A CRITERION FOR THE STOCHASTIC ORDERING OF RANDOM SUMS

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Abstract

It is shown that vectors $(S_{M_1}, \ldots, S_{M_n})$ and $(S'_{M'_1}, \ldots, S'_{M'_n})$ of random sums of positive random variables are stochastically ordered by upper orthant dependence, lower orthant dependence or by the supermodular ordering whenever their corresponding random number of terms $(M_1, \ldots, M_n)$ and $(M'_1, \ldots, M'_n)$ are themselves ordered in this fashion. Applications in actuarial science are considered.

COPULAS; LOWER ORTHANT DEPENDENCE; POSITIVE QUADRANT DEPENDENCE; PROBABILITY OF RUIN; UPPER ORTHANT DEPENDENCE; RANDOM SUMS; SUPERMODULAR ORDERING.

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1. Introduction

Let $(M_1, \ldots, M_n)$ be a vector of integer-valued random variables, and given sequences $(X_{1k}), \ldots, (X_{nk})$ of positive random variables, consider the random sums

$$S_{M_i} = \sum_{k=1}^{M_i} X_{ik}, \quad 1 \leq i \leq n. \tag{1}$$

In actuarial science, for example, S_{M_i} might represent the total amounts paid by an insurer for the M_i claims generated by his/her i-th class of business in a given time period. In queueing theory, the same random sums might stand for the total service time for random numbers $M_1, \ldots, M_n$ of clients lining up in n parallel queues.

In these two areas and possibly in other fields of application, it may be of interest to examine the impact of the degree of dependence between the M_i's on functions of the S_{M_i}'s. To be more specific, suppose that $\mathbf{M} = (M_1, \ldots, M_n)$ and $\mathbf{M}' = (M'_1, \ldots, M'_n)$ are independent vectors that have the same univariate marginals but different joint distributions H and H'. Let $S_{M_1}, \ldots, S_{M_n}$ be defined as in (1) and put

$$S'_{M'_i} = \sum_{k=1}^{M'_i} X'_{ik}, \quad 1 \leq i \leq n. \tag{2}$$

where for all $1 \leq i \leq n$, the sequences (X_{ik}) and (X'_{ik}) are identically distributed and independent of $\mathbf{M}$ and $\mathbf{M}'$. No assumption is made on the dependence structure between the elements of any particular sequence (X_{ik}).

The vectors $(S_{M_1}, \ldots, S_{M_n})$ and $(S'_{M'_1}, \ldots, S'_{M'_n})$ then have identical univariate marginals, but how would $E(S_{M_1} \cdots S_{M_n})$ compare with $E(S'_{M'_1} \cdots S'_{M'_n})$ for instance? Or what relation might there be between the distributions of the sums $S_{M_1} + \cdots + S_{M_n}$ and $S'_{M'_1} + \cdots + S'_{M'_n}$?

The purpose of this note is to provide a partial answer to these questions, which were originally motivated by recent work by Cossette & Marceau (1999) who examined the impact on riskiness and finite-time ruin probabilities of increased dependence between two or several classes of business. In the bivariate case, the proposed solution exploits the notion of positive quadrant dependence (PQD) due to Lehmann (1966), whose definition is recalled in Section 2. The main result, which is proved there, states that the pair (S_{M_1}, S_{M_2}) is more PQD than the pair $(S'_{M'_1}, S'_{M'_2})$ whenever (M_1, M_2) is itself more PQD than (M'_1, M'_2).

Three possible multivariate extensions of this observation are given in Section 3, based on generalizations of PQD called the upper orthant, the lower orthant, and the supermodular orderings. Examples of applications in actuarial risk theory are provided in Section 4.

For stochastic comparisons involving finite sums of possibly dependent random variables, the readers may refer to Müller (1997) and Bäuerle & Müller (1998), who also give actuarial applications. Additional material on the comparison of random sums may also be found in the work of Jean-Marie & Liu (1992). Throughout this paper as in theirs, increasingness and positivity are understood to be non-strict, except when explicitly stated otherwise.

2. Ordering two pairs of random sums

In general, random pairs $\mathbf{U} = (U_1, U_2)$ and $\mathbf{U}' = (U'_1, U'_2)$ with the same marginals are said to be ordered by PQD, denoted $\mathbf{U} \prec_{pqd} \mathbf{U}'$, whenever

$$P(U_1 > u_1, U_2 > u_2) \leq P(U'_1 > u_1, U'_2 > u_2) \tag{3}$$

for all reals u_1 and u_2. As stated in Theorem 4.G.1 of Shaked & Shantikumar (1994), inequality (3) occurs if, and only if,

$$E\{\phi_1(U_1)\phi_2(U_2)\} \leq E\{\phi_1(U'_1)\phi_2(U'_2)\}$$

for all positive increasing functions ϕ_1 and ϕ_2 (for which the expectations exist, of course; this caveat will not be repeated).

An observation made just below that theorem also implies that if φ is a two-place function such that

$$\Delta_{x_1}^{x_1'}\Delta_{x_2}^{x_2'}\,\varphi \equiv \varphi(x_1', x_2') - \varphi(x_1', x_2) - \varphi(x_1, x_2') + \varphi(x_1, x_2) \geq 0$$

for all $x_1 \leq x_1'$, $x_2 \leq x_2'$, then

$$(U_1, U_2) \prec_{pqd} (U_1', U_2') \Rightarrow E\{\varphi(U_1, U_2)\} \leq E\{\varphi(U_1', U_2')\}. \tag{4}$$

It is explained below how these facts may be combined to obtain the following result.

Proposition 1. *Let (M_1, M_2) and (M_1', M_2') be two pairs of integer-valued random variables, and let (S_{M_1}, S_{M_2}) and $(S'_{M_1'}, S'_{M_2'})$ be two pairs of random sums whose components are defined as in* (1) *and* (2) *in terms of identically distributed sequences (X_{ik}) and (X'_{ik}) of positive random variables. Then*

$$(M_1, M_2) \prec_{pqd} (M_1', M_2') \Rightarrow (S_{M_1}, S_{M_2}) \prec_{pqd} (S'_{M_1'}, S'_{M_2'}).$$

Proof. Given arbitrary positive increasing functions ϕ_1 and ϕ_2, one needs only show that

$$E\{\phi_1(S_{M_1})\phi_2(S_{M_2})\} \leq E\{\phi_1(S'_{M_1'})\phi_2(S'_{M_2'})\}. \tag{5}$$

To this end, define

$$\varphi(m_1, m_2) \equiv E\left\{\phi_1(S_{m_1})\phi_2(S_{m_2})\right\} = E\left\{\phi_1(S'_{m_1})\phi_2(S'_{m_2})\right\}$$

and note that for arbitrary integers $m_i \leq m_i'$, one has $S_{m_i'} \geq S_{m_i}$, $i = 1, 2$, because the X_{ij}'s are positive. Since the ϕ_i's are increasing by hypothesis, it follows that

$$\Delta_{m_1}^{m_1'}\Delta_{m_2}^{m_2'}\,\varphi = E\left[\left\{\phi_1(S_{m_1'}) - \phi_1(S_{m_1})\right\}\left\{\phi_2(S_{m_2'}) - \phi_2(S_{m_2})\right\}\right] \geq 0,$$

whence it follows from (4) that

$$E\{\varphi(M_1, M_2)\} \leq E\{\varphi(M_1', M_2')\}$$

which is the same as (5).

Remark 1. As is apparent from its proof, this result is valid as soon as the X_{ij} and X'_{ij} are positive and the distribution of the M_i and M_i are independent from them. In particular, the summands need not be sequences of mutually independent and identically distributed random variables.

The following consequence of Proposition 1 is of interest for actuarial applications, as will be seen in Section 4.

Corollary 1. *Let* (M_1, M_2), (M_1', M_2'), (S_{M_1}, S_{M_2}) *and* $(S'_{M_1'}, S'_{M_2'})$ *be as in Proposition 1. Let also* $S = S_{M_1} + S_{M_2}$ *and* $S' = S'_{M_1'} + S'_{M_2'}$. *Then* $S \prec_{cx} S'$ *in the stop-loss order. In other words,*

$$E\{\phi(S)\} \leq E\{\phi(S')\} \tag{6}$$

for all convex functions ϕ.

Proof. Since $(S_{M_1}, S_{M_2}) \prec_{pqd} (S'_{M_1'}, S'_{M_2'})$ by Proposition 1, the conclusion follows from the main result of Dhaene & Goovaerts (1996).

3. Ordering two vectors of random sums

As is well known, condition (3) is equivalent to

$$P(U_1' \leq u_1, U_2' \leq u_2) \geq P(U_1 \leq u_1, U_2 \leq u_2)$$

for all reals u_1 and u_2. In higher dimensions, however, these two requirements are different and hence yield different generalizations of Lehmann's notion of PQD, whose definitions are recalled below.

Definition 1. *Let* $\mathbf{U} = (U_1, \ldots, U_n)$ *and* $\mathbf{U}' = (U_1', \ldots, U_n')$ *be two random vectors.*

a) $\mathbf{U} \prec_{uo} \mathbf{U}'$ *in the upper orthant dependence ordering if*

$$P(U_1 > u_1, \ldots, U_n > u_n) \leq P(U_1' > u_1, \ldots, U_n' > u_n) \tag{7}$$

for all reals $u_1, \ldots, u_n$ *or if*

$$E\left\{\prod_{i=1}^{n} \phi_i(U_i)\right\} \leq E\left\{\prod_{i=1}^{n} \phi_i(U_i')\right\}$$

for all positive increasing functions $\phi_1, \ldots, \phi_n$.

b) $\mathbf{U} \prec_{\ell o} \mathbf{U}'$ *in the lower orthant dependence ordering if*

$$P(U_1 \leq u_1, \ldots, U_n \leq u_n) \leq P(U_1' \leq u_1, \ldots, U_n' \leq u_n) \tag{8}$$

for all reals $u_1, \ldots, u_n$ *or if*

$$E\left\{\prod_{i=1}^{n} \phi_i(U_i)\right\} \leq E\left\{\prod_{i=1}^{n} \phi_i(U_i')\right\}$$

for all positive decreasing functions $\phi_1, \ldots, \phi_n$.

c) $\mathbf{U} \prec_{co} \mathbf{U}'$ *in the concordance ordering if* $\mathbf{U} \prec_{uo} \mathbf{U}'$ *and* $\mathbf{U} \prec_{\ell o} \mathbf{U}'$ *hold simultaneously.*

Remark 2. The notation used in the above definition and henceforth is that of Joe (1990), who first considered the concordance ordering. Observe, however, that condition (8) is noted $\mathbf{U}' \prec_{\ell o} \mathbf{U}$ by Shaked & Shantikumar (1994).

Another extension of the bivariate PQD ordering which implies both $\prec_{uo}$ and $\prec_{\ell o}$ (and hence $\prec_{co}$; cf. Shaked & Shantikumar 1997) is given below and characterized in terms of successive applications to an arbitrary function $\varphi(x_1, \ldots, x_n)$ of the difference operators $\Delta_{x_i}^{x_i'}$ defined for $1 \leq i \leq n$ by

$$\Delta_{x_i}^{x_i'} \varphi = \varphi(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n) - \varphi(x_1, \ldots, x_n)$$

for given $x_1, \ldots, x_n$ and $x_i \leq x_i'$.

Definition 2. *One writes* $\mathbf{U} \prec_{sm} \mathbf{U}'$ *between two random vectors* $\mathbf{U} = (U_1, \ldots, U_n)$ *and* $\mathbf{U}' = (U_1', \ldots, U_n')$ *if, and only if,*

$$E\{\phi(U_1, \ldots, U_n)\} \leq E\{\phi(U_1', \ldots, U_n')\} \tag{9}$$

for all real-valued functions $\phi(x_1, \ldots, x_n)$ *such that* $\Delta_{x_i}^{x_i'} \Delta_{x_j}^{x_j'} \phi \geq 0$ *for any integers* $1 \leq i < j \leq n$ *and reals* $x_i \leq x_i'$ *and* $x_j \leq x_j'$. *Such functions are called* supermodular, *whence the symbol* $\prec_{sm}$.

With these notions, which obviously coincide in the case $n = 2$, it is then possible to state the following extension of Proposition 1.

Proposition 2. *Let* $\mathbf{M} = (M_1, \ldots, M_n)$, $\mathbf{M}' = (M_1', \ldots, M_n')$ *be two vectors of integer-valued random variables, and let* $\mathbf{S}_\mathbf{M} = (S_{M_1}, \ldots, S_{M_n})$ *and* $\mathbf{S}'_{\mathbf{M}'} = (S'_{M_1'}, \ldots, S'_{M_n'})$, *whose components are defined as in* (1) *and* (2) *in terms of identically distributed sequences* (X_{ik}) *and* (X'_{ik}) *of positive random variables. The following implications hold true:*

(i) $\mathbf{M} \prec_{uo} \mathbf{M}' \Rightarrow \mathbf{S}_{\mathbf{M}} \prec_{uo} \mathbf{S}'_{\mathbf{M}'}$;

(ii) $\mathbf{M} \prec_{\ell o} \mathbf{M}' \Rightarrow \mathbf{S}_{\mathbf{M}} \prec_{\ell o} \mathbf{S}'_{\mathbf{M}'}$;

(iii) $\mathbf{M} \prec_{co} \mathbf{M}' \Rightarrow \mathbf{S}_{\mathbf{M}} \prec_{co} \mathbf{S}'_{\mathbf{M}'}$.

(iv) $\mathbf{M} \prec_{sm} \mathbf{M}' \Rightarrow \mathbf{S}_{\mathbf{M}} \prec_{sm} \mathbf{S}'_{\mathbf{M}'}$.

Proof. It proceeds along the same lines as the proof of Proposition 1. For parts (i) and (ii), simply define

$$\varphi(m_1, \ldots, m_n) \equiv E\left\{\prod_{i=1}^{n} \phi_i(S_{m_i})\right\}$$

and use the fact that

$$\Delta_{m_1}^{m_1'} \cdots \Delta_{m_n}^{m_n'} \varphi = E\left[\prod_{i=1}^{n} \left\{\phi_i(S_{m_i'}) - \phi_i(S_{m_i})\right\}\right]$$

is positive when the ϕ_i's are increasing but only for n even when the ϕ_i's are decreasing. As implied by the discussion on page 143 of Shaked & Shantikumar (1994), one then has

$$\mathbf{M} \prec_{uo} \mathbf{M}' \quad \text{or} \quad \mathbf{M} \prec_{\ell o} \mathbf{M}' \Rightarrow E\{\varphi(\mathbf{M})\} \leq E\{\varphi(\mathbf{M}')\}$$

and hence

$$E\left\{\prod_{i=1}^{n} \phi_i(S_{M_i})\right\} \leq E\left\{\prod_{i=1}^{n} \phi_i(S_{M_i'})\right\} = E\left\{\prod_{i=1}^{n} \phi_i(S'_{M_i'})\right\},$$

which is the desired conclusion.

Since the concordance ordering is the conjunction of the upper and lower orthant dependence orderings, statement (iii) is immediate from (i) and (ii).

For part (iv), the argument is similar but hinges on the fact that the function

$$\varphi(m_1, \ldots, m_n) \equiv E\left\{\phi\left(S_{m_1}, \ldots, S_{m_n}\right)\right\},$$

is supermodular whenever ϕ is itself supermodular by hypothesis. To see this, consider $\varphi(m_1, \ldots, m_n)$ for fixed $m_1, \ldots, m_n$. Given $i < j$, $m_i < m_i'$ and $m_j < m_j'$, observe that

$$\Delta_{m_i}^{m_i'} \Delta_{m_j}^{m_j'} \varphi = E\left\{E\left(\Delta_{S_{m_i}}^{S_{m_i}+a} \Delta_{S_{m_j}}^{S_{m_j}+b} \phi \Big| \sum_{k=m_i}^{m_i'} X_{ik} = a, \sum_{k=m_j}^{m_j'} X_{jk} = b\right)\right\}$$

and that the expression inside this double expectation is positive for all possible values $S_{m_i} = s_{m_i}$, $1 \le i \le n$, because ϕ is supermodular by hypothesis and $a \ge 0$, $b \ge 0$ in view of the positivity of the X_{ij}'s. Thus the argument is complete.

The following is a multivariate version of Corollary 1, whose usefulness will be illustrated in the application section.

COROLLARY 2. *Let* $\mathbf{M}$, $\mathbf{M}'$, $\mathbf{S_M}$ *and* $\mathbf{S}'_{\mathbf{M}'}$ *be as in Proposition 2. Let also*

$$S = \sum_{i=1}^{n} S_{M_i} \quad \text{and} \quad S' = \sum_{i=1}^{n} S'_{M_i}.$$

(i) *If* $\mathbf{M} \prec_{sm} \mathbf{M}'$, *then* $S \prec_{cx} S'$ *in the stop-loss, or convex ordering* $\prec_{cx}$.

(ii) *If* $\mathbf{M} \prec_{uo} \mathbf{M}'$, *then*

$$E\left(e^{tS}\right) \le E\left(e^{tS'}\right), \quad t \ge 0 \tag{10}$$

that is, $\mathbf{S_M}$ *precedes* $\mathbf{S}'_{\mathbf{M}'}$ *in the exponential ordering (Goovaerts et al. 1990, Definition 4.3.1, p. 44).*

(iii) *If* $\mathbf{M} \prec_{\ell o} \mathbf{M}'$, *then*

$$E\left(e^{-tS}\right) \le E\left(e^{-tS'}\right), \quad t \ge 0 \tag{11}$$

that is, $\mathbf{S_M}$ *precedes* $\mathbf{S}'_{\mathbf{M}'}$ *in the Laplace ordering.*

Proof. (i) That $\mathbf{S_M} \prec_{sm} \mathbf{S}'_{\mathbf{M}'} \Rightarrow S \prec_{cx} S'$ constitutes Theorem 3.1 in Müller (1997). It follows from the well known fact that a function $\varphi(x_1, \ldots, x_n) = \psi(x_1 + \cdots + x_n)$ is supermodular if and only is ψ convex (Lorentz 1953). See also Müller (1997).

Part (ii) is immediate from the definition of $\mathbf{S_M} \prec_{sm} \mathbf{S}'_{\mathbf{M}'}$ and the fact that (10) compares products of positive increasing functions. The proof of part (iii) is similar.

REMARK 3. While $\mathbf{M} \prec_{sm} \mathbf{M}' \Rightarrow \mathbf{M} \prec_{co} \mathbf{M}'$, the latter is not sufficient to imply $S \prec_{cx} S'$. Counter-examples to that effect are provided by Müller (1997, Example 1) and Hu & Wu (1999, Example 4.1). Needless to say, $\mathbf{M} \prec_{co} \mathbf{M}'$ implies both (10) and (11).

4. Illustrations

This section presents a number of applications of the above results to the collective risk model in actuarial science, in which a portfolio consists of $n \geq 2$ classes of business whose respective number of claims are random variables $M_1, \ldots, M_n$. In this model, the claim amounts X_{ik} in class i are typically assumed to form a sequence of independent and identically distributed random variables with distribution F_i, $i = 1, \ldots, n$; they are also assumed to be independent of the M_k's.

In this context, it is of interest to examine the impact of the dependence between the M_k's on the riskiness of the portfolio, as measured, e.g., by the stop-loss premium or by the adjustment coefficient. To begin with, assume that there are only $n = 2$ classes of business and that the joint distribution of the pair (M_1, M_2) with marginals G_1 and G_2 is expressed as

$$P(M_1 \leq m_1, M_2 \leq m_2) = C_\alpha \left\{G_1(m_1), G_2(m_2)\right\} \tag{12}$$

in terms of a one-parameter copula C_α, that is, a distribution function with uniform marginals on the unit square. Suppose also that

$$\alpha < \alpha' \Rightarrow C_\alpha(u_1, u_2) \leq C_{\alpha'}(u_1, u_2) \tag{13}$$

for all $0 \leq u_1, u_2 \leq 1$, so that the family (C_α) is ordered by positive quadrant dependence. If S and S' are defined as in equations (1) and (2), it follows from Corollary 1 that their corresponding stop-loss premiums can then be ordered as

$$\pi_S(d) \equiv E_\alpha \left\{\max(S - d, 0)\right\} \leq \pi_{S'}(d) \equiv E_{\alpha'} \left\{\max(S' - d, 0)\right\} \tag{14}$$

since $\phi(s) = \max(s - d, 0)$ is a convex function for all deductibles $d \geq 0$.

Condition (13) is regarded in the statistical literature as a basic requirement which insures that α can be interpreted as a parameter of dependence. It is met by virtually all families of bivariate copula models used in practice (cf. Hutchinson & Lai 1990, Joe 1997 or Nelsen 1999 for extensive lists). The dominance relation described by equation (14) is thus verified in all such models. This conclusion is illustrated below in but one special case.

Example 1. For $\alpha > 0$, consider the copula

$$C_\alpha(u_1, \ldots, u_n) = \left(\sum_{i=1}^{n} u_i^{-\alpha} - n + 1\right)^{-1/\alpha},$$

usually attributed to Clayton (1978) but whose origin can actually be tracked back at least to Kimeldorf & Sampson (1975) in the case $n = 2$. The first systematic study of this class of distributions was published by Cook & Johnson (1981), who interpret the parameter α as a measure of the strength of the dependence between the u_k's. In the bivariate case, this is supported by relation (13), which implies that nonparametric measures of dependence are monotone increasing functions of α. For example, Genest & MacKay (1986) show that Kendall's tau is equal to $\alpha/(\alpha+2)$, which increases monotonically from 0 to 1 as α from 0 to ∞.

Now suppose that M_1 and M_2 are connected by Clayton's copula C_α through relation (12) and that S_{M_i} is defined as in (1) with non-negative random claims X_{ik}, $i = 1, 2$. In view of Corollary 1, the stop-loss premium $\pi_S(d)$ associated with $S = S_{M_1} + S_{M_2}$ will thus increase monotonically as a function of α for arbitrary fixed $d \geq 0$. This is illustrated in Figure 1 in the case where M_1 and M_2 are negative binomial random variables with unit mean and variance 2. In this figure, the X_{1i}'s are taken to be mutually independent and identically distributed Pareto random variables with mean 5 and variance 125, and the X_{2i}'s are similarly assumed to be log-normal with $\mu = 1/2$ and $\sigma = 2\{\log(5) - 1/2\}^{1/2}$. Displayed are the stop-loss premiums corresponding to the choices $\alpha = 0, 1, 4$ and $\alpha \to \infty$. M_1 and M_2 are stochastically independent when $\alpha = 0$ and comonotonic (i.e., $M_1 = M_2$ with probability one) in the limiting case $\alpha \to \infty$.

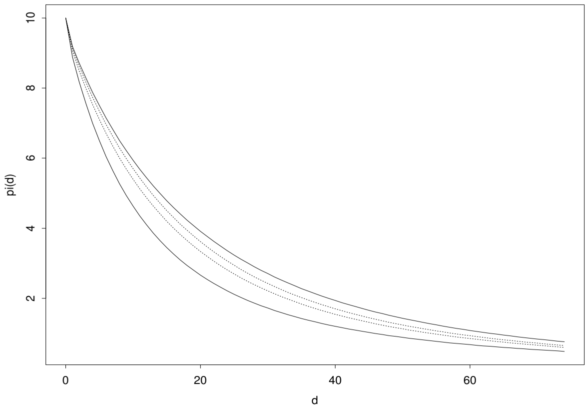

FIGURE 1: *Stop-loss premium for $S = S_{M_1} + S_{M_2}$ when M_1 and M_2 are negative binomial random variables with unit mean and variance 2 whose joint copula belongs to the Clayton family with parameter $\alpha = 0, 1, 4$ and $\alpha \to \infty$ from left to right. The summands in S_{M_1} are taken to be mutually independent and identically distributed Pareto random variables with mean 5 and variance 125, and the summands in S_{M_2} are similarly assumed to be log-normal with $\mu = 1/2$ and $\sigma = 2\{\log(5) - 1/2\}^{1/2}$.*

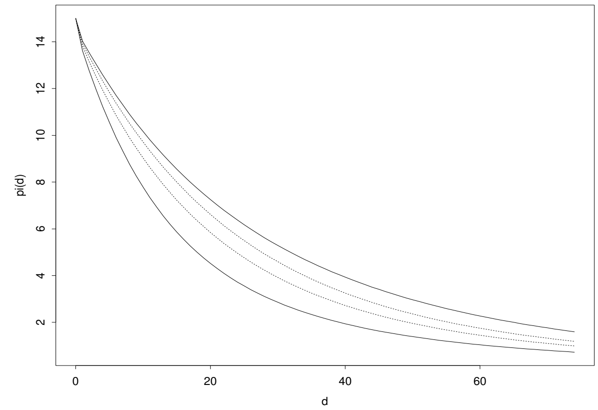

FIGURE 2: *Stop-loss premium for* $S = S_{M_1} + S_{M_2} + S_{M_3}$ *when* M_1, M_2, *and* M_3 *are negative binomial random variables with unit mean and variance 2 whose joint copula belongs to the Clayton family with parameter* $\alpha = 0, 1, 4$ *and* $\alpha \to \infty$ *from left to right. The summands in* S_{M_1} *and* S_{M_3} *are taken to be mutually independent and identically distributed Pareto random variables with mean 5 and variance 125, and the summands in* S_{M_2} *are similarly assumed to be log-normal with* $\mu = 1/2$ *and* $\sigma = 2\{\log(5) - 1/2\}^{1/2}$.

As can be seen from Figures 2 and 3, the stop-loss premiums $\pi_S(d)$ continue to be ordered as a function of α when there are 3 and 4 classes of business whose vector $\mathbf{M} = (M_1, \ldots, M_n)$ of numbers of claims has a distribution constructed from a copula belonging to the Clayton family. In these examples, the M_i's were again taken to be negative binomial variates with unit mean and variance 2; the distributions selected for the X_{ik}'s are specified in the figure captions. Additional illustrations of the same phenomenon are provided by Cossette *et al.* (1999) when the M_i's are Bernoulli random variables. Indeed, extensive experimentation carried out by the authors indicates that this ordering of the stop-loss premiums continues to prevail for all values of $n \geq 2$ and for a variety of distributions for the M_i's and the X_{ik}'s.

While this conclusion would be immediate from Corollary 2 if the Clayton family of copulas were ordered by the supermodular relation, the authors have not yet been able to establish that fact for $n > 3$. When $n = 3$, however, the ordering can easily be proved by first conditioning on u_3 (say) and checking that the resulting bivariate family of distributions

$$\begin{aligned} C_\alpha(u_1, u_2 \mid u_3) &\equiv P_\alpha(U_1 \leq u_1, U_2 \leq u_2 \mid U_3 = u_3) \\ &= \left\{(u_1u_3)^{-\alpha} + (u_2u_3)^{-\alpha} + u_3^{-2\alpha} - 2u_3^{-\alpha}\right\}^{-(\alpha+1)/\alpha} \end{aligned}$$

is ordered by PQD. The conclusion follows, since the supermodular ordering is closed under mixtures (Theorem 2.2d of Shaked & Shantikumar 1997).

Unfortunately, this method of proof does not extend readily to higher dimensions, and in the present state of development of the literature concerning the supermodular ordering, there are no simple criteria available for checking that a multivariate family of distributions is ordered by $\prec_{sm}$. As a result, very few situations have been identified so far in which Corollary 2 could be applied directly. One such example is provided by the equicorrelated multivariate normal copula, as recently shown by Bäuerle (1997). Additional ones are given in Examples 2 and 3.

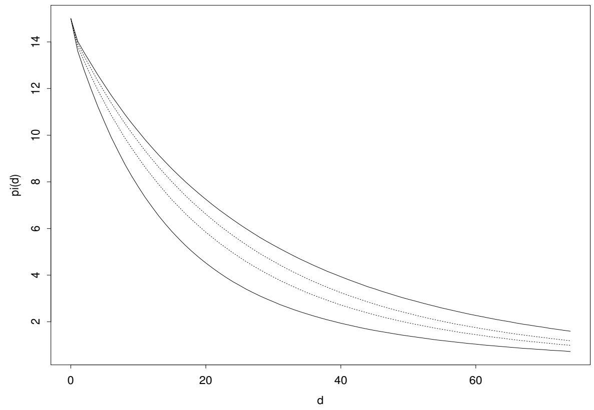

FIGURE 3: *Stop-loss premium for* $S = S_{M_1} + \cdots + S_{M_4}$ *when* $M_1, \ldots, M_4$ *are negative binomial random variables with unit mean and variance 2 whose joint copula belongs to the Clayton family with parameter* $\alpha = 0, 1, 4$ *and* $\alpha \to \infty$ *from left to right. The summands in* S_{M_1} *and* S_{M_3} *are taken to be mutually independent and identically distributed Pareto random variables with mean 5 and variance 125, and the summands in* S_{M_2} *and* S_{M_4} *are similarly assumed to be log-normal with* $\mu = 1/2$ *and* $\sigma = 2\{\log(5) - 1/2\}^{1/2}$.

EXAMPLE 2. For $0 \le \alpha \le 1$, consider the so-called Fréchet family, defined as a convex combination between the independence copula and the upper Fréchet bound characterizing comonotonic variables, viz.

$$C_\alpha(u_1, \ldots, u_n) = (1-\alpha)u_1 \cdots u_n + \alpha \min(u_1, \ldots, u_n). \tag{15}$$

Observe that if ϕ is a supermodular function of $\mathbf{U} = (u_1, \ldots, u_n)$, then

$$E_\alpha\{\phi(\mathbf{U})\} = (1-\alpha)E_0\{\phi(\mathbf{U})\} + \alpha E_1\{\phi(\mathbf{U})\}.$$

Thus if $\alpha \le \alpha'$, one has

$$E_{\alpha'}\{\phi(\mathbf{U})\} - E_\alpha\{\phi(\mathbf{U})\} = (\alpha' - \alpha)\left[E_1\{\phi(\mathbf{U})\} - E_0\{\phi(\mathbf{U})\}\right] \ge 0$$

if and only if $E_1\{\phi(\mathbf{U})\} \geq E_0\{\phi(\mathbf{U})\}$, which follows immediately from Theorem 2.5 of Shaked & Shantikumar (1997). Accordingly, $C_\alpha \prec_{sm} C_{\alpha'}$. Thus if

$$P(M_1 \leq m_1, \ldots, M_n \leq m_n) = C_\alpha \{G_1(m_1), \ldots, G_n(m_n)\}$$

with marginals $G_1, \ldots, G_n$ and C_α defined by (15), the stop-loss premium $\pi_S(d)$ associated to $S = S_{M_1} + \cdots + S_{M_n}$ will increase monotonically in α for every fixed value of $d \geq 0$, irrespective of the distribution of the individual claims X_{ik}.

EXAMPLE 3. Consider a family of univariate distribution functions $H_{\theta,\eta}$ that is closed under convolution for fixed η, so that $\theta_1 \in \Theta$ and $\theta_2 \in \Theta$ imply that $\theta_1 + \theta_2 \in \Theta$ and $H_{\theta_1,\eta} * H_{\theta_2,\eta} = H_{\theta_1+\theta_2,\eta}$. Suppose in particular that the numbers $M_1, \ldots, M_n$ of claims in n classes of business are modelled as $M_i = N_i + N_0$ in terms of independent, integer-valued N_i distributed as H_{θ_i,η_i}. For example, one might take N_i to be Poisson with mean θ_i or negative binomial with parameters θ_i and η, so that

$$P(N_i = k) = \frac{\Gamma(\theta_i + k)}{\Gamma(\theta_i) k!} \left(\frac{1}{1+\eta_i}\right)^{\theta_i} \left(\frac{\eta_i}{1+\eta_i}\right)^k, \quad k = 0, 1, \ldots$$

Now suppose that $M_i' = N_i' + N_0'$ is defined in a similar way, but with N_i' distributed as H_{θ_i',η_i}. If $\theta_0' - \theta_0 \in \Theta$ and $\theta_i + \theta_0 = \theta_i' + \theta_0'$ for $1 \leq i \leq n$, then as will be shown presently, the stop-loss premium associated with the vector $\mathbf{M} = (M_1, \ldots, M_n)$ is smaller than that which is associated with the vector $\mathbf{M}' = (M_1', \ldots, M_n')$. To this end, define

$$g_i(z, w) = H^{-1}_{\theta_i,\eta_i}(z) + H^{-1}_{\theta_0,\eta_i}(w)$$

and

$$\tilde{g}_i(u, v, w) = H^{-1}_{\theta_i+\theta_0-\theta_0',\eta_i}(u) + H^{-1}_{\theta_0'-\theta_0,\eta_i}(v) + H^{-1}_{\theta_0,\eta_i}(w).$$

It is clear that $\tilde{g}_i(u, v, w)$ is non-decreasing in v and that if $Z_1, \ldots, Z_n$, $U_1, \ldots, U_n$, V and W are $2n+2$ mutually independent uniform random variables, then $g_i(Z_i, w)$ has the same distribution as $\tilde{g}_i(U_i, V, w)$ for arbitrary $0 \leq w \leq 1$. Furthermore, observe that $\mathbf{M}$ and $\mathbf{M}'$ are distributed as $(g_1(Z_1, W), \ldots, g_n(Z_n, W))$ and $(\tilde{g}_1(U_1, V, W), \ldots, \tilde{g}_n(U_n, V, W))$, respectively. One then has $\mathbf{M} \prec_{sm} \mathbf{M}'$ as a consequence of Theorem 3.1 of Bäuerle (1997).

Finally, this section concludes on an example in which the weaker ordering $\prec_{uo}$ suffices to compare, through their adjustment coefficients, the riskiness of two portfolios involving several classes of business.

Example 4. Following Cossette & Marceau (1999), consider a more general Poisson shock model in which the number of claims in the i-th book of business is modelled as

$$M_i = \sum_{A:i\in A} N_A,$$

where for each non-empty subset A of $\{1,\ldots,n\}$, N_A is a Poisson random variable with mean λ_A. Assuming that the N_A's are mutually independent, the vector $\mathbf{M} = (M_1,\ldots,M_n)$ then has a multivariate Poisson distribution. In the case $n=2$ for example, one has $M_1 = N_1 + N_{12}$, $M_2 = N_2 + N_{12}$ and $\mathrm{Cov}(M_1,M_2) = \mathrm{Var}(N_{123})$, while in the case $n=3$, one finds

$$\begin{aligned} M_1 &= N_1 + N_{12} + N_{13} + N_{123} \\ M_2 &= N_2 + N_{12} + N_{23} + N_{123} \\ M_3 &= N_3 + N_{13} + N_{23} + N_{123} \end{aligned}$$

and

$$\begin{aligned} \mathrm{Cov}(M_1,M_2) &= \mathrm{Var}(N_{12}) + \mathrm{Var}(N_{123}) \\ \mathrm{Cov}(M_1,M_3) &= \mathrm{Var}(N_{13}) + \mathrm{Var}(N_{123}) \\ \mathrm{Cov}(M_2,M_3) &= \mathrm{Var}(N_{23}) + \mathrm{Var}(N_{123}). \end{aligned}$$

Now consider another multivariate Poisson vector $\mathbf{M}'$ with underlying variables N'_A. The conditions under which $\mathbf{M} \prec_{uo} \mathbf{M}'$ or $\mathbf{M} \prec_{\ell o} \mathbf{M}'$ have been identified by Brown & Rinott (1988, Theorems 1.1 and 1.2); cf. also Wong (1997). For $\prec_{uo}$, one must have

$$\sum_{A:B\subset A} \lambda_A \le \sum_{A:B\subset A} \lambda'_A \tag{16}$$

for all non-empty subsets B of $\{1,\ldots,n\}$. For $\prec_{\ell o}$, the requirement is

$$\sum_{A:A\cap B\neq\emptyset} \lambda_A \ge \sum_{A:A\cap B\neq\emptyset} \lambda'_A \tag{17}$$

for all non-empty subsets B of $\{1,\ldots,n\}$. In the case where $n=2$ and equal marginals are demanded, equation (16) reduces to $\lambda_{12} \le \lambda'_{12}$ while in the case $n=3$, the conditions for $\prec_{uo}$ are

$$\lambda_{12} + \lambda_{123} \le \lambda'_{12} + \lambda'_{123}, \quad \lambda_{13} + \lambda_{123} \le \lambda'_{13} + \lambda'_{123}, \quad \lambda_{23} + \lambda_{123} \le \lambda'_{23} + \lambda'_{123}$$

and $\lambda_{123} \le \lambda'_{123}$.

Assuming that (16) holds true, it is of course immediate from Proposition 2 (or Proposition 1 when $n=2$) that the vectors $\mathbf{S_M} = (S_{M_1},\ldots,S_{M_n})$ and $\mathbf{S}'_{\mathbf{M}'} =$

$(S'_{M'_1}, \ldots, S'_{M'_n})$ of total claims are ordered in the same way as $\mathbf{M}$ and $\mathbf{M}'$. Part (ii) of Corollary 2 further implies that the sums S and S' of the components of $\mathbf{S_M}$ and $\mathbf{S'_{M'}}$ are ordered by the exponential ordering, viz.,

$$f(r) \equiv E\left(e^{rS}\right) \leq E\left(e^{rS'}\right) \equiv g(r), \quad r \geq 0. \tag{18}$$

As shown by Cossette & Marceau (1999, Theorem 4), this implies an ordering of the adjustment coefficients R and R' corresponding to the two models. Indeed if c denotes the premium income collected in a given period, R and R' are defined as the solutions of the equations $f(r) = h(r)$ and $g(r') = h(r')$ where $h(r) = e^{cr}$. But since $f(0) = g(0) = h(0) = 1$ and $h'(0) = c \geq f'(0) = g'(0) = E(S) = E(S')$ by hypothesis, h will necessarily cross g before it meets f, whence $R' \leq R$. In other words, if $\mathbf{M} \prec_{sm} \mathbf{M}'$, then the portfolio associated with $\mathbf{M}'$ is riskier than that which is associated with $\mathbf{M}$.

Remark 4. It goes without saying that relation (18) would continue to hold if the vectors $\mathbf{M}$ and $\mathbf{M}'$ of claims in the various classes of business were connected by underlying copulas C_α and $C_{\alpha'}$ such that $\alpha \leq \alpha' \Rightarrow C_\alpha \prec_{co} C_{\alpha'}$. This result, which follows from Corollary 2 (ii) and Remark 3, remains valid whatever the marginal distributions of the M_i's.

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References

Baccelli, F., & A. M. Makowski(1989). Multidimensional stochastic ordering and associated random variables. *Operations Research*, 37, 478–487.

Bäuerle, N. (1997). Inequalities for stochastic models via supermodular orderings. *Communications in Statistics, Stochastic Models*, 13, 181–201.

Bäuerle, N., & A. Müller (1998). Modeling and comparing dependencies in multivariate risk portfolios. *ASTIN Bulletin* 28, 59–76.

Brown, L. D., & Y. Rinott (1988). Inequalities for multivariate infinitely divisible processes. *The Annals of Probability*, 16, 642–657.

Clayton, D. G. (1978). A model for association in bivariate life tables and its application in epidemiological studies of familial tendency in chronic disease incidence. *Biometrika*, 65, 141–151.

Cook, R. D., & M. E. Johnson (1981). A family of distributions for modelling non-elliptically symmetric multivariate data. *Journal of the Royal Statistical Society Series B*, 43, 210–218.

Cossette, H., & É. Marceau (1999). The discrete-time risk model with correlated classes of business. *Insurance: Mathematics and Economics*, to appear.

Cossette, H., Marceau, É., Gaillardetz, P., & J. Rioux (1999). Dependence in the individual risk model. Submitted for publication.

Dhaene, J. and M. J. Goovaerts (1996). Dependency of risks and stop-loss order. *ASTIN Bulletin* 26, 201-212.

Genest, C., & R. J. MacKay (1986). Copules archimédiennes et familles de lois bidimensionnelles dont les marges sont données. *The Canadian Journal of Statistics*, 14, 145–159.

Goovaerts, M. J., Kaas, R., van Heerwaarden, & Bauwenlinckx (1990). *Effective Actuarial Methods*. North-Holland, Amsterdam.

Hu, T., & Wu, Z. (1999). On dependence of risks and stop-loss premiums. *Insurance: Mathematics and Economics*, 24, 323–332.

Hutchinson, T. P., & Lai, C. D. (1990). *Continuous Bivariate Distributions, Emphasising Applications*. Rumsby Scientific Publishing, Adelaide.

Jean-Marie, A., & Z. Liu (1992). Stochastic comparisons for queueing models via random sums and intervals. *Advances in Applied Probability*, 24, 960–985.

Joe, H. (1990). Multivariate concordance. *Journal of Multivariate Analysis*, 35, 12–30.

Kimeldorf, G., & A. R. Sampson (1975). Uniform representation of bivariate distributions. *Communications in Statistics*, 4, 617–627.

Lehmann, E. L. (1966). Some concepts of dependence. *The Annals of Mathematical Statistics*, 37, 1137–1153.

Lorentz, G. G. (1953). An inequality for rearrangements. *The American Mathematical Monthly*, 60, 176–179.

Müller, A. (1997). Stop-loss order for portfolios of dependent risks. *Insurance: Mathematics and Economics*, 21, 219–223.

Nelsen, R. B. (1999). *An Introduction to Copulas.* Lecture Notes in Statistics No. 139. Springer-Verlag, New York.

Shaked, M., & J. G. Shantikumar (1994). *Stochastic Orders and Their Applications.* Academic Press, New York.

Shaked, M., & J. G. Shantikumar (1997). Supermodular stochastic orders and positive dependence of random vectors. *Journal of Multivariate Analysis*, 61, 86–101.

Sklar, A. (1959). Fonctions de répartition à n dimensions et leurs marges. *Publications de l'Institut de statistique de l'Université de Paris*, 8, 229–231.

Wong, (1997). Preservation of multivariate stochastic orders under multivariate Poisson shock models. *Journal of Applied Probability*, 34, 1009–1020.