5. Some extensions, including the case of an aggregate deductible and more than one reinstatement are discussed in section 6. Section 7 gives a conclusion. Throughout the paper we will use a numerical example, based on the following assumptions :

The X_i, N and T are independent.

N : Poisson with mean λ

X : truncated Pareto with parameters A, B and α

$$\begin{aligned} F_X(x) &= 0 \quad x \leq A \\ F_X(x) &= \frac{A^{-\alpha} - x^{-\alpha}}{A^{-\alpha} - B^{-\alpha}} \quad A < x < B \\ F_X(x) &= 1 \quad x \geq B \end{aligned}$$

where $F_X(x)$ denotes the cumulative density function of the random variable X.

T : Beta distributed with parameters a and b

$$f_T(t) = \frac{1}{B(a,b)} t^{a-1}(1-t)^{b-1} \quad 0 \leq t \leq 1 \quad a > 0, b > 0$$

where $f_T(t)$ denotes the density function of the random variable T and $B(a,b)$ is the Beta function :

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$$

Note that in the particular case where $a = b = 1$ the Beta distribution degenerates to the Uniform distribution.
We will take $\lambda = 1$ or 2, $A = 20$, $B = 50$, $\alpha = 1.5$, $a = 0.5, 1, 5$ and $b = 0.5, 1, 5$.

We will study 4 reinsurance excess of loss covers :

- 30 xs 20
- 10 xs 20
- 10 xs 30
- 10 xs 40

As we will use convolutions or the algorithm of Panjer (1981), we need a discretization of the random variable X. The discrete distribution X_{dis} has been obtained by the minimization of the Kolmogorov distance between the exact distribution and the approximated one (see Walhin and Paris (1998)). The minimization has been handled with the following constraints

- X_{dis} has the following support : $20, 21, \cdots, 49, 50$
- $F_{X_{dis}}(B) = 1$
- $F_{X_{dis}}(x) \geq 0 \quad \forall x = 20, \cdots, 50$
- $\mathbb{E}X_{dis} = \mathbb{E}X$
- $\mathbb{V}arX_{dis} = \mathbb{V}arX$
- $\mathbb{E}\min(10, \max(0, X_{dis} - 20)) = \mathbb{E}\min(10, \max(0, X - 20))$
- $\mathbb{E}\min(10, \max(0, X_{dis} - 30)) = \mathbb{E}\min(10, \max(0, X - 30))$

in order to keep the first two moments as well as the expectations on each layer we will study. The expectation on the layer 10 xs 40 is automatically kept by linear combination of the defined constraints.
The Kolmogorov distance between the exact and the discretized distribution is 0.0472.

2. Pro rata capita : exact formula

An initial premium P is paid. Reinstatement premiums, functions of P, might be paid. This random part of the premium income is

$$P\sum_{j=1}^{k}\frac{c_j}{L}\min(L,\max(0,S_R-(j-1)L))$$

The reinsurer covers $(k+1)$ times the layer :

$$\min(S_R,(k+1)L)$$

The premium P is the solution to the problem of equating the mean premium and the mean aggregate claims :

$$\mathbb{E}[P(1+\sum_{i=1}^{k}c_j\min(1,\max(0,\frac{S_R-(j-1)L}{L}))]=\mathbb{E}\min(S_R,(k+1)L)$$

This immediately gives

$$P=\frac{\mathbb{E}\min(S_R,(k+1)L)}{1+\sum_{j=1}^{k}\frac{c_j}{L}\mathbb{E}\min(L,\max(0,S_R-(j-1)L))}$$

This is the premium given by the expected value premium principle studied in Sundt (1991). Sundt(1991) also studies the standard deviation principle while Walhin and Paris (1999) study the PH transform principle.
In this paper we will only concentrate on the expected value premium principle.
The following table gives the premium for the 30 xs 20 cover with uniform price $c_j \equiv c$ of the reinstatement.

c/k	0	1	2	3	4
0%	8.75	9.48	9.51	9.52	9.52
50%		8.28	8.21	8.21	8.21
100%		7.34	7.23	7.22	7.22
150%		6.59	6.45	6.45	6.45

Table 1: 30 xs 20 : pure premium

3. The rate on line method

The rate on line is defined as

$$ROL = \frac{\mathbb{E}N\mathbb{E}R}{L}$$

It is the premium of an unlimited free reinstatements treaty covering the layer $[D, D+L]$ divided by the lenght of the layer L.
Let us assume that we do not know the distribution S_R. Instead, we know the rate on line. The rate on line method assumes that there are only total losses, i.e. losses hitting the layer completely . For such losses, the frequency is the ROL.
Let us define two new random variables :

$R' =$ L with probability 1

N' has the same distribution as N but with mean ROL

It is then easy to show that the formula for P becomes

$$P = L\frac{\sum\limits_{i=0}^{k} \mathbb{P}(N' > i)}{1 + \sum\limits_{i=1}^{k} c_i \mathbb{P}(N' > i-1)}$$

This formula is obviously easier to evaluate than the exact one.
Our numerical example gives the following approximate premiums :

c/k	0	1	2	3	4
0%	8.15	9.38	9.50	9.51	9.52
50%		8.26	8.22	8.21	8.21
100%		7.37	7.24	7.22	7.22
150%		6.66	6.47	6.45	6.45

Table 2: 30 xs 20 : approximate pure premium

We note that, in case of free reinstatements, the exact premium is always higher than the approximate one. This is obviously not a safe approximation and it can be shown on numerical examples that the error may be larger than 10%.
We will now prove that the conjecture mentionned above is always true.

Proposition 1 *Let N be such that its probability generating function $\psi_N(t) = W(\theta(t-1))$ where W is a function independent of the parameter θ.*
Then the approximate premium is always lower than the exact premium if the reinstatements are free.

Let us define the random variable R'' :

x	$\mathbb{P}[R'' = x]$
0	$1 - \frac{\mathbb{E}R}{L}$
L	$\frac{\mathbb{E}R}{L}$

By a theorem of Panjer and Willmot (1984) we have that

$$\begin{aligned} \psi_{S_{R''}} &= \psi_N(\psi_{R''};\theta) \\ &= \psi_N(1 - \frac{\mathbb{E}R}{L} + \frac{\mathbb{E}R}{L}\psi_{R'};\theta) \\ &= \psi_N(\psi_{R'};\frac{\mathbb{E}R}{L}\theta) \\ &= \psi_{S_{R'}} \end{aligned}$$

from which we conclude that $S_{R''} = S_{R'}$.
From the crossing condition (see Goovaerts et al. (1990) for a reference), it is clear that $R'' >_{sl} R$:

$$SL(R'', t) \geq SL(R, t) \quad \forall t$$

where $SL(R, t)$ denotes the stop loss premium for a risk R with retention t :

$$SL(R, t) = \int_t^\infty (x - t) dF_R(x)$$

By Bühlmann et al. (1977), the stop loss order is preserved under compounding :

$$SL(S_{R''}, t) \geq SL(S_R, t) \quad \forall t$$

With free reinstatements we have

$$\begin{aligned} P &= \mathbb{E}S_R - SL(S_R, (t+1)L) \\ P' &= \mathbb{E}S'_R - SL(S'_R, (t+1)L) \end{aligned}$$

Combined with the fact that $\mathbb{E}S_R = \mathbb{E}S_{R'}$ we find

$$P' \leq P$$

■

In particular the proposition is interesting when N is Poisson ($W(x) = e^x$ with $\theta = \lambda$) and Negative Binomial ($W(x) = (1-x)^{-r}$ with $\theta = \beta$).

The following tables give the fraction $\frac{\text{approximated premium}}{\text{exact premium}}$ for the layers 30 xs 20, 10 xs 20, 10 xs 30 and 10 xs 40 with uniform price (c) of the reinstatement.

c/k	0	1	2	3	4
0%	0.93174	0.98889	0.99898	0.99993	0.99999
50%		0.99757	1.00049	1.00007	1.00000
100%		1.00438	1.00165	1.00017	1.00001
150%		1.00988	1.00256	1.00026	1.00001

Table 3: 30 xs 20 : approximated / exact premium

c/k	0	1	2	3	4
0%	0.98554	0.99379	0.99837	0.99970	0.99996
50%		0.99659	0.99983	1.00010	1.00003
100%		0.99849	1.00073	1.00034	1.00008
150%		0.99986	1.00135	1.00050	1.00011

Table 4: 10 xs 20 : approximated / exact premium

c/k	0	1	2	3	4
0%	0.99100	0.99834	0.99983	0.99999	1.00000
50%		0.99924	1.00001	1.00001	1.00000
100%		0.99997	1.00016	1.00002	1.00000
150%		1.00058	1.00028	1.00003	1.00000

Table 5: 10 xs 30 : approximated / exact premium

c/k	0	1	2	3	4
0%	0.98841	0.99957	0.99999	1.00000	1.00000
50%		0.99990	1.00000	1.00000	1.00000
100%		1.00022	1.00002	1.00000	1.00000
150%		1.00052	1.00003	1.00000	1.00000

Table 6: 10 xs 40 : approximated / exact premium

We notice that the fraction increases with the price (c) of the reinstatements. This result can be shown analytically.
We also see that the fraction tends to unity with the number of reinstatements (k). This is logical as when $k \to \infty$, the claims distribution (R) is not important. The formula for unlimited uniformly paid reinstatements is

$$P = \frac{\mathbb{E}N\mathbb{E}R}{1 + \frac{c}{L}\mathbb{E}N\mathbb{E}R}$$

where we only use $\mathbb{E}R$.

Now let us assume that there is an aggregate deductible (AD), i.e. the reinsurer is liable for the aggregate claims in excess of AD. It is not difficult to extend the formulae and the following example shows that the approximation is not controlled and may give very bad results.
Let us assume an aggregate deductible (AD=15) for the cover 30 xs 20. We find

c/k	0	1	2	3	4
0%	1.23881	1.27606	1.28599	1.28773	1.28796
50%		1.26636	1.27298	1.27403	1.27415
100%		1.25740	1.26113	1.26157	1.26160
150%		1.24909	1.25029	1.25021	1.25015

Table 7: 30 xs 20 with $AD = 15$: approximated / exact premium

The results are bad and this is not surprising as we know from Bühlmann et al. (1977) that

$$SL(S'_R, AD) > SL(S_R, AD)$$

i.e. the unlimited free reinstatement premium is higher for the approximate case.

4. Pro rata temporis : exact formula

The pro rata temporis clause is essentially used for catastrophe reinsurance where typically only one reinstatement is offered by the reinsurer. In this section we will only consider this particular case .
We now have to take into account the time remaining until the renewal after each claim. Thus we introduce a new random variable T, the remaining time. We also have to study the random variable Y_i, part of the ith claim leading to a reinstatement premium. The Y_i are defined as

$$\begin{aligned} Y_1 &= R_1 \\ Y_i &= \min[\max(0, L - \sum_{l=1}^{(i-1)} Y_l), R_i] \quad , i \geq 2 \end{aligned}$$

Let us recall that the R_i are assumed to be independent. So we have

$$f_{R_1,\cdots,R_r}(x_1,\cdots,x_r) = f_{R_1}(x_1) \times \cdots \times f_{R_r}(x_r)$$

The Y_i are associated with the order statistic of T : $T_{(1:r)} \leq \ldots \leq T_{(r:r)}$.
From now $i \in \{1,\cdots,r\}$ will also denote the index of the ith order statistic: $T_{(i:r)}$ when we consider r claims.

The exact premium for an excess of loss reinsurance with pro rata capita and pro rata temporis reinstatements is :

$$P = \frac{\mathbb{E}\min(S_R, 2L)}{1 + \sum_{n=1}^{r} \mathbb{P}(N=n) \sum_{i=1}^{n} \frac{c}{L}\mathbb{E}(1 - T_{(i:n)})\mathbb{E}Y_i} \tag{4}$$

Unfortunately, evaluating $\mathbb{E}Y_i$ even for i small is very time consuming. Indeed, the random variables $Y_1,\cdots,Y_r$ are highly correlated. So, evaluating $\mathbb{E}Y_i$ requires an i-multiple integral (or sum in the discrete case).
The expectations $\mathbb{E}T_{(i:r)}$ are not too complicated. Indeed, it is not difficult to show that the distribution of the i-th order statistic of T is given by

$$F_{T_{(i:r)}}(u) = \sum_{l=i}^{r} \frac{r!}{l!(r-l)!} F_T^l(u)(1 - F_T(u))^{r-l}$$

Then we immediately have

$$\mathbb{E}(T_{i:r}) = \int_0^\infty \left(1 - \sum_{l=i}^{r} \frac{r!}{l!(r-l)!} F_T^l(u)(1 - F_T(u))^{t-l}\right) du$$

For the practical evaluation of a premium, we have to assume a maximum number of claims r. This is not a problem as these treaties are essentially used for natural perils for which we

do not expect a high frequency.
For our numerical example, we will use a truncated Poisson random variable. We choose to truncate at $r = 4$. The truncation is such that $\mathbb{P}(N \geq 4)$ accumulates at 4.
In order to make different modelizations of the time pattern, we work with a Beta distribution and we let its parameters vary.
We find

i	1	2	3	4
$\mathbb{E}Y_i$	9.52	8.30	5.84	3.42

Table 8: $\mathbb{E}Y_i$ for the 30 xs 20 cover

r/i	1	2	3	4
1	0.5	–	–	–
2	0.58593	0.41407	–	–
3	0.62889	0.5	0.37110	–
4	0.65630	0.54666	0.45333	0.34369

Table 9: $\mathbb{E}(1 - T_{(i:r)})$ for $a = b = 5$

	P $a = b = 5$	P $a = 5, b = 0.5$	P no pro rata temporis
$c = 50\%$	8.80	9.32	8.25
$c = 100\%$	8.23	9.20	7.32
$c = 150\%$	7.73	9.07	6.57

Table 10: Premiums for the 30 xs 20 cover

Obviously the premiums with no pro rata temporis clause are smaller as there will not be any discount for remaining time for the potential reinstatement premiums.

5. Pro rata temporis : a natural approximation

It is clear that a trivial simplification of the formula (4) is to consider that each claim occurs at time $\mathbb{E}T$. Then we assume a mean time remaining equal to $\mathbb{E}(1-T)$ for each claim. This is a very natural approximation.

$$\begin{aligned} P &= \frac{\mathbb{E}\min(S_R, 2L)}{1+\mathbb{E}(1-T)\sum_{n=1}^{r}\mathbb{P}(N=n)\sum_{i=1}^{n}\frac{c}{L}\mathbb{E}Y_i} \\ &= \frac{\mathbb{E}\min(S_R, 2L)}{1+\mathbb{E}(1-T)\frac{c}{L}\mathbb{E}\min(L, \max(0, S_R-L))} \end{aligned}$$

Clearly this formula is far easier to use than the exact formula.
Most of the time, the reinstatements are payable at 100%. We will henceforth work with $c = 100\%$. We have made some comparisons between the exact and the approximate formulae. The tables give the fraction $\frac{\text{exact}}{\text{approximated}}$ for λ and a and b varying as well as the different layers. We find

	$a=b=5$	$a=0.5, b=5$	$a=5, b=0.5$	$a=0.5, b=0.5$	$a=b=1$
$\lambda=0.5$	0.9995	0.9996	0.9996	0.9988	0.9990
$\lambda=1$	0.9980	0.9988	0.9986	0.9953	0.9961
$\lambda=2$	0.9929	0.9959	0.9948	0.9836	0.9865

Table 11: 30 xs 20 : exact / approximated

	$a=b=5$	$a=0.5, b=5$	$a=5, b=0.5$	$a=0.5, b=0.5$	$a=b=1$
$\lambda=0.5$	0.9968	0.9981	0.9977	0.9925	0.9938
$\lambda=1$	0.9899	0.9946	0.9920	0.9766	0.9806
$\lambda=2$	0.9734	0.9867	0.9767	0.9395	0.9497

Table 12: 10 xs 20 : exact / approximated

	$a=b=5$	$a=0.5, b=5$	$a=5, b=0.5$	$a=0.5, b=0.5$	$a=b=1$
$\lambda=0.5$	0.9994	0.9996	0.9996	0.9987	0.9989
$\lambda=1$	0.9980	0.9988	0.9986	0.9955	0.9963
$\lambda=2$	0.9941	0.9967	0.9956	0.9863	0.9887

Table 13: 10 xs 30 : exact / approximated

	$a=b=5$	$a=0.5, b=5$	$a=5, b=0.5$	$a=0.5, b=0.5$	$a=b=1$
$\lambda=0.5$	0.9999	0.9999	0.9999	0.9999	0.9999
$\lambda=1$	0.9999	0.9999	0.9999	0.9997	0.9998
$\lambda=2$	0.9996	0.9998	0.9997	0.9992	0.9993

Table 14: 10 xs 40 : exact / approximated

It seems clear that the approximated formula is always higher than the exact formula, which is a conservative approximation.
Moreover the error does not seem to be large. This is logical.
In the case of 0 claim, there is no error.
In the case of 1 claim, $\mathbb{E}T = \mathbb{E}T_{(1:1)}$ and so there is no error either.
Thus, the error occurs in case $[N \geq 2]$, which is rather exceptional in catastrophe treaties.

We note that the case $10xs20$ is the worst, which is logical. Indeed, for that layer, EY_1 will be the largest because it is the layer for which the probability of having a total loss is the higher. If, moreover, $\mathbb{E}(1-T_{(1:r)})$ is quite different from $\mathbb{E}(1-T)$ (case $a=b=0.5$), the approximation is bad. This is almost the worst conceivable case for our approximation.

We will now prove that the above conjecture is always true.

Lemma 1 *For the order statistic $T_{(1:r)}, \cdots, T_{(r:r)}$, we have*

$$\sum_{i=1}^{r} \mathbb{E}T_{(i:r)} = r\mathbb{E}T$$

This result immediately comes from the almost sure equality :

$$T_{(1:r)} + \cdots + T_{(r,r)} = T_1 + \cdots + T_r \quad \text{a.s.}$$

■

Proposition 2 *With pro rata temporis reinstatement premiums, the approximated premium is always larger than the exact premium.*

We have to prove that

$$\sum_{n=1}^{r} \mathbb{P}(N=n) \sum_{i=1}^{n} \frac{c}{L} \mathbb{E}(1-T_{(i:n)}) \mathbb{E}Y_i < \mathbb{E}(1-T) \sum_{n=1}^{r} \mathbb{P}(N=n) \sum_{i=1}^{n} \frac{c}{L} \mathbb{E}Y_i$$

Let us show that the inequality is true for each term of the sum in n :

$$\sum_{i=1}^{n} \mathbb{E}(1 - T_{(i:n)}) \mathbb{E}Y_i < \mathbb{E}(1 - T) \sum_{i=1}^{n} \mathbb{E}Y_i$$

Let γ be such that

$$\mathbb{E}T_{(\gamma:n)} \leq \mathbb{E}T \leq \mathbb{E}T_{(\gamma+1:n)}$$

We have

$$\sum_{i=1}^{\gamma} \mathbb{E}T_{(i:n)} \mathbb{E}Y_i + \sum_{i=\gamma+1}^{n} \mathbb{E}T_{(i:n)} \mathbb{E}Y_i > \sum_{i=1}^{\gamma} \mathbb{E}T \mathbb{E}Y_i + \sum_{i=\gamma+1}^{n} \mathbb{E}T \mathbb{E}Y_i$$

which is equivalent to

$$\sum_{i=\gamma+1}^{n} (\mathbb{E}T_{(i:n)} - \mathbb{E}T) \mathbb{E}Y_i > \sum_{i=1}^{\gamma} (\mathbb{E}T - \mathbb{E}T_{(i:n)}) \mathbb{E}Y_i$$

and the last inequality is always true because
i) $\mathbb{E}Y_1 \geq \mathbb{E}Y_2 \geq \cdots \geq \mathbb{E}Y_n$
ii) $\sum_{i=\gamma+1}^{n} (\mathbb{E}T_{(i:n)} - \mathbb{E}T) = \sum_{i=1}^{\gamma} (\mathbb{E}T - \mathbb{E}T_{(i:n)})$ because of lemma 1. ■

In the next section we will extend the formulae to the case of an aggregate deductible with pro rata temporis reinstatements and to the case of two reinstatements at different prices.

7. Pro rata temporis : extensions

In the case of an aggregate deductible AD, the random variables R_i are transformed into the random variables Z_i :

$$\begin{aligned} W_1 &= \max(0, R_1 - AD) \\ W_i &= \max(0, R_i - \max(0, (AD - \sum_{l=1}^{i-1} R_l))) \quad , \quad i \geq 2 \\ Z_1 &= W_1 \\ Z_i &= \min[\max(0, L - \sum_{l=1}^{(i-1)} Z_l), W_i] \quad , \quad i \geq 2 \\ P &= \frac{\mathbb{E}\min(S_R, 2L)}{1 + \sum_{n=1}^{r} \mathbb{P}(N=n) \sum_{i=1}^{n} \frac{c}{L}\mathbb{E}(1 - T_{(i:n)})\mathbb{E}Z_i} \end{aligned}$$

For $\lambda = 1$ and $c = 100\%$, we have evaluated the ratio $\frac{\text{exact}}{\text{approximated}}$ for two aggregate deductibles. We find

	$a=b=5$	$a=0.5, b=5$	$a=5, b=0.5$	$a=b=5$	$a=b=1$
$AD=15$	1.0036	1.0021	1.0025	1.0085	1.0070
$AD=45$	1.0005	1.0003	1.0003	1.0013	1.0010

Table 15: 30 xs 20 : exact / approximated

We can see that the approximation is not conservative. This is due to the fact that the Z_i are no longer ordered. Note that for some AD, the approximation is conservative.
Note also that even if the approximation is not conservative, it is better than without an aggregate deductible. This is due to the fact that $Z_i < Y_i$. So the error decreases.

Now let us extend the formulae to the case of two reinstatements payable at prices c_1 and c_2. The case of multiple reinstatements is similar but has no practical interest. This is the reason why we limit ourselves to the case of two reinstatements.
The random variables Y_i are transformed into the random variables Z_i. Note that the random variables Z_i take into account the price of the reinstatement.

$$
\begin{aligned}
Z_1 &= R_1 \\
Z_2 &= c_1(L-R_1)+c_2(R_1+R_2-L) \quad \text{if } R_1+R_2 \geq L \\
& c_1 R_2 \quad \text{else} \\
Z_k &= c_1 R_k \quad \text{if } \sum_{i=1}^{k} R_i \leq L \\
& c_2 \max(0, 2L - \sum_{i=1}^{k-1} R_i) \quad \text{if } \sum_{i=1}^{k} R_i \geq 2L \\
& c_1 \max(0, L - \sum_{i=1}^{k-1} R_i) + c_2[\min(L, \sum_{i=1}^{k-1} R_i) + R_k - L] \quad \text{else} \\
P &= \frac{\mathbb{E}\min(S_R, 2L)}{1+\sum_{n=1}^{r}\mathbb{P}(N=n)\sum_{i=1}^{n}\frac{1}{L}\mathbb{E}(1-T_{(i:n)})\mathbb{E}Z_i}
\end{aligned}
$$

As a numerical example, we take $a=b=1$ and $\lambda=1$. We find

c_1	0	0	1	2	1	1	2	2
c_2	1	2	0	0	1	2	1	2
ratio	0.97	0.97	0.99	1.02	0.99	1.00	1.02	1.02

Table 16: 10 xs 20 : exact / approximated

We observe that the approximation cannot be controlled.

6. Conclusion

We have studied throughout this paper two approximations used to price specific clauses for reinsurance treaties.
The rate on line method approximates the premium when there are free or paid reinstatements pro rata capita. We saw that in case of free reinstatements, the approximation is always too low. It is not conservative. As to paid reinstatements, there is no general rule. In conclusion, I wish to remark that this method should not be used with free reinstatements and, in general, the exact method should be preferred since it is not too difficult to set an exact price with the use of recursive techniques like the algorithm of Panjer (1981).
In case of reinstatements which are payable pro rata temporis, we saw that the exact calculation is extremely difficult even with a low frequency of claims. The good news is that the natural simplification is conservative in the sense that it always gives a higher premium than the exact one in the case of one paid reinstatement, which is the most common one. In case of one reinstatement and an aggregate deductible, the approximation is not conservative but better in relative terms. The case of two (or more) reinstatements, which is not common in catastrophe treaties, is not well controlled by the approximation.

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