# The effect of excess-of-loss reinsurance with reinstatements on the cedent's portfolio

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## Abstract

The adjustment coefficient for the cedent's retained risk after excess-of-loss reinsurance with reinstatements is calculated.
Therefore we need a multivariate aggregate claims distribution. This distribution is easily given by a multivariate extension of Panjer's recursion.
Numerical examples show the interest for the cedent to calculate the adjustment coefficient for its portfolio when buying excess of loss reinsurance with reinstatements.
An optimal organization of the calculations is discussed.

## Keywords

Excess-of-loss reinsurance, aggregate claims distribution, cedent, paid or free reinstatements, loading, net premium principle, multivariate Panjer's recursion, multivariate De Pril's recursion.

## 1. Introduction

Let us assume a portfolio with the following characteristics :

the number of claims is a random variable $N$

the severity of the claims is a random variable $X$

Then

$$S = X_1 + \cdots + X_N$$

is the aggregate claims distribution of the portfolio for the period considered.
We will assume that the $X_i$'s are iid and independent of $N$.
In order to limit the heterogeneity of his portfolio, the Insurer may ask for an excess-of-loss reinsurance.
In such a treaty, the Reinsurer will pay the part of every claim exceeding a deductible $D$ up

to a limit $D+L$. $L$ is called the layer.
We have for each claim :

$$A_i = \begin{cases} X_i & \text{if} \quad X_i < D \\ D & \text{if} \quad D \leq X_i \leq D+L \\ X_i - L & \text{if} \quad X_i > D+L \end{cases}$$

the retained part of a risk for the cedent. $R_i = X_i - A_i$ is the part of the risk for which the Reinsurer is liable.
The aggregate parts for both the Insurer and the Reinsurer are

$$S_A = A_1 + \cdots + A_N$$
$$S_R = R_1 + \cdots + R_N$$

Note that there may be more than one layer as it will be seen in the subsequent sections.
In this paper we will be interested in the probability of ruin in discrete time. Let $U_n$ denotes the surplus of the insurer at time $n$. We have

$$U_n = u + nc - (S_1 + \cdots + S_n)$$

where $u$ denotes the initial surplus of the insurer
$c$ denotes the premium earned each year
$S_i$ denotes the aggregate claims for the year $i$ $\quad 1 \leq i \leq n$
We assume that the $S_i$ are iid

It is well known that the probability of ruin is bounded (Lundberg's inequality)

$$\mathbb{P}[ruin] \leq e^{-Ru} \tag{1}$$

where $R$, the adjustment coefficient, is given by the only strictly positive solution of

$$e^{-cr}\mathbb{E}[e^{rS}] = 1 \tag{2}$$

In view of (1), we are interested in getting the higher adjustment coefficient. However the Insurer is also interested in having the higher expected gain

$$\mathbb{E}G = c - \mathbb{E}S \tag{3}$$

Bowers et al (1986) studied some reinsurance arrangements, among others the excess-of-loss treaty, from the point of view of the expected gain and the adjustment coefficient.
The aim of this paper is to make the same kind of study for excess-of-loss arrangements with reinstatements. Indeed, in practice, the Reinsurer will often limit his liability to a fixed number of times the layer.
If the layer may be consumed $k+1$ times, we say that there are $k$ reinstatements. The latter may be free or paid. In case of paid reinstatements, the reinstatements premiums are a fraction $c_i$ , $1 \leq i \leq k$ of the initial reinsurance premium $p_0$ in proportion of the part of the layer hit by the claims.
Sundt(1990) has given the methodology to price excess-of-loss treaties with reinstatements.
If there are $k$ reinstatements in our excess-of-loss arrangement, the part of the cedent becomes

$$S_{Ced} = S_A + \max(0, S_R - (k+1)L) + p_0 \sum_{i=1}^{k} c_i \min(1, \max(0, \frac{S_R - (i-1)L}{L})) \tag{4}$$

We assume implicitely that the reinstatement premiums are considered as claims because we put them in $S_{Ced}$.
Now, if we want to study the adjustment coefficient of the cedent's portfolio, we have to compute the distribution of $S_{Ced}$ which is a function of $S_A$ and $S_R$ that are not independent. We therefore need the joint distribution of $(S_A, S_R)$ in order to find the distribution of $S_{Ced}$. We have

$$(S_A, S_R) = (\sum_{i=1}^{N} A_i, \sum_{i=1}^{N} R_i)$$

By restricting the $X_i$'s to be arithmetic and $N$ to belong to the Panjer's class of counting distributions, we will extend Panjer's formula for this bivariate case in order to obtain the bivariate distribution of aggregate claims.
The paper is organized as follows : section 2 deals with the bivariate extension of Panjer's algorithm. Section 3 compares different reinstatements from the point of view of the cedent for given portfolios. Section 4 gives the multivariate extension of Panjer's algorithm. Section 5 completes section 3 by considering excess-of-loss arrangements with multiple layers. Section 6 shows how to organize the calculations. Section 7 gives the conclusion.

## 2. Bivariate Panjer's recursion

Bivariate extensions of Panjer's algorithm have been recently studied in Hesselager (1996) where one can find the bivariate aggregate claim distribution of $(S_1, S_2)$ with

$$S_1 = \sum_{i=1}^{N_0+N_1} X_i$$

$$S_2 = \sum_{i=1}^{N_0+N_2} Y_i$$

where the $N_i$ , $0 \leq i \leq 2$ belong to Panjer's class of counting distributions and the $X_i$ are independent of the $Y_i$.
If one takes the particular case that $N_1$ and $N_2$ are degenerate to 0, we have almost the solution to the problem raised in the introduction behave that we have to relax the hypothesis of independence between $X_i$ and $Y_i$. This is done in the following proposition.
We will adopt the following notation :

$$\sum_{x_1,\ldots,x_k}^{s_1,\ldots,s_k} g(x_1,\ldots,x_k) = \sum_{x_1=0}^{s_1} \ldots \sum_{x_k=0}^{s_k} g(x_1,\ldots,x_k) - g(0,\ldots 0)$$

**Proposition 1**

*Let*

$$(S_1, S_2) = (\sum_{i=1}^{N} X_i, \sum_{i=1}^{N} Y_i)$$

*with* $(X_i, Y_i)$ *iid, arithmetic and independent of* $N$
$X_i$ *and* $Y_i$ *are not necessarily independent*
$N$ *belonging to Panjer's class, i.e. such that*

$$\frac{p(n)}{p(n-1)} = a + \frac{b}{n} \quad , n \geq 1$$

*Then, if* $\Psi_N(z)$ *denotes the probability generating function of* $N$*, we have*

$$\begin{aligned}
(5) f_{(S_1,S_2)}(0,0) &= \Psi_N(f_{(X_1,X_2)}(0,0)) \\
(6) f_{(S_1,S_2)}(j,k) &= \frac{1}{1 - a f_{(X_1,X_2)}(0,0)} \sum_{m}^{j} \sum_{n}^{k} [a + b\frac{m}{j}] f_{(S_1,S_2)}(j-m, k-n) f_{(X_1,X_2)}(m,n) \\
& \quad j \geq 1 \\
(7) f_{(S_1,S_2)}(j,k) &= \frac{1}{1 - a f_{(X_1,X_2)}(0,0)} \sum_{m}^{j} \sum_{n}^{k} [a + b\frac{n}{k}] f_{(S_1,S_2)}(j-m, k-n) f_{(X_1,X_2)}(m,n) \\
& \quad k \geq 1
\end{aligned}$$

*Proof*
Let $\Psi_{(X_1,X_2)}(z_1, z_2)$ and $\Psi_{(S_1,S_2)}(z_1, z_2)$ be the probability generating functions of $(X_1, X_2)$ and $(S_1, S_2)$ respectively.
From

$$f_{(S_1,S_2)}(j,k) = \sum_{n=0}^{\infty} p(n) f^{*n}_{(X_1,X_2)}(j,k)$$

we get

$$\Psi_{(S_1,S_2)}(z_1, z_2) = \sum_{n=0}^{\infty} p(n) \Psi^{n}_{(X_1,X_2)}(z_1, z_2)$$

from which (5) follows immediately.
By hypothesis, one has

$$np(n) = a(n-1)p(n-1) + (a+b)p(n-1)$$

If $\Psi^{(z_i)}_{X_1,X_2}(z_1, z_2)$ is the partial derivative of $\Psi_{X_1,X_2}(z_1, z_2)$ with respect to $z_i$ and if we multiply on both sides by $\Psi^{n-1}_{(X_1,X_2)}(z_1, z_2) z_1 \Psi^{(z_1)}_{(X_1,X_2)}(z_1, z_2)$ and sum over $n = 1 \to \infty$, we get

$$z_1 \Psi^{(z_1)}_{(S_1,S_2)}(z_1, z_2) = a \Psi_{(X_1,X_2)}(z_1, z_2) z_1 \Psi^{(z_1)}_{(S_1,S_2)}(z_1, z_2) + (a+b) z_1 \Psi^{(z_1)}_{(X_1,X_2)}(z_1, z_2) \Psi_{(S_1,S_2)}(z_1, z_2)$$

which gives, returning in the initial space

$$\begin{aligned} jf_{(S_1,S_2)}(j,k) &= a\sum_{m=0}^{j}\sum_{n=0}^{k} f_{(X_1,X_2)}(m,n)(j-m)f_{(S_1,S_2)}(j-m,k-n) \\ &\quad +(a+b)\sum_{m=0}^{j}\sum_{n=0}^{k} f_{(X_1,X_2)}(m,n)mf_{(S_1,S_2)}(j-m,k-n) \end{aligned}$$

(6) is easily derived after an algebraic manipulation.
(7) follows similarly which completes the proof. □

Of course more complicated bivariate recursions can be derived in the same way, for example with the Sundt and Jewell (1981) family of counting distributions :

$$\frac{p(n)}{p(n-1)} = a + \frac{b}{n} \quad n = m, m+1, \cdots \quad m \geq 1$$

As a subproduct of this bivariate recursion, we can also derive a formula for the convolutions of bivariate arithmetic distributions, extending De Pril's (1985) formula.
Let

$$(X,Y)^{*^n} = (X_1 + \cdots + X_n, Y_1 + \cdots + Y_n)$$

for which one has

$$f^{*^{n+1}}_{(X,Y)}(x,y) = \sum_{j=0}^{x}\sum_{k=0}^{y} f^{*^n}_{(X,Y)}(j,k) f_{(X,Y)}(x-j,y-k)$$

**Proposition 2**

*Let $(X_i, Y_i)$ be defined on the non-negative integers and $f_{X,Y}(0,0) > 0$.*
*Then the following recursion holds :*

$$\begin{aligned} (8)\, f^{*^n}_{(X,Y)}(0,0) &= f^n_{X,Y}(0,0) \\ (9)\, f^{*^n}_{(X,Y)}(x,y) &= \frac{1}{f_{(X,Y)}(0,0)}(\sum_{j}^{x}\sum_{k}^{y}\left[\frac{n+1}{x}j-1\right] f^{*^n}_{(X,Y)}(x-j,y-k)f_{(X,Y)}(j,k)\; x \geq 1 \\ (10)\, f^{*^n}_{(X,Y)}(x,y) &= \frac{1}{f_{(X,Y)}(0,0)}(\sum_{j=0}^{x}\sum_{k=0}^{y}\left[\frac{n+1}{y}k-1\right] f^{*^n}_{(X,Y)}(x-j,y-k)f_{(X,Y)}(j,k)\; y \geq 1 \end{aligned}$$

*Proof :*

From the proposition 1 and by assuming

$$(R_1, R_2) = (W_1 + \cdots + W_K, Z_1 + \cdots + W_K)$$

with $K \sim Bi(n, 1 - f_{(X,Y)}(0,0))$
$(W,Z)$ such that $f_{(W,Z)}(x,y) = \frac{f_{(X,Y)}(x,y)}{1-f_{(X,Y)}(0,0)}$
we immediately get the desired result because $(R_1, R_2) = (S_1, S_2)$. □

Note that this recursion needs an atom on $(0,0)$ or may be generalised if there is an atom on $(c,c)$ with no mass on the other points $(x,y)$ such that $x \leq c$ and $y \leq c$.

## 3. Comparison of an excess-of-loss arrangement with different reinstatements

Let us take a numerical example. We assume that the claims severity is given by

| $X$ | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 10 | 12 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|
| $f_X(x)$ | 0.2 | 0.15 | 0.15 | 0.2 | 0.06 | 0.06 | 0.06 | 0.05 | 0.04 | 0.03 |

The number of claims is Poisson distributed with mean $\lambda = 3$.
The reinsurance layer is $4\,xs\,6$, i.e. the Reinsurer pays for the part of the claims exceeding 6 with a maximum payment of 4 per claim.
Let us assume that the cedent loading is 50% and the Reinsurer loading is 100%.
We calculate the premiums according to the net premium principle.
The following tables gives the reinsurance premium in function of the number of reinstatements ($k$) as well as the percentage ($c_i$) of the reinstatements premiums :

| $c_i$ | $k = 0$ | 1 | 2 | 3 |
|---|---|---|---|---|
| 0% | 1.4592 | 1.7550 | 1.7955 | 1.7996 |
| 50% | | 1.4843 | 1.4724 | 1.4697 |
| 100% | | 1.2859 | 1.2479 | 1.2420 |
| 150% | | 1.1343 | 1.0828 | 1.0754 |
| $1st$ : 100% $2nd$ : 0% | | | 1.3155 | |
| $1st$ : 0% $2nd$ : 100% | | | 1.6718 | |

These premiums are easily obtained with the methodology of Sundt (1990).
Note the interesting comportment of the reinsurance premium when the reinstatements are payable : it diminishes with the number of reinstatements for this particular example.
The following table gives the adjustment coefficient and the expected gain for the different reinsurance structures described hereabove.

| $c$ | $k = 0$ | 1 | 2 | 3 |
|---|---|---|---|---|
| 0% | 0.1019 | 0.1142 | 0.1223 | 0.1252 |
| 50% | | 0.1064 | 0.1070 | 0.1065 |
| 100% | | 0.1008 | 0.0972 | 0.0953 |
| 150% | | 0.0965 | 0.0906 | 0.0880 |
| $1st$ : 100% $2nd$ : 0% | | | 0.1064 | |
| $1st$ : 0% $2nd$ : 100% | | | 0.1068 | |
| $\mathbb{E}G$ | 4.9758 | 4.6799 | 4.6395 | 4.6353 |

The expected gain diminishes with the number of reinstatements because the loading of the Reinsurer is higher than the loading of the cedent. We can make the following comments :

1 if we have the choice only between paid reinstatements at 100%, then clearly we will choose 1 reinstatement because this situation gives the higher adjustment coefficient with the higher expected gain.

2 if we need 2 or 3 reinstatements, that are payable, we will always choose 2 for the same reason.

3 for a given number of reinstatements, that are constant, the adjustement coefficient decreases when the percentage increases

The remarks 1 and 2 are not general and show the interest for the cedent to make the calculations in order to choose the best treaty according to the two criterions : best expected gain ; best adjustment coefficient.
The remark 3 can be shown to be always true :

**Proposition 3**

*Let*

$$\begin{aligned} S_{\text{ced},0} &= S_A + \max(0, S_R - (k+1)L) + c_0 p_0 \min(k, \frac{S_R}{L}) + p_0 \\ S_{\text{ced},1} &= S_A + \max(0, S_R - (k+1)L) + c_1 p_1 \min(k, \frac{S_R}{L}) + p_1 \end{aligned}$$

*$p_i$ and $R_i$ are associated with $c_i$ according to $S_{ced,i}$.*
*The $p_i$ are calculated with the net premium principle.*
*If*

$$c_0 < c_1$$

*Then*

$$R_0 > R_1$$

*Proof*

Let

$$\begin{aligned} g_0(S_R)|S_A &= S_A + \max(0, S_R - (k+1)L) + c_0 p_0 \min(k, \frac{S_R}{L}) + p_0 \\ g_1(S_R)|S_A &= S_A + \max(0, S_R - (k+1)L) + c_1 p_1 \min(k, \frac{S_R}{L}) + p_1 \\ f(S_R) &= \min(k, \frac{S_R}{L}) \end{aligned}$$

We use Ohlin's (1969) lemma with the random variable $S_R|S_A$. The hypotheses of Ohlin's lemma are verified :

$$\begin{aligned} g_0(x)|S_A & \quad \text{is increasing with } x \\ g_1(x)|S_A & \quad \text{is increasing with } x \\ \mathbb{E}[g_0(S_R)|S_A] &= \mathbb{E}[g_1(S_R)|S_A] \end{aligned}$$

We have

$$\begin{aligned} g_0(S_R)|S_A &\leq g_1(S_R)|S_A \quad S_R \geq x_0 \\ g_0(S_R)|S_A &\geq g_1(S_R)|S_A \quad S_R \leq x_0 \end{aligned}$$

with

$$x_0 = f^{-1}\left(\frac{p_0 - p_1}{c_1 p_1 - c_0 p_0}\right)$$

Then by Ohlin's lemma we have

$$\mathbb{E}[e^{rg_0(S_R)}|S_A] \leq \mathbb{E}[e^{rg_1(S_R)}|S_A]$$

Multiplying both sides by $\mathbb{P}[S_A = s_A]$ and summing on $s_A$ gives the desired result. $\square$

## 4. Multivariate extension of Panjer's recursion

When we have two layers, the aggregate claims of the cedent writes :

$$\begin{aligned} S_{\text{Ced}} = S_A &+ \max(0, S_{R_1} - (k_1+1)L_1) \\ &+ \max(0, S_{R_2} - (k_2+1)L_2) \\ &+ p_1 \sum_{i=1}^{k_1} c_{1i} \min(1, \max(0, \frac{S_{R_1} - (i-1)L_1}{L_1})) \\ &+ p_2 \sum_{i=1}^{k_2} c_{2i} \min(1, \max(0, \frac{S_{R_2} - (i-1)L_2}{L_2})) \end{aligned}$$

with

$$\begin{aligned} R_{1i} &= \min(L_1, \max(0, X_i - D_1)) \\ R_{2i} &= \min(L_2, \max(0, X_i - (D_1 + L_1)) \\ A_i &= X_i - R_{1i} - R_{2i} \end{aligned}$$

In practice there may be until 4 layers in an excess-of-loss treaty. So we need to extend the methodology to the multivariate case.
The bivariate recursion is easily extended to the following multivariate recursion using the same arguments than in section 2.

**Proposition 4**

*Let*

$$(S_1, \cdots, S_n)) = (\sum_{i=1}^{N} X_{1i}, \cdots, \sum_{i=1}^{N} X_{ni})$$

*with* $(X_{1i}, \cdots, X_{ni})$ *iid, arithmetic and independent of* $N$
*The components of the vector* $X$ *are not necessarily independent*
$N$ *belonging to Panjer's class, i.e. such that*

$$\frac{p(n)}{p(n-1)} = a + \frac{b}{n} \quad , n \geq 1$$

*Then*

$$\begin{aligned}
f_S(0) &= \Psi_N(f_X(0)) \\
f_S(s_1, \dots s_k) &= \frac{1}{(1 - a f_X(0))} \sum_{x_1, \dots, x_k}^{s_1, \dots, s_k} [a + b\frac{x_1}{s_1}] f_S(s_1 - x_1, \dots, s_k - x_k) f_X(x_1, \dots, x_k) \; s_1 \geq 1 \\
&\dots \\
f_S(s_1, \dots s_k) &= \frac{1}{(1 - a f_X(0))} \sum_{x_1, \dots, x_k}^{s_1, \dots, s_k} [a + b\frac{x_k}{s_k}] f_S(s_1 - x_1, \dots, s_k - x_k) f_X(x_1, \dots, x_k) \; s_k \geq 1
\end{aligned}$$

## 5. Excess-of-loss arrangements with multiple layers

In this section we take the same numerical example than in section 3.
We have a second layer : $4\, xs\, 10$.
Let us assume one free reinstatement for each layer. We find :

| | |
|---|---|
| $p_1 = 3.5101$ | $r = 0.1242$ |
| $p_2 = 1.1971$ | $\mathbb{E}G = 4.0813$ |

Now let us assume a paid reinstatement at 100% for each layer. We have three cotations. Reinsurer 1 gives the cotation by application of the net premium principle. You might observe cotation 2 and 3 on the market.

| | Cotation 1 | Cotation 2 | Cotation 3 |
|---|---|---|---|
| $p_1$ | 2.5719 | 2.8 | 2.4 |
| $p_2$ | 1.0494 | 0.8 | 1.24 |
| $p_1 + p_2$ | 3.6213 | 3.60 | 3.64 |
| $r$ | 0.1050 | 0.1040 | 0.1057 |
| $\mathbb{E}G$ | 4.0813 | 4.0545 | 4.0985 |

We can make the following remarks :

- the adjustement coefficient is better when the reinstatement are free ; this is connected with the proposition 3.
- with the paid reinstatements, if we look only at $p_1 + p_2$ we would choose the cotation 2. However this is a nonsense because cotation 2 has the worst expected gain as well as the worst adjustment coefficient. At the opposite, the best treaty for the cedent is given by the cotation 3 even if for this cotation $p_1 + p_2$ is the highest. This shows that it is a non sense to aggregate the reinsurance premiums of different layers when we have to compare different cotations. The structure of the agreement is such that the expected gain and the adjustement coefficient should be kept in mind.

An interesting exercise is to compare the influence of the structure of the layers. Hereunder we give the figures for the double layer structure (T1 and T2) described before as well as the figures for a simple layer structure (T : $8\,xs\,6$). The Reinsurers applies the net premium principle. We find

| | 2 rec at 100% on T1<br>1 rec at 100% on T2 | 2 rec free on T1<br>1 rec free on T2 | 3 rec at 100% on T1<br>1 rec at 100% on T2 | 3 rec free on T1<br>1 rec free on T2 |
|---|---|---|---|---|
| $p_1$ | 2.49591 | 3.59103 | 2.48419 | 3.59928 |
| $p_2$ | 1.04941 | 1.19715 | 1.04373 | 1.19992 |
| $p_1 + p_2$ | 3.54532 | 4.78818 | 3.52792 | 4.79920 |
| $r$ | 0.10093 | 0.13261 | 0.09846 | 0.13568 |
| $\mathbb{E}G$ | 4.04103 | 4.04103 | 4.03551 | 4.03550 |

| | 1 rec at 100% on T | 1 rec free on T | 2 rec at 100% on T | 2 rec free on T |
|---|---|---|---|---|
| $p$ | 3.75916 | 4.76885 | 3.69682 | 4.79867 |
| $r$ | 0.10783 | 0.13030 | 0.10301 | 0.13586 |
| $\mathbb{E}G$ | 4.05149 | 4.05143 | 4.03651 | 4.03640 |

This table shows that if the reinstatements are payable at 100%, it is better for the cedent to work with one layer while with free reinstatements, a choice has to be made.
The combination of two reinstatements (paid at 100% or free) on the low layer and one on the high layer are less interesting than two reinstatements on one large layer.
Once again we remark that the reinsurance premiums are not a good criterion in order to see what is the better cover for the cedent.

## 6. Organization of the calculations

In this section we first study the number of multiplications required to find the multivariate distribution $(S_1, \cdots, S_n)$ :

$$\sum_{x_1=0}^{s_1} \cdots \sum_{x_n=0}^{s_n} (3(\min(x_1, m_1) + 1) \cdots (\min(x_n, m_n) + 1) + 1) \tag{11}$$

where $(m_1, \cdots, m_n)$ are the maximal values taken by $(X_1, \cdots, X_n)$
$(s_1, \cdots, s_n)$ are the maximal values for which $(S_1, \cdots, S_n)$ are evaluated
An upper bound for (11) is

$$3(s_1+1)\cdots(s_n+1)(m_1+1)\cdots(m_n+1) \tag{12}$$

For our trivariate example, we had

| | | | |
|---|---|---|---|
| $X_1$ | : | part of the cedent | $m_1 = 6$ |
| $X_2$ | : | part of the reinsurer, low layer | $m_2 = 4$ |
| $X_3$ | : | part of the reinsurer, high layer | $m_3 = 4$ |
| $S_A$ | : | aggregate part for which the cedent is liable | $s_1 = 64$ |
| $S_{R1}$ | : | aggregate part for which the first layer reinsurer is liable | $s_2 = 44$ |
| $S_{R2}$ | : | aggregate part for which the second layer reinsurer is liable | $s_3 = 34$ |

The number of multiplications needed was :

| | |
|---|---|
| Exact | 46 290 821 |
| Upper bound | 53 849 250 |

for a precision given by

$$\sum_{i=0}^{64}\sum_{j=0}^{44}\sum_{k=0}^{34}\mathbb{P}(S_1 = i, S_{R1} = j, S_{R2} = k) = 0.999999905971$$

Of course it is possible to take advantage of the particular structure of the distribution of the random vector $(X_1, X_2, X_3)$.
For our particular case, the number of multiplications is given by

$$\sum_{x_1=1}^{s_1}[3(1+\min(x_1,m_1)+1] + \sum_{x_1=m_1}^{s_1}\sum_{x_2=0}^{s_2}[3((\min(x_1,m_1)+(1+\min(x_2,m_2)))+1]$$

$$+\sum_{x_1=m_1}^{s_1}\sum_{x_2=m_2}^{s_2}\sum_{x_3=1}^{s_3}[3((\min(x_1,m_1)+(1+\min(x_2,m_2)+(1+\min(x_3,m_3))+1]$$

An upper bound is given by

$$s_1(3m_1+4)+(s_1-m_1+1)(s_2+1)(3(m_1+m_2)+4)$$

$$+(s_1-m_1+1)(s_2-m_2+1)s_3 3(m_1+m_2+m_3)+7)$$

With this organization, the number of multiplications needed was :

| | |
|---|---|
| Exact | 4 074 885 |
| Upper bound | 4 121 732 |

Of course, in the particular case Poisson ($a = 0$), the recursion is simplified. The number of multiplications is given by :

$$\sum_{x_1=1}^{s_1}[3\min(x_1,m_1)+1] + \sum_{x_1=m_1}^{s_1}\sum_{x_2=1}^{s_2}[3\min(x_2,m_2)+1] + \sum_{x_1=m_1}^{s_1}\sum_{x_2=m_2}^{s_2}\sum_{x_3=1}^{s_3}[3\min(x_3,m_3)+1]$$

An upper bound is given by

$$s_1(3m_1+1)+(s_1-m_1+1)s_2(3m_2+1)+(s_1-m_1+1)(s_2-m_2+1)s_3(3m_3+1)$$

With this organization, the number of multiplications needed was :

| | |
|---|---|
| Exact | 1 059 513 |
| Upper bound | 1 104 162 |

Now for our purpose that consists in evaluating the adjustment coefficient of a ceding company when it buys an excess-of-loss cover with reinstatements, let us remind that we only need the moment generating function of $S_{Ced}$.
If we restrict ourselves to a counting distribution that is Poisson distributed, we have the following property :
Let $X$ be the distribution of the severity of claims
Let $N$ be Poisson distributed
Let $D_1$ be the lower deductible We immediately have

$$\begin{aligned} S &= S_1 + S_2 \\ S &= Y_1 + \cdots Y_{N_1} + Z_1 + \cdots Z_{N_2} \end{aligned}$$

with $Y_1 = X|X \leq D_1$
$N_1 = (N|X \leq D_1)$
$Y_2 = X|X > D_1$
$N_2 = (N|X > D_1)$
$S_1 \perp S_2$

The mgf of $S_{Ced}$ is then the product of the mgf of $S_1$ and of the mgf a function of $(Z_1, \cdots, Z_{N_2})$. Then we gain time of calculation at two levels :

- The distribution of $S_1$ needs not to be calculated
- The multivariate distribution based on $(Z_1, \cdots, Z_{N_2})$ will be evaluated more rapidly because the mean of $N_2$ is lower that the one of $N$. Note that most of the time the difference will be very important.

If, as it was the case in our trivariate example, the high claims liable to the cedent are completely cut by the reinsurance, then the multivariate distribution based on $(Z_1, \cdots Z_{N_2})$ is simplified because the $A$ which is the part of the cedent is degenerated at $D_1$, which reduce the dimensionality of the problem.
With the following hypotheses

| | | |
|---|---|---|
| $A$ | : | part of the cedent ; degenerated at 6 |
| $R1$ | : | part of the reinsurer, low layer ; $m_2 = 4$ |
| $R2$ | : | part of the reinsurer, high layer ; $m_3 = 4$ |
| $S_A^Z$ | : | aggregate claims distribution for which the cedent is liable ; $s_1 = 48 = 8 \times 6$ |
| $S_1^Z$ | : | aggregate claims distribution for which the first layer reinsurer is liable ; $s_2 = 30$ |
| $S_2^Z$ | : | aggregate claims distribution for which the second layer reinsurer is liable ; $s_3 = 25$ |

The frequency of the claims touching the reinsurance layers is $\lambda = 0.54$.
Using this technique, the number of multiplications needed was

$$\sum_{x_1=1}^{8} \sum_{x_2=0}^{s_2} \sum_{x_3=0}^{s_3} (1 + \min(x_2, m_2))(1 + \min(x_3, m_3)) + 1$$

An upper bound is given by

$$8(s_2 + 1)(s_3 + 1)((m_2 + 1)(m_3 + 1) + 1)$$

With this organization, the number of multiplications needed was :

| | |
|---|---|
| Exact | 145 648 |
| Upper bound | 167 648 |

for a precision given by

$$\sum_{i=0}^{48} \sum_{j=0}^{30} \sum_{k=0}^{25} \mathbb{P}(S_1^Z = i, S_2^Z = j, S_3^Z = k) = 0.999999989229$$

## 7. Conclusion

In this paper we have derived a multivariate Panjer's algorithm in order to find multivariate compound distributions depending on the same counting random variable.
This algorithm has been shown to be of interest when one is interested in finding the distribution of aggregate claims for which a ceding company is liable when it buys excess-of-loss reinsurance with reinstatements.
The technique described authorizes the cedent to make calculations in order to compare the expected gain and the adjustement coefficient for different covers.
We remarked that

- working on the basis of the net premium principle, free reinstatements are always better than paid reinstatements
- comparing the reinsurance premiums is not a good criterion
- the technique is interesting in order to compare covers with different types of layers

Some tips have been given in order to reduce the time of calculations when the counting distribution is Poisson distributed, showing once again the great interest of the Poison distribution for this kind of problems.

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