Stochastic s-(increasing) convexity

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Abstract. For many applications, especially in actuarial sciences, a problem of great practical interest is the stochastic comparison of mixed distributions and of compound sums. This question is investigated here with respect to the class of stochastic s-(increasing) convex orderings introduced recently. The analysis is based on a central property, called stochastic s-(increasing) convexity, for families of parametric distributions.

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1 Introduction

Let $\{X_\theta,\ \theta \in \Theta\}$ be a family of random variables valued in a subset $\mathcal{S} \subseteq \mathbb{R}$, and indexed by a single parameter real $\theta \in \Theta \subseteq \mathbb{R}$. The associated family of distributions is denoted by $\mathcal{P}(\Theta) = \{P_\theta,\ \theta \in \Theta\}$.

Now, given any function $\phi : \mathcal{S} \to \mathbb{R}$, let us define the function ϕ^* as

$$\phi^* : \Theta \to \mathbb{R}\ ;\ \theta \mapsto E[\phi(X_\theta)]. \tag{1}$$

In the present paper, our main purpose is to investigate whether some specific property of the function ϕ can be transmitted to the function ϕ^*. More precisely, the property under study is the notion of s-(increasing) convex functions, which was introduced by Popoviciu (1933) in the theory of approximation and developed later in many research works. We also refer, e.g., to the books by Roberts and Varberg (1973) and Pečarić, Proschan and Tong (1992). Note that the 1-convex functions correspond to the increasing functions, and the 2-(increasing) convex functions to the (increasing) convex functions in the usual sense.

Recently, these classes of s-increasing convex functions have been used in probability and statistics in order to compare two random variables, or their distributions, in the so-called stochastic s-(increasing) convex orderings. We mention, e.g., Lefèvre and Utev (1996), Denuit and Lefèvre (1997), Denuit, Lefèvre and Shaked (1998), Denuit, Lefèvre and Utev (1999a,b) and Denuit, Lefèvre and Mesfioui (1999). In fact, these stochastic orderings will play, here again, an important role (see Section 4). For that reason, a succinct presentation of the theory is given in Section 2.

Coming back to (1), if the s-(increasing) convexity is transmitted from ϕ to ϕ^*, the family of distributions $\mathcal{P}(\Theta)$ is said to be stochastically s-(increasing) convex. This property of stochastic s-(increasing) convexity is analyzed in some details in Section 3. We will obtain some characterizations and sufficient conditions, and we will point out that a number of classical parametric distributions enjoy the property. Note that some results given in the actuarial literature will be so generalized. Moreover, we observe that the special cases where $s = 1$ and 2 have been examined by Shaked and Shanthikumar (1988), (1994, Chapter VI).

In Section 4, we apply some of the previous results for comparing mixed distributions and compound sums. This problem has many applications in theory and practice, especially in actuarial sciences. For illustration, we will derive lower and upper bounds, in the stochastic 2 and 3-convex orderings, for the distribution of the aggregate claim amounts over some period of time, when the claim frequency is mostly unknown, the only information available being its first two moments, with its largest value, and possibly the property that its density is decreasing.

To close, we indicate that in Denuit, Lefèvre and Utev (1999b), all this analysis based on the s-(increasing) convex functions has been developed in the more general framework of the generalized (increasing) convex functions with respect to an arbitrary Tchebycheff system of functions (in the sense of, e.g., Karlin and Studden (1966)). As underlined in that paper, however, such an extension can have the disadvantage to conceal certain properties that are closely related to the stochastic s-(increasing) convexity.

2 Stochastic s-(increasing) convex orderings

2.1 Integral stochastic orderings

Most stochastic orderings that allow to compare two random variables X and Y valued in $\mathcal{S} \subseteq \mathbb{R}$, are defined (or, at least, can be defined) by reference to a class $\mathcal{F}$ of measurable functions $\phi : \mathcal{S} \to \mathbb{R}$, namely

$$X \preceq_{\mathcal{F}}^{\mathcal{S}} Y \Leftrightarrow E\phi(X) \leq E\phi(Y) \text{ for all the functions } \phi \in \mathcal{F} \tag{2}$$

for which the expectations exist. Such orderings are usually referred to as integral stochastic orderings and obviously rely on the classical expected utility theory (viewing $\mathcal{F}$ as a class of utility functions). A rather general theory of these orderings can be found in Marshall (1991) and Müller (1997).

As particular cases of (2), the stochastic dominance is obtained when $\mathcal{F}$ represents the class of the increasing functions ϕ, and the stochastic (increasing) convex ordering when $\mathcal{F}$ is the class of the (increasing) convex functions ϕ (see, e.g., Shaked and Shanthikumar (1994)). We notice that the stop-loss order discussed in actuarial sciences corresponds to the stochastic increasing convex ordering (see, e.g., Kaas, Van Heerwaarden and Goovaerts (1994)).

2.2 Cones of s-(increasing) convex functions

Following Popoviciu (1933), given any positive integer s, the class $\overline{\mathcal{U}}^{\mathcal{S}}_{s-cx}$ of the s-convex functions on $\mathcal{S}$ is defined by

$$\overline{\mathcal{U}}^{\mathcal{S}}_{s-cx} = \{\phi : \mathcal{S} \to \mathbb{R} \,|\, [x_0, x_1, \ldots, x_s]\phi \geq 0 \;\; \forall \;\; x_0 < x_1 < \ldots < x_s \in \mathcal{S}\},$$

where $[\ldots]$ denotes the classical divided difference operator, which is defined recursively for $\phi : \mathcal{S} \to \mathbb{R}$ and $x_0 \neq x_1 \neq \ldots \neq x_s$ in $\mathcal{S}$ by

$$[x_0, x_1, \ldots, x_k]\phi = \frac{[x_1, x_2, \ldots, x_k]\phi - [x_0, x_1, \ldots, x_{k-1}]\phi}{x_k - x_0}, \quad k = 1, 2, \ldots, s-1,$$

starting from $[x_i]\phi = \phi(x_i)$ for $i = 0, 1, \ldots, s$. Note that the s-convexity is defined for any subset $\mathcal{S}$ of the real line $\mathbb{R}$ and does not require that $\mathcal{S}$ is an interval. The name "s-convex" comes from the fact that the functions in $\overline{\mathcal{U}}^{\mathcal{S}}_{2-cx}$, that is, the 2-convex functions, are convex in the usual sense when $\mathcal{S}$ is an interval.

The class $\overline{\mathcal{U}}^{\mathcal{S}}_{s-icx}$ of the s-increasing convex functions with domain $\mathcal{S}$ is then defined as the class of all the functions being simultaneously k-convex for $k = 1, 2, \ldots, s$, i.e.

$$\overline{\mathcal{U}}^{\mathcal{S}}_{s-icx} = \bigcap_{k=1}^{s} \overline{\mathcal{U}}^{\mathcal{S}}_{k-cx}.$$

2.3 Induced stochastic s-(increasing) convex orderings

The s-convex ordering and the s-increasing convex ordering are the integral stochastic orderings generated by the cone of the s-convex functions and by the cone of the s-increasing convex functions, respectively. To be more specific, given two random variables X and Y valued in $\mathcal{S}$, X is said to be smaller than Y in the s-convex ordering, denoted henceforth by $X \preceq^{\mathcal{S}}_{s-cx} Y$, when (2) holds with $\mathcal{F} = \overline{\mathcal{U}}^{\mathcal{S}}_{s-cx}$. Similarly, X is said to be smaller than Y in the s-increasing convex ordering, denoted by $X \preceq^{\mathcal{S}}_{s-icx} Y$, when (2) holds with $\mathcal{F} = \overline{\mathcal{U}}^{\mathcal{S}}_{s-icx}$.

As particular cases, we have that $\preceq^{\mathcal{S}}_{1-(i)cx}$ is just the stochastic dominance and $\preceq^{\mathcal{S}}_{2-(i)cx}$ is the usual (increasing) convex ordering. We mention that the stop-loss order of degree $s-1$ in actuarial sciences corresponds here to $\preceq_{s-icx}^{\mathbb{R}^+}$ (see, e.g., Kaas and Hesselager (1995)).

Now, the subset $\mathcal{S}$ in (2) can be taken arbitrarily. For particular choices of $\mathcal{S}$, however, the s-convex orderings can be characterized by replacing $\overline{\mathcal{U}}_{s-cx}^{\mathcal{S}}$ by some dense subclass in it. First, let us consider $\mathcal{S}=[a,b]$, $a<b\in\mathbb{R}$, b being possibly infinite. In this case, it is shown in Denuit, Lefèvre and Shaked (1998) that $\preceq_{s-cx}^{[a,b]}$ can be defined through (2) with for $\mathcal{F}$ the class

$$\mathcal{M}_{s-cx}^{[a,b]}=\{\pm(x-a)^k,\ \ k=1,2,\ldots,s-1;\ \ (x-t)_+^{s-1},\ \ t\in[a,b]\}, \tag{3}$$

where $x_+=x$ if $x>0$ and 0 otherwise; for $s=1$, by convention $(x-t)_+^0$ is the indicator function of $[t,+\infty[$. Alternatively, $\preceq_{s-cx}^{[a,b]}$ can be generated by the cone

$$\mathcal{U}_{s-cx}^{[a,b]}=\left\{\phi:[a,b]\to\mathbb{R}\,\middle|\,\phi^{(s)}\geq 0\text{ on }[a,b]\right\}, \tag{4}$$

where $\phi^{(s)}$ denotes the s-th derivative of the function ϕ. Denoting by $\mathcal{C}^s([a,b])$ the class of the functions $\phi:[a,b]\to\mathbb{R}$ which are s times continuously differentiable, it can be proved that

$$\mathcal{U}_{s-cx}^{[a,b]}=\overline{\mathcal{U}}_{s-cx}^{[a,b]}\cap\mathcal{C}^s([a,b]).$$

From an analytical point of view $\mathcal{M}_{s-cx}^{[a,b]}$ consists of the extreme points of the class $\overline{\mathcal{U}}_{s-cx}^{[a,b]}$ while $\mathcal{U}_{s-cx}^{[a,b]}$ contains the "regular" s-convex functions.

On the other hand, when $\mathcal{S}=\mathcal{D}_n\equiv\{0,1,\ldots,n\}$, where n is some positive integer (by convention, $\mathcal{D}_\infty$ stands for the set $\mathbb{N}$), it can be shown (see Denuit and Lefèvre (1997) and Denuit, Lefèvre and Utev (1999a)) that

$$\overline{\mathcal{U}}_{s-cx}^{\mathcal{D}_n}=\{\phi:\mathcal{D}_n\to\mathbb{R}|\Delta^s\phi(i)\geq 0\text{ on }\mathcal{D}_{n-s}\}, \tag{5}$$

where Δ^s denotes the usual s-th degree forward difference operator, which is defined recursively for a function $\phi:\mathcal{D}_n\to\mathbb{R}$ and $k\in\mathcal{D}_n$ by $\Delta^k\phi(i)=\Delta^{k-1}\phi(i+1)-\Delta^{k-1}\phi(i)$ for all $i\in\mathcal{D}_n$ such that $i+k\in\mathcal{D}_n$, i.e. for $i=0,1,\ldots,n-k$ (by convention, $\Delta^1\phi\equiv\Delta\phi$ and $\Delta^0\phi\equiv\phi$). Moreover, $\preceq_{s-cx}^{\mathcal{D}_n}$ can also be defined through (2) with for $\mathcal{F}$ the class

$$\begin{aligned}\mathcal{M}_{s-cx}^{\mathcal{D}_n}=\Bigg\{\pm\binom{x}{i},\ \ i=1,2,\ldots,s-1;&\\ \binom{x-i}{s-1},\ \ i=1,2,\ldots,n-s+1\Bigg\},&\end{aligned} \tag{6}$$

where $\binom{x}{i}=0$ when $x<i$. Note that (5) and (6) can be seen as discrete analogs of (4) and (3), respectively.

A similar study can be carried out for the s-increasing convex order. More precisely, the analogs for $\preceq^{\mathcal{S}}_{s-icx}$ of (3), (4), (5) and (6) above are respectively given by

$$\mathcal{M}^{[a,b]}_{s-icx} = \{(x-a)^k, \;\; k=1,2,\ldots,s-1; \;\; (x-t)_+^{s-1}, \;\; t\in[a,b]\}, \tag{7}$$

$$\mathcal{U}^{[a,b]}_{s-icx} = \left\{\phi:[a,b]\to I\!R \,\middle|\, \phi^{(k)}\geq 0 \text{ on } [a,b] \text{ for } k=1,2,\ldots,s\right\}, \tag{8}$$

$$\overline{\mathcal{U}}^{\mathcal{D}_n}_{s-icx} = \{\phi:\mathcal{D}_n\to I\!R | \Delta^k\phi(i)\geq 0 \text{ on } \mathcal{D}_{n-k} \text{ for } k=1,2,\ldots,s\}, \tag{9}$$

$$\mathcal{M}^{\mathcal{D}_n}_{s-icx} = \left\{\binom{x}{i}, \;\; i=1,2,\ldots,s-1; \right.$$
$$\left. \binom{x-i}{k-1}, \;\; i=1,2,\ldots,n-k+1 \text{ for } k=1,2,\ldots,s\right\}. \tag{10}$$

3 Stochastic s-convexity

3.1 Definition

Let us turn to our central question, namely the possible transmission of the properties of s-(increasing) convexity from ϕ to ϕ^* defined in (1).

Definition 1. If the s-convexity (resp. the s-increasing convexity) is transmitted from ϕ to ϕ^*, i.e.

$$\text{if } \phi^*\in\overline{\mathcal{U}}^{\Theta}_{s-cx} \text{ (resp. } \overline{\mathcal{U}}^{\Theta}_{s-icx}\text{) whenever } \phi\in\overline{\mathcal{U}}^{\mathcal{S}}_{s-cx} \text{ (resp. } \overline{\mathcal{U}}^{\mathcal{S}}_{s-icx}\text{)},$$

provided that the expectation defining ϕ^* exists, then the family $\mathcal{P}(\Theta)$ (or $\{X_\theta, \;\; \theta\in\Theta\}$) is said to be stochastically s-convex (resp. stochastically s-increasing convex).

We observe that the stochastic 1-(increasing) convexity coincides with the stochastic increasingness, while the stochastic 2-(increasing) convexity is the usual stochastic (increasing) convexity (see, e.g., Shaked and Shanthikumar (1988, 1994)).

3.2 Characterizations

We are going to show that when $\mathcal{S}$ is an interval or an arithmetic grid, it suffices to check the transmission of the s-convexity, or of the s-increasing convexity, for the subclasses $\mathcal{U}^{\mathcal{S}}_{s-cx}$ and $\mathcal{M}^{\mathcal{S}}_{s-cx}$ of $\overline{\mathcal{U}}^{\mathcal{S}}_{s-cx}$, or for the subclasses $\mathcal{U}^{\mathcal{S}}_{s-icx}$ and $\mathcal{M}^{\mathcal{S}}_{s-icx}$ of $\overline{\mathcal{U}}^{\mathcal{S}}_{s-icx}$.

Proposition 1. *For $\mathcal{S} = [a,b]$, $a < b \in I\!R$, b possibly infinite, $\mathcal{P}(\Theta)$ is stochastically s-convex if, and only if,*

$$\phi^* \in \overline{\mathcal{U}}^{\Theta}_{s-cx} \text{ for all } \phi \in \mathcal{U}^{[a,b]}_{s-cx}, \tag{11}$$

or, equivalently, if, and only if,

$$\phi^* \in \overline{\mathcal{U}}^{\Theta}_{s-cx} \text{ for all } \phi \in \mathcal{M}^{[a,b]}_{s-cx}. \tag{12}$$

$\mathcal{P}(\Theta)$ is stochastically s-increasing convex if, and only if,

$$\phi^* \in \overline{\mathcal{U}}^{\Theta}_{s-icx} \text{ for all } \phi \in \mathcal{U}^{[a,b]}_{s-icx}, \tag{13}$$

or, equivalently, if, and only if,

$$\phi^* \in \overline{\mathcal{U}}^{\Theta}_{s-icx} \text{ for all } \phi \in \mathcal{M}^{[a,b]}_{s-icx}. \tag{14}$$

Proof. Let us begin with (11) and (13). It is well-known that the class $\overline{\mathcal{U}}^{[a,b]}_{s-cx}$ of the s-convex functions consists in all the pointwise limits of the sequences of functions with a non-negative derivative of degree s, that is those functions in $\mathcal{U}^{[a,b]}_{s-cx}$. Similarly, the elements of $\overline{\mathcal{U}}^{[a,b]}_{s-icx}$ are the pointwise limits of sequences in $\mathcal{U}^{[a,b]}_{s-icx}$. This yields (11) and (13) since $\overline{\mathcal{U}}^{\Theta}_{s-cx}$ and $\overline{\mathcal{U}}^{\Theta}_{s-icx}$ are closed in the topology of the pointwise convergence. Let us pass to (12) and (14). As shown in Popoviciu (1933), any $\phi \in \overline{\mathcal{U}}^{[a,b]}_{s-cx}$ can be obtained as the uniform limit of a sequence of functions $\{\varphi_n,\ n \geq s\}$ which are of the form

$$\varphi_n(x) = \phi(a) + \sum_{j=0}^{s-1} \alpha_j (x-a)^j + \sum_{j=0}^{n-s} \beta_j (x-t_j)_+^{s-1},$$

where $\alpha_0, \alpha_1, \ldots, \alpha_{s-1}$ are real constants, $\beta_0, \beta_1, \ldots, \beta_{n-s}$ are non-negative constants (so that $\varphi_n \in \overline{\mathcal{U}}^{[a,b]}_{s-cx}$), and $t_0 < t_1 < \ldots < t_{n-s} \in [a,b]$. Similarly, any $\phi \in \overline{\mathcal{U}}^{[a,b]}_{s-icx}$ is the uniform limit of a sequence of functions $\{\tilde{\varphi}_n,\ n \geq s\}$ which are of the form

$$\tilde{\varphi}_n(x) = \phi(a) + \sum_{j=0}^{s-1} \tilde{\alpha}_j (x-a)^j + \sum_{j=0}^{n-s} \tilde{\beta}_j (x-t_j)_+^{s-1},$$

where $\tilde{\alpha}_0, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_{s-1}$ and $\tilde{\beta}_0, \tilde{\beta}_1, \ldots, \tilde{\beta}_{n-s}$ are non-negative constants (so that $\tilde{\varphi}_n \in \overline{\mathcal{U}}^{[a,b]}_{s-icx}$), and $t_0 < t_1 < \ldots < t_{n-s} \in [a,b]$. Thus (12) and (14) follow directly.

Proposition 2. *For $\mathcal{S} = \mathcal{D}_n$, n possibly infinite, $\mathcal{P}(\Theta)$ is stochastically s-convex if, and only if,*

$$\phi^* \in \overline{\mathcal{U}}^{\Theta}_{s-cx} \text{ for all } \phi \in \mathcal{M}^{\mathcal{D}_n}_{s-cx},$$

while $\mathcal{P}(\Theta)$ is stochastically s-increasing convex if, and only if,

$$\phi^* \in \overline{\mathcal{U}}^{\Theta}_{s-icx} \textit{ for all } \phi \in \mathcal{M}^{\mathcal{D}_n}_{s-icx}.$$

Proof. We recall (see, e.g., Denuit and Lefèvre (1997)) that any function $\phi : \mathcal{D}_n \to I\!R$ can be expanded as

$$\phi(j) = \phi(0) + \binom{j}{1}\Delta\phi(0) + \binom{j}{2}\Delta^2\phi(0) + \ldots + \binom{j}{s-1}\Delta^{s-1}\phi(0) + \sum_{i=1}^{n-s+1}[\Delta^s\phi(i-1)]\binom{j-i}{s-1}, \quad j \in \mathcal{D}_n.$$

The results announced are then straightforward.

3.3 A sufficient condition

Denuit, Lefèvre and Utev (1999b) have pointed the special role taken by the families of distributions that are totally positive. Specifically, let us assume that there exists a sigma-finite dominating measure ν for the family of probability distributions $\mathcal{P}(\Theta)$, and let f_θ denote the associated density function for P_θ. The family $\mathcal{P}(\Theta)$ is said to be totally positive of degree s (TP_s, in short) if the kernel obtained by considering the density $f_\theta(x)$ as a function of two variables (the parameter θ and the variable x) is TP_s, that is when the determinant of the matrix $\{f_{\theta_i}(x_j)\}_{i,j=1,2,\ldots,k}$ is non-negative for all $\theta_1 < \theta_2 < \ldots < \theta_k \in \Theta$ and $x_1 < x_2 < \ldots < x_k \in \mathcal{S}$, with $k = 1, 2, \ldots, s$; see, e.g., Karlin (1968).

Property 1. Let $\mathcal{P}(\Theta)$ be a TP_s family. If, in addition, for $k = 1, 2, \ldots, s-1$,

$$EQ_k(X_\theta) \equiv \int_{x\in\mathcal{S}} Q_k(x) f_\theta(x) d\mu(x) = \widetilde{Q}_k(\theta), \quad \theta \in \Theta,$$

where Q_k is an arbitrary real polynomial of exact degree k with positive highest coefficient, and $\widetilde{Q}_k$ is an associated real polynomial of the same type, then $\mathcal{P}(\Theta)$ is stochastically s-convex and stochastically s-increasing convex.

3.4 Families with a scale or location parameter

We start with the situation where the parameter θ involved in X_θ is a scale parameter.

Property 2. Let Y be a non-negative random variable, and define the random variables X_θ as

$$X_\theta = \theta Y \text{ with } \theta \in I\!R.$$

Then $\{X_\theta, \ \theta \in I\!R\}$ is stochastically s-convex and stochastically s-increasing convex, for any $s \geq 1$.

Proof. Let $\phi \in \mathcal{U}^{I\!R}_{s-cx}$. We then get

$$\frac{d^s}{d\theta^s}\phi^*(\theta) = \frac{d^s}{d\theta^s}E\phi(\theta Y) = E\left[Y^s\phi^{(s)}(\theta Y)\right] \geq 0,$$

hence the result.

That property is no longer valid, in general, when Y is a real-valued random variable Y. An exception is given below.

Property 3. Let Y be a real-valued random variable with a symmetric density about 0, and define $X_\theta = \theta Y$ with $\theta \in I\!R$. Then $\{X_\theta,\ \theta \in I\!R\}$ is stochastically $2s$-convex, for any $s \geq 1$, while $\{X_\theta,\ \theta \in I\!R^+\}$ is stochastically $2s$-convex and stochastically $2s$-increasing convex, for any $s \geq 1$.

Proof. For $\phi \in \mathcal{U}^{I\!R}_{2s-cx}$, we have

$$\frac{d^{2s}}{d\theta^{2s}}\phi^*(\theta) = E\left[Y^{2s}\phi^{(2s)}(\theta Y)\right] \geq 0,$$

so that $\{X_\theta,\ \theta \in I\!R\}$ is stochastically $2s$-convex for all s. For $\phi \in \mathcal{U}^{I\!R}_{2s-icx}$, we obtain that for $k = 1, 2, \ldots, s$,

$$\frac{d^{2k}}{d\theta^{2k}}\phi^*(\theta) = E\left[Y^{2k}\phi^{(2k)}(\theta Y)\right] \geq 0,$$

and as $EY^{2k-1} = 0$ by hypothesis,

$$\frac{d^{2k-1}}{d\theta^{2k-1}}\phi^*(\theta) = E\left\{Y^{2k-1}[\phi^{(2k-1)}(\theta Y) - \phi^{(2k-1)}(0)]\right\},$$

where $[\phi^{(2k-1)}(\theta x) - \phi^{(2k-1)}(0)]x^{2k-1} \geq 0$ for all real x and all $\theta \geq 0$ since $\phi^{(2k)} \geq 0$.

Let us now examine the case where the parameter θ involved in X_θ is a location parameter. The proof of the following result is immediate.

Property 4. Let Y be a real-valued random variable, and define the random variables X_θ as

$$X_\theta = Y + \theta \text{ with } \theta \in I\!R.$$

Then $\{X_\theta,\ \theta \in I\!R\}$ is stochastically s-convex and stochastically s-increasing convex, for any $s \geq 1$.

3.5 Partial and compound sums

We consider here that X_θ corresponds to some partial or compound sum of random variables. The following properties have been established in Denuit, Lefèvre and Utev (1999b).

Property 5. Let $\{Y_n, \ n \geq 1\}$ be a sequence of non-negative independent and identically distributed random variables and define the random variables X_θ as

$$X_\theta = \sum_{n=1}^{\theta} Y_n \text{ with } \theta \in \mathbb{N}.$$

Then $\{X_\theta, \ \theta \in \mathbb{N}\}$ is stochastically s-convex and stochastically s-increasing convex, for any $s \geq 1$.

Property 6. Let $\{Y_n, \ n \geq 1\}$ be a sequence of non-negative independent and identically distributed random variables. Let $\{N_\theta, \ \theta \in \Theta\}$ be a family of non-negative integer-valued random variables, independent of the Y_n's, and define the random variables X_θ as

$$X_\theta = \sum_{n=1}^{N_\theta} Y_n \text{ with } \theta \in \Theta.$$

If $\{N_\theta, \ \theta \in \Theta\}$ is stochastically s-increasing convex, then $\{X_\theta, \ \theta \in \Theta\}$ also is.

We mention that the result of Property 6 only holds for the stochastic s-increasing convexity and not for the stochastic s-convexity. The reason is that the proof uses the fact that the composition of two s-increasing convex functions is itself an s-increasing convex function, and that this does not hold for the s-convexity. The property below is closely related.

Property 7. Let $\{X_\theta, \ \theta \in \mathbb{R}\}$ be a stochastically s-increasing convex family of random variables. Then, for any s-increasing convex measurable function φ, the family of random variables $\{\varphi(X_\theta), \ \theta \in \mathbb{R}\}$ is also stochastically s-increasing.

3.6 Standard parametric families

Table 1 points out the possible stochastic s-convexity and s-increasing convexity for standard parametric families. This easily follows from the properties derived in this Section, or by a direct argument.

Examples of parametric distributions that are not stochastically s-convex are rather frequent. It can be checked, for instance, that if X_θ is an exponential random variable with mean $\theta \in \Theta$, then $\{\sqrt{X_\theta}, \ \theta \in \Theta\}$ is not stochastically s-convex (although being TP_s).

4 Stochastic comparisons of mixtures and compound sums

4.1 s-(increasing) convex ordering between mixtures

The property of stochastic s-(increasing) convexity provides a natural way for comparing mixed distributions through their mixing kernel, in the stochastic s-(increasing) convex ordering.

Probability Distribution	Stochastic s-(increasing) convexity
Poisson with mean λ	s-cx and s-icx in $\lambda \in I\!R^+$
Binomial(n,p) with mean np	s-cx and s-icx in $n \in I\!N$ s-cx and s-icx in $p \in [0,1]$
Negative Binomial(n,p) with mean np	s-cx and s-icx in $n \in I\!N$ s-cx and s-icx in $p \in [0,1]$
Discrete Uniform on $\mathcal{D}_n$	s-cx and s-icx in $n \in I\!N$
Continuous Uniform on $[0,\theta]$	s-cx and s-icx in $\theta \in I\!R^+$
Negative Exponential with mean λ	s-cx and s-icx in $\lambda \in I\!R^+$
Erlang(n,λ)	s-cx and s-icx in $n \in I\!N$ s-cx and s-icx in $\lambda \in I\!R^+$
Normal$(0,\sigma)$	$2s$-cx and $2s$-icx in $\sigma \in I\!R^+$
Normal(μ,σ_0)	s-cx and s-icx in $\mu \in I\!R$
LogNormal$(0,\sigma)$	$2s$-icx in $\sigma \in I\!R^+$
LogNormal(μ,σ_0)	s-icx in $\mu \in I\!R$
Chi-square(n)	s-cx and s-icx in $n \in I\!N$
Compound Poisson Sum(λ) with a.s.≥ 0 summands	s-icx in $\lambda \in I\!R^+$
Compound Binomial Sum(n,p) with a.s.≥ 0 summands	s-icx in $n \in I\!N$ s-icx in $p \in [0,1]$
Compound Neg. Binomial Sum(n,p) with a.s.≥ 0 summands	s-icx in $n \in I\!N$ s-icx in $p \in [0,1]$
Compound Discrete Uniform$(\mathcal{D}_n)$ with a.s.≥ 0 summands	s-icx in $n \in I\!N$

Table 1. Stochastic s-convexity (s-cx) and s-increasing convexity (s-icx) of some standard parametric distributions, for any $s \geq 1$.

Given the random variables $\{X_\theta,\ \theta \in \Theta\}$, let X_Λ denote a random variable distributed as a mixture of these X_θ's with mixing kernel Λ, i.e.

$$P[X_\Lambda \leq x] = \int_{\theta\in\Theta} P[X_\theta \leq x] dP[\Lambda \leq \theta], \quad x \in I\!R.$$

Property 8. If the family of distributions $\mathcal{P}(\Theta)$ is stochastically s-(increasing) convex, then

$$\Lambda_1 \preceq^{\Theta}_{s-(i)cx} \Lambda_2 \Rightarrow X_{\Lambda_1} \preceq^{\mathcal{S}}_{s-(i)cx} X_{\Lambda_2}. \tag{15}$$

Proof. For any function $\phi : \mathcal{S} \to I\!R$, we can write, by (1),

$$E[\phi(X_\Lambda)] = E[\phi^*(\Lambda)].$$

Now, let $\phi \in \overline{\mathcal{U}}^{\mathcal{S}}_{s-(i)cx}$. Since $\mathcal{P}(\Theta)$ is stochastically s-(increasing) convex, we know that $\phi^* \in \overline{\mathcal{U}}^{\Theta}_{s-(i)cx}$. Therefore, $\Lambda_1 \preceq^{\Theta}_{s-(i)cx} \Lambda_2$ implies that

$$E[\phi^*(\Lambda_1)] \leq E[\phi^*(\Lambda_2)],$$

or equivalently,

$$E[\phi(X_{\Lambda_1})] \leq E[\phi(X_{\Lambda_2})],$$

as required.

Combining Table 1 and Property 8 directly yields the stochastic comparisons between mixtures given in Tables 2-3.

Probability distributions	Orderings among random parameters	Induced orderings between mixed distributions
Poisson(λ)	$\Lambda_1 \preceq_{s-(i)cx}^{\mathbb{R}^+} \Lambda_2$	Poisson(Λ_1) $\preceq_{s-(i)cx}^{\mathbb{N}}$ Poisson(Λ_2)
Binomial(n,p)	$N_1 \preceq_{s-(i)cx}^{\mathbb{N}_0} N_2$	Binomial(N_1,p) $\preceq_{s-(i)cx}^{\mathbb{N}}$ Binomial(N_2,p)
	$\pi_1 \preceq_{s-(i)cx}^{[0,1]} \pi_2$	Binomial(n,π_1) $\preceq_{s-(i)cx}^{\mathcal{D}_n}$ Binomial(n,π_2)
Negative Binomial(n,p)	$N_1 \preceq_{s-(i)cx}^{\mathbb{N}_0} N_2$	Neg. Bin.(N_1,p) $\preceq_{s-(i)cx}^{\mathbb{N}}$ Neg. Bin.(N_2,p)
	$\pi_1 \preceq_{s-(i)cx}^{[0,1]} \pi_2$	Neg. Bin.(n,π_1) $\preceq_{s-(i)cx}^{\mathbb{N}}$ Neg. Bin.(n,π_2)
Discrete Uniform(n)	$N_1 \preceq_{s-(i)cx}^{\mathbb{N}_0} N_2$	Disc. Unif.(N_1) $\preceq_{s-(i)cx}^{\mathbb{N}}$ Disc. Unif.(N_2)
Continuous Uniform $[0,\theta]$	$\Theta_1 \preceq_{s-(i)cx}^{\mathbb{R}^+} \Theta_2$	Cont. Unif.$[0,\Theta_1]$ $\preceq_{s-(i)cx}^{\mathbb{R}^+}$ Cont. Unif.$[0,\Theta_2]$
Negative Exponential(λ)	$\Lambda_1 \preceq_{s-(i)cx}^{\mathbb{R}_0^+} \Lambda_2$	Neg. Exp.(Λ_1) $\preceq_{s-(i)cx}^{\mathbb{R}^+}$ Neg. Exp.(Λ_2)
Erlang(n,λ)	$N_1 \preceq_{s-(i)cx}^{\mathbb{N}} N_2$	Erlang(N_1,λ) $\preceq_{s-(i)cx}^{\mathbb{R}^+}$ Erlang(N_2,λ)
	$\Lambda_1 \preceq_{s-(i)cx}^{\mathbb{R}_0^+} \Lambda_2$	Erlang(n,Λ_1) $\preceq_{s-(i)cx}^{\mathbb{R}^+}$ Erlang(n,Λ_2)

Table 2. Stochastic s-(increasing) convex orderings between mixed distributions, for any $s \geq 1$.

We note that various comparison properties derived in the actuarial literature (see, e.g., Goovaerts and De Vylder (1984), Hesselager (1998) and Hesselager, Wang and Willmot (1998)) correspond to particular cases of the above results.

Probability distributions	Orderings among random parameters	Induced orderings between mixed distributions
Normal$(0,\sigma)$	$S_1 \preceq^{\mathbb{R}^+}_{2s-(i)cx} S_2$	Normal$(0,S_1)$ $\preceq^{\mathbb{R}}_{2s-(i)cx}$Normal$(0,S_2)$
Normal(μ,σ_0)	$M_1 \preceq^{\mathbb{R}}_{s-(i)cx} M_2$	Normal(M_1,σ_0) $\preceq^{\mathbb{R}}_{s-(i)cx}$Normal$(M_2,\sigma_0)$
LogNormal$(0,\sigma)$	$S_1 \preceq^{\mathbb{R}^+}_{2s-icx} S_2$	LogNormal$(0,S_1)$ $\preceq^{\mathbb{R}^+}_{2s-icx}$LogNormal$(0,S_2)$
LogNormal(μ,σ_0)	$M_1 \preceq^{\mathbb{R}}_{s-icx} M_2$	LogNormal(M_1,σ_0) $\preceq^{\mathbb{R}^+}_{s-icx}$LogNormal$(M_2,\sigma_0)$
Chi-square(n)	$N_1 \preceq^{\mathbb{N}}_{s-(i)cx} N_2$	Chi-square(N_1) $\preceq^{\mathbb{R}^+}_{s-(i)cx}$Chi-square$(N_2)$
Comp. Poisson Sum(λ) with a.s.≥ 0 summands	$\Lambda_1 \preceq^{\mathbb{R}_0^+}_{s-icx} \Lambda_2$	C.P.S.(Λ_1) $\preceq^{\mathbb{R}^+}_{s-icx}$C.P.S.$(\Lambda_2)$
Comp. Bin. Sum(n,p) with a.s.≥ 0 summands	$N_1 \preceq^{\mathbb{N}_0}_{s-icx} N_2$	C.B.S.(N_1,p) $\preceq^{\mathbb{R}^+}_{s-icx}$C.B.S.$(N_2,p)$
	$\pi_1 \preceq^{[0,1]}_{s-icx} \pi_2$	C.B.S.(n,π_1) $\preceq^{\mathbb{R}^+}_{s-icx}$C.B.S.$(n,\pi_2)$
Comp. Neg. Bin. Sum(n,p) with a.s.≥ 0 summands	$N_1 \preceq^{\mathbb{N}_0}_{s-icx} N_2$	C.N.B.S.(N_1,p) $\preceq^{\mathbb{R}^+}_{s-icx}$C.N.B.S.$(N_2,p)$
	$\pi_1 \preceq^{[0,1]}_{s-icx} \pi_2$	C.N.B.S.(n,π_1) $\preceq^{\mathbb{R}^+}_{s-icx}$C.N.B.S.$(n,\pi_2)$

Table 3. Stochastic s-(increasing) convex orderings between mixed distributions, for any $s \geq 1$ (continued from Table 2).

4.2 s-(increasing) convex ordering between compound sums

Using Properties 5 and 8, we are able to compare compound sums, through their number of random variables, in the stochastic s-(increasing) convex ordering.

Property 9. Let $\{Y_n,\ n \geq 1\}$ be a sequence of non-negative independent and identically distributed random variables, and let N_1 and N_2 be two integer-valued random variables, independent from that sequence. Then,

$$N_1 \preceq^{\mathbb{N}}_{s-(i)cx} N_2 \Rightarrow \sum_{n=1}^{N_1} Y_n \preceq^{\mathbb{R}^+}_{s-(i)cx} \sum_{n=1}^{N_2} Y_n.$$

Moreover, let $\{Z_n,\ n \geq 1\}$ be another sequence of non-negative independent and identically distributed random variables such that $Z_n \preceq^{\mathbb{R}^+}_{s-(i)cx} Y_n$ for all

n. Then,

$$N_1 \preceq^{\mathbb{N}}_{s-(i)cx} N_2 \Rightarrow \sum_{n=1}^{N_1} Z_n \preceq^{\mathbb{R}^+}_{s-(i)cx} \sum_{n=1}^{N_2} Y_n.$$

4.3 An application in actuarial sciences

For illustration, let us consider the actuarial context in which $S = \sum_{n=1}^{N} Y_n$ represents the aggregate claim amounts over some period of time. Here N denotes the number of claims and the Y_n's are their severities. As usual, it is assumed that the Y_n's and N satisfy the conditions stipulated in Property 9.

Now, quite often in practice, the insurance has thousands of realizations for the Y_n's, so that a statistical model for the claim severities can be built adequately. On the contrary, the company has only one realization of N for each period, so that data on N are generally scarce. This leads us to examine the situations where the distribution of the Y_n's can be viewed as known, while the distribution of N is mostly unknown. Specifically, we will assume that the only information available on N are its first two moments, with its largest value.

On basis of this partial information, our goal is then to find stochastic bounds, lower and upper, for the aggregate claims S under interest. By Property 9, s-convex bounds on N will allow to construct s-convex bounds on S. Thus, the problem reduces to obtaining such lower and upper bounds on N. This has been done in Denuit and Lefèvre (1997) and Denuit, Lefèvre and Mesfioui (1999), yielding the results below.

Property 10. Suppose that the number of claims N is valued in $\mathcal{D}_n$ and has mean μ_1. Define $\xi_0 \in \mathcal{D}_{n-1}$ such that $\xi_0 < \mu_1 \leq \xi_0 + 1$. Then,

$$S_{\min}^{(2)} \preceq^{\mathbb{R}^+}_{2-cx} S \preceq^{\mathbb{R}^+}_{2-cx} S_{\max}^{(2)}, \tag{16}$$

where

$$S_{\min}^{(2)} = \begin{cases} \sum_{k=1}^{\xi_0} Y_k, & \text{with the probability } \xi_0 + 1 - \mu_1, \\ \sum_{k=1}^{\xi_0+1} Y_k, & \text{with the probability } \mu_1 - \xi_0, \end{cases}$$

and

$$S_{\max}^{(2)} = \begin{cases} 0, & \text{with the probability } 1 - \frac{\mu_1}{n}, \\ \sum_{k=1}^{n} Y_k, & \text{with the probability } \frac{\mu_1}{n}. \end{cases}$$

Suppose that, in addition, N has second moment μ_2. Define ξ_1 and $\xi_2 \in \mathcal{D}_{n-1}$ such that

$$\xi_1 < \frac{\mu_2}{\mu_1} \leq \xi_1 + 1 \text{ and } \xi_2 < \frac{n\mu_1 - \mu_2}{n - \mu_1} \leq \xi_2 + 1.$$

Then,

$$S_{\min}^{(3)} \preceq^{\mathbb{R}^+}_{3-cx} S \preceq^{\mathbb{R}^+}_{3-cx} S_{\max}^{(3)}, \tag{17}$$

where

$$S_{\min}^{(3)} = \begin{cases} 0, & \text{with the probability } p_1 = 1 - p_2 - p_3, \\ \sum_{k=1}^{\xi_1} Y_k, & \text{with the probability } p_2 = \frac{(\xi_1+1)\mu_1-\mu_2}{\xi_1}, \\ \sum_{k=1}^{\xi_1+1} Y_k, & \text{with the probability } p_3 = \frac{\mu_2-\xi_1\mu_1}{1+\xi_1}, \end{cases}$$

and

$$S_{\max}^{(3)} = \begin{cases} \sum_{k=1}^{\xi_2} Y_k, & \text{with the probability } q_1 = \frac{(1+\xi_2)(n-\mu_1)+\mu_2-n\mu_1}{n-\xi_2}, \\ \sum_{k=1}^{\xi_2+1} Y_k, & \text{with the probability } q_2 = \frac{(n+\xi_2)\mu_1-\mu_2-n\xi_2}{n-1-\xi_2}, \\ \sum_{k=1}^{n} Y_k, & \text{with the probability } q_3 = 1 - q_1 - q_2. \end{cases}$$

Note that $S_{\min}^{(2)}$ and $S_{\min}^{(3)}$ are defined independently of n, while $S_{\max}^{(2)}$ and $S_{\max}^{(3)}$ require the value of n. From (16), we deduce, for instance, bounds on the stop-loss premium for S, namely

$$E\left(S_{\min}^{(2)} - d\right)_+ \leq E(S-d)_+ \leq E\left(S_{\max}^{(2)} - d\right)_+,$$

for any deductible d.

When, moreover, the density of N is known to be decreasing, the bounds (16) and (17) can be improved as follows.

Property 11. Suppose that the number of claims N is valued in $\mathcal{D}_n$, has a decreasing density and has mean μ_1. Define $\rho_0 \in \mathcal{D}_n$ such that $\rho_0 < 2\mu_1 \leq \rho_0 + 1$. Then,

$$S_{\min}^{(2)*} \preceq_{2-cx}^{\mathbb{R}^+} S \preceq_{2-cx}^{\mathbb{R}^+} S_{\max}^{(2)*}, \tag{18}$$

where

$$S_{\min}^{(2)*} = \sum_{k=1}^{N_{\min}^{(2)*}} Y_k \text{ with } f_{N_{\min}^{(2)*}}(k) = \begin{cases} \frac{2(\rho_0+1-\mu_1)}{(\rho_0+1)(\rho_0+2)}, & \text{for } k = 0, 1, \ldots, \rho_0, \\ \frac{2\mu_1-\rho_0}{\rho_0+2}, & \text{for } k = \rho_0 + 1, \\ 0, & \text{for } k \geq \rho_0 + 2, \end{cases}$$

and

$$S_{\max}^{(2)*} = \sum_{k=1}^{N_{\max}^{(2)*}} Y_k \text{ with } f_{N_{\max}^{(2)*}}(k) = \begin{cases} 1 - \frac{2\mu_1}{n+1}, & \text{for } k = 0, \\ \frac{2\mu_1}{n(n+1)}, & \text{for } k = 1, \ldots, n. \end{cases}$$

Suppose that, in addition, N has second moment μ_2. Define ρ_1 and $\rho_2 \in \mathcal{D}_{n-1}$ such that

$$\rho_1 < \frac{3\mu_2 - \mu_1}{2\mu_1} \leq \rho_1 + 1 \text{ and } \rho_2 < \frac{2n\mu_1 - 3\mu_2 + \mu_1}{n - 2\mu_1} \leq \rho_2 + 1.$$

Then,

$$S_{\min}^{(3)*} \preceq_{3-cx}^{\mathbb{R}^+} S \preceq_{3-cx}^{\mathbb{R}^+} S_{\max}^{(3)*}, \tag{19}$$

where

$$S_{\min}^{(3)*} = \sum_{k=1}^{N_{\min}^{(3)*}} Y_k \text{ with } f_{N_{\min}^{(3)*}}(k) = \begin{cases} \tilde{p}_1 + \frac{\tilde{p}_2}{\rho_1+1} + \frac{\tilde{p}_3}{\rho_1+2}, & \text{for } k = 0, \\ \frac{\tilde{p}_2}{\rho_1+1} + \frac{\tilde{p}_3}{\rho_1+2}, & \text{for } k = 1, \ldots, \rho_1, \\ \frac{\tilde{p}_3}{\rho_1+2}, & \text{for } k = \rho_1 + 1, \\ 0, & \text{for } k \geq \rho_1 + 2, \end{cases}$$

where $\tilde{p}_1 = 1 - \tilde{p}_2 - \tilde{p}_3$,

$$\tilde{p}_2 = \frac{2(\rho_1 + 1)\mu_1 - 3\mu_2 + \mu_1}{\rho_1} \text{ and } \tilde{p}_3 = \frac{3\mu_2 - \mu_1 - 2\rho_1\mu_1}{\rho_1 + 1},$$

and

$$S_{\max}^{(3)*} = \sum_{k=1}^{N_{\max}^{(3)*}} Y_k \text{ with } f_{N_{\max}^{(3)*}}(k) = \begin{cases} \frac{\tilde{q}_1}{\rho_2+1} + \frac{\tilde{q}_2}{\rho_2+2} + \frac{\tilde{q}_3}{n+1}, & \text{for } k = 0, 1, \ldots, \rho_2 \\ \frac{\tilde{q}_2}{\rho_2+2} + \frac{\tilde{q}_3}{n+1}, & \text{for } k = \rho_2 + 1, \\ \frac{\tilde{q}_3}{n+1}, & \text{for } k = \rho_2 + 2, \ldots, n, \end{cases}$$

where $\tilde{q}_3 = 1 - \tilde{q}_1 - \tilde{q}_2$,

$$\tilde{q}_1 = \frac{(1 + \rho_2)(n - 2\mu_1) + 3\mu_2 - \mu_1 - 2n\mu_1}{n - \rho_2}$$

and

$$\tilde{q}_2 = \frac{2(n + \rho_2)\mu_1 - 3\mu_2 + \mu_1 - n\rho_2}{n - 1 - \rho_2}.$$

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