Multivariate Cumulative Sum Control Charts Based on Projection Pursuit

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Abstract

A natural multivariate extension of the two-sided cumulative sum chart is proposed via the projection pursuit method. A modification is given for improving its response to large mean shifts. Simulation studies show that, by and large, the proposed charts perform very similar to the competing charts (Crosier's MCUSUM, MC1 and MEWMA) in terms of the average run length, standard deviation of the run length, average delayed run length and standard deviation of the delayed run length. A distinctive advantage of the proposed charts is that they are more effective against the inertia in reacting to mean shifts. However, the proposed charts are more difficult to compute.

Key words and phrases: Multivariate control chart, CUSUM control chart, projection pursuit, average run length.

Short running title: Multivariate CUSUM Chart

1. INTRODUCTION

In many quality control problems, several related process characteristics are of interest. Monitoring these characteristics independently can be very misleading (see, for example, Alt, 1985; Montgomery, 1991, p. 322-324). Thus, multivariate quality control is necessary in this setting. In the past decade, several kinds of multivariate control charts for the process mean have appeared. Most of them are the generalizations of the corresponding univariate procedures.

Among the univariate procedures, Shewhart's $\bar{x}$ chart supplemented with run rules has been widely used (see Nelson, 1984, 1985). Two very effective alternatives to the $\bar{x}$

chart may be used when detection of small sustained shifts is important (see Montgomery 1991, p. 279): the two-sided cumulative sum (CUSUM) chart and the exponentially weighted moving average (EWMA) chart. As a multivariate counterpart of the $\bar{x}$ chart, the χ^2-chart was first suggested and used by Hotelling (1947) in the testing of bombsights. It is well-known that the χ^2-chart is relatively insensitive to small mean shifts. This disadvantage raises the problem of how to obtain multivariate extensions of run rules, the two-sided CUSUM and EWMA charts. Woodall and Ncube (1985) suggested a multiple CUSUM chart by using a series of the CUSUM control charts on original characteristics or on principal component axes depending on the type of shift in the mean that is considered to be important to detect. Pignatiello and Runger (1990) showed that the Woodall-Ncube multiple CUSUM chart does not have good average run length (ARL) properties when the process mean shifts along several characteristics simultaneously. To lessen the sensitivity of the multiple univariate CUSUM chart to directions, they recommended using the univariate CUSUM charts aimed at several uniformly selected directions. Obviously, the more the directions the less the sensitivity. But, at the same time they found that the resulting control chart is hard to manage when there are three or more characteristics. Hawkins (1993) indicated that under some circumstances separate controls on the regression-adjusted variables by the CUSUM charts can both improve the speed of detection and make the chart signal more easily interpretable. Chan and Li (1995) proposed some run rules for the bivariate data patterns.

The above problem also motivates attempts to extend the univariate CUSUM and EWMA statistics to multivariate data. One difficulty encountered with generalizing the CUSUM statistics is that there are two cumulative sums for each variable (see Crosier, 1986). Crosier (1986, 1988) and Pignatiello and Runger (1990) proposed some two-sided methods which require only one cumulative sum, then they generalized these new statistics to the higher dimension. As a result, they obtained two multivariate CUSUM charts: the MCUSUM and MC1 charts. Note that neither of these two charts is the natural multivariate extension of the univariate CUSUM chart. In contrast, the multivariate extension of the EWMA (MEWMA) chart is natural and straightforward (see Lowry et al., 1992). In Lowry et al. (1992), the authors also showed that the MCUSUM, MC1 and MEWMA charts preserve the shorter ARL's of the corresponding univariate procedures for detecting small mean shifts while some of these charts can build up an arbitrarily large amount of inertia. For example, if the MEWMA statistic is on one side of the central line when a shift in the other direction occurs, there may be a delayed detection of the shift.

In this article, we develop a natural multivariate extension of the CUSUM chart, namely PPCUSUM, via the projection pursuit approach. The basic principle for constructing the PPCUSUM chart is given in Section 2. The method of improving the speed of the PPCUSUM chart in detecting larger mean shifts for the PPCUSUM chart is developed in Section 3. The procedure for operating the PPCUSUM chart is given in Section 4. Then illustrative examples are presented in Section 5. The PPCUSUM chart has two advantages over the competing charts (Crosier's MCUSUM, MC1 and MEWMA). First, it is more effective in coping with the inertia problems (see Section 6). Second, the shift-time (the time in which the shift occurs) are often relatively easy to detect by the change in the slope of the plotted points in the chart. There is, however, a drawback of using the PPCUSUM chart: it requires more computation than the competing charts. Simulation studies in Section 6 show that the PPCUSUM chart and its modification can perform roughly the same as the MC1 and MEWMA charts in detecting mean shifts in a multivariate normal distribution. In the same section, we prove that the PPCUSUM chart can be viewed as the limit of the Pignatiello-Runger multiple CUSUM chart (see Pignatiello and Runger, 1990, p. 183-184) as the number of the uniformly selected directions is increasing to infinity. We point out that the PPCUSUM chart can perform better than the Pignatiello-Runger multiple CUSUM chart and that it is relatively easy for implementation.

2. AN ALTERNATIVE MULTIVARIATE CUSUM CHART

Suppose that the $p-$quality characteristics, say X, of a process are normally distributed with mean μ and covariance matrix Σ_0. The aim of a multivariate control chart is to detect any shift in the process mean from a target value μ_0 by taking successive independent samples of size n from the process. The performance of a multivariate control chart is often measured by the average run length (ARL) and the standard deviation of the run length (SRL). For the $i-$th sample ($i = 1, 2, \cdots$), consider the standardized sample mean $\bar{y}_i = \sqrt{n}\Sigma_0^{-1/2}(\bar{x}_i - \mu_0)$, where $\bar{x}_i$ is the sample mean of the $i-$th sample. In practice it may be necessary to estimate both μ_0 and Σ_0 from past data, but for simplicity we assume here that μ_0 and Σ_0 are known. In the following let $||\cdot||$ denote the Euclidean norm of a vector. Let k be the reference value of the CUSUM chart. For a direction a

with $||a|| = 1$, define the CUSUM statistics

$$\begin{aligned} C_1^a &= \max\{0, a^\tau \bar{y}_1 - k\}, \\ C_2^a &= \max\{0, C_1^a + a^\tau \bar{y}_2 - k\}, \\ &\vdots \\ C_i^a &= \max\{0, C_{i-1}^a + a^\tau \bar{y}_i - k\}, \quad 1 \leq i < +\infty. \end{aligned}$$

Healy (1987) pointed out that if the process mean shifts along a direction b_0, the CUSUM chart for the projections, $\{b_0^\tau \Sigma_0^{-1} \bar{x}_i, i = 1, 2, \cdots\}$ (or $\{a_0^\tau \bar{y}_i, i = 1, 2, \cdots\}$ with $a_0 = \Sigma_0^{-1/2} b_0$), will give the optimal ARL performance. Without loss of generality, we choose b_0 such that $||a_0|| = 1$. Note that a control chart can be viewed as repeated significance tests (see Alt, 1988). Hence, Healy's result implies that when the first i samples are obtained, one should use the statistic $C_i^{a_0}$ to test whether there is a shift along the direction a_0. In practice a_0 is unknown and, therefore, such test cannot be applied directly.

The projection pursuit method is a powerful tool of constructing an estimate of a_0. It often views a_0 as the most interesting direction in terms of some unknown index (objective function) and is implemented by the following steps (see, for example, Huber, 1985). The first step is to select a sample index (projection pursuit index) which is approximately proportional to the unknown index, and quantitatively measures how interesting a direction is. The second step is to obtain the most interesting direction in terms of the index just selected and draw a further inference from the projected data in that direction.

In the following paragraph, we first use the projection pursuit method to derive an estimate, say $\hat{a}_0$, of a_0 when the first i samples are obtained, and approximate $C_i^{a_0}$ by $C_i^{\hat{a}_0}$. Then, we obtain a control chart by plotting $C_i^{\hat{a}_0}$ against the sample number i. This chart, called PPCUSUM, is a projection pursuit approximation of Healy's optimal CUSUM chart when the underlying direction of shift is unknown.

Note that $\Sigma_0^{-1/2}(\mu - \mu_0) = a_0 ||\Sigma_0^{-1/2}(\mu - \mu_0)||$ by the definition of a_0. The method of estimating a_0 is based on the fact that as a function of a, the deviation between $a^\tau \mu$ and $a^\tau \mu_0$ attains the maximum at a_0. That is, a_0 is the most interesting direction if we take such deviation function as our index. However that deviation function depends on the unknown parameter μ and, thus, should be replaced by some sample index. The CUSUM statistic C_i^a is a natural candidate for such sample index because C_i^a is the maximum likelihood ratio test of the significance of the deviation between $a^\tau \mu$ and $a^\tau \mu_0$, and because by the probability theory we can show that C_i^a is approximately proportional to $|a^\tau \mu - a^\tau \mu_0|$ almost surely when $|a^\tau \mu - a^\tau \mu_0| > k$, i is large (the details are not shown here). After choosing the sample index C_i^a, we should estimate a_0 by the direction which gives the maximum value of C_i^a. The CUSUM $C_i^{a_0}$ can be estimated by $C_i^{\hat{a}_0} = \max_{||a||=1} C_i^a$. Thus the PPCUSUM chart is equivalent to a control chart which indicates an off-target signal as soon as $\max_{||a||=1} C_i^a > h$, where h is the upper control limit.

Note that the estimate $\hat{a}_0$ is automatically adjusted with respect to the successive samples. So the above idea of using Healy's result to contruct a multivariate control chart is different from those in the literature where the CUSUM charts are often used along several predetermined directions simultaneously (see, for example, Hawkins, 1993).

Note that it is possible to use the normal model based likelihood test to introduce our procedure. Then the framework is simpler. The main motivation of using the projection pursuit method is that we show: (i) essentially our procedure is an extension of both the Pignatiello-Runger chart and Healy's chart; (ii) the procedure can be easily extended to the other settings (for instance, monitoring multivariate covariance and using some robust or nonparametric quality indices) because the projection pursuit does not require a normal model (see Liu and Singh, 1993; Chan and Zhang 1996, for more details).

For each i, let

$$\begin{aligned}
C_{ii} &= ||\bar{y}_i|| - k = \sqrt{n}((\bar{x}_i^\tau - \mu_0)\Sigma_0^{-1}(\bar{x}_i - \mu_0))^{1/2} - k, \\
C_{ij} &= ||\bar{y}_i + \cdots + \bar{y}_j|| - (i-j+1)k \\
&= \sqrt{n}\{(\bar{x}_j - \mu_0 + \cdots + \bar{x}_i - \mu_0)^\tau \Sigma_0^{-1}(\bar{x}_j - \mu_0 + \cdots + \bar{x}_i - \mu_0)\}^{1/2} \\
&\quad -(i-j+1)k, 1 \le j < i, \\
C_i &= \max\{0, C_{ii}, \cdots, C_{i1}\}.
\end{aligned}$$

In the following we show that $\max_{||a||=1} C_i^a = C_i$, $i = 1, 2, \cdots$. This makes the calculation of $\max_{||a||=1} C_i^a$ simple.

We first note that, for each i,

$$C_i^a = \max\{0, a^\tau \bar{y}_i - k, a^\tau(\bar{y}_{i-1} + \bar{y}_i) - 2k, \cdots, a^\tau(\bar{y}_1 + \cdots + \bar{y}_i) - ik\}$$

by induction. Then, it is obvious that for each fixed i and $0 \leq j \leq i-1$,

$$\begin{aligned}&\max_{||a||=1} C_i^a \geq \max_{||a||=1}\{a^\tau(\bar{y}_{i-j} + \cdots + \bar{y}_i) - (j+1)k\}\\ &= ||\bar{y}_{i-j} + \cdots + \bar{y}_i|| - (j+1)k\end{aligned}$$

which implies $\max_{||a||=1} C_i^a \geq C_i$. On the other hand, for each a, $||a|| = 1$,

$$C_i^a \leq \max\{0, ||\bar{y}_i|| - k, \cdots, ||\bar{y}_1 + \cdots + \bar{y}_i|| - ik\} = C_i$$

which gives the inequality $\max_{||a||=1} C_i^a \leq C_i$.

By the definition, all one needs to do is recursively add the y-vectors together in calculating $\{C_i\}$. It seems that the MEWMA is easier to compute than the PPCUSUM (see Runger and Prabhu, 1996).

To conclude this section, we discuss the problem of how to interpret the off-target signal in the PPCUSUM chart. Let

$$S_0 = 0, \quad S_i = \sum_{j=1}^{i} \bar{y}_j$$

for $i = 1, 2, \cdots$. Then

$$C_i = \max\{0, ||S_i - S_{i-1}|| - k, ||S_i - S_{i-2}|| - 2k, \cdots, ||S_i - S_0|| - ik\}$$

for $i = 1, 2, \cdots$. This implies that when an off-target signal appears in the PPCUSUM at the $i-$th time period, there will exist i_1 such that

$$||S_i - S_{i_1}|| - (i - i_1)k = \max_{||a||=1} C_i^a > h, \quad i > i_1.$$

We use $\Sigma_0^{1/2}(S_i - S_{i_1})/||S_i - S_{i_1}||$ to estimate the direction of shift in the mean of X, where $(S_i - S_{i_1})/||S_i - S_{i_1}||$ gives the maximum value of C_i^a. We make a CUSUM control chart for the projections of the transformed samples $\{\bar{y}_i, i = 1, 2, \cdots\}$ in that estimated direction. Then we can visually detect the shift-time by the change in the slope of the plotted points. It is helpful to choose those of relatively larger absolute values from p components of $\Sigma_0^{1/2}(S_i - S_{i_1})/||S_i - S_{i_1}||$ to diagnose the causes of the signal. We can also use the charts suggested by Pignatiello and Runger (1990) and Hawkins (1993) to interpret the signal.

3. MODIFICATION OF THE PPCUSUM CHART

Similar to the statistics discussed in Chan and Zhang (1996b), when the process mean is on-target, the statistic C_i plotted in the PPCUSUM chart will be within the decision interval $[0, h]$ and keep away from the upper control limit h for $1 \leq i \leq m$ and a large m. Our simulation experience shows that in this setting $(1-\lambda)^{i-t_i+1}h$ is very small, where $0 < \lambda < 1$ is a suitably chosen constant and

$$t_i = \min\{j : 1 \leq j < i, C_{ij} = C_i\}.$$

If we add $(1-\lambda)^{i-t_i+1}h$ to each C_i, $1 \leq i \leq m$, the resulting sequence should be still within the decision interval $[0, h]$. However, when a large mean shift occurs, there is a higher chance that C_{ii} takes a larger value. Thus $\{C_i\}$ will soon cross the upper control limit. Our simulation experience indicates that $(1-\lambda)^{i-t_i+1}h$ is not very small in this setting. Consequently, adding $(1-\lambda)^{i-t_i+1}h$ to C_i will cause C_i to cross the upper control limit more quickly.

We are thus led to modifying the PPCUSUM chart by adding $(1-\lambda)^{i-t_i+1}h$ to the statistic C_i. The new statistic is denoted by CM_i and the modified chart is denoted by FPCUSUM. By the above discussion, it seems possible to use the FPCUSUM to improve the response of the PPCUSUM to large mean shifts. This point is validated by the numerical results in Tables 6.1 and 6.3. For example, when $p = 2$, for a shift μ with $d = d(\mu, \mu_0) = ((\mu - \mu_0)\Sigma_0^{-1}(\mu - \mu_0))^{1/2} = 3.0$, PPCUSUM has the ARL of 2.51 while the FPCUSUM has the smaller ARL of 1.88. It can be shown that the FPCUSUM chart has the fast initial response feature (the details are omitted).

4. PROCEDURE FOR OPERATING THE NEW CHARTS

We begin with discussing the choice of the parameter λ of the FPCUSUM2 chart. For $p = 5$ ($p = 2$), the ARL and SRL performances of the FPCUSUM chart under $\lambda = 0.3$, 0.4, 0.6 ($\lambda = 0.45, 0.55$) are compared using the Monte Carlo simulation. The number of the replications in this simulation is 12000. The results, reported in Table 4.1, suggest that in general if we want to detect large mean shifts quickly, then a small λ is preferable, and that if both the small and large mean shifts are considered, the relatively good choice of λ is as follows: $\lambda = 0.55$ when $p = 2, 3, 4$ and $\lambda = 0.6$ when $p = 5$.

Let μ_1 be the off-target vector of interest. Then, like the univariate case, the parameter k of the PPCUSUM and FPCUSUM charts is usually chosen about the half of the Mahalanobis distance $d(\mu_1, \mu_0) = ((\mu_1 - \mu_0)^\tau \Sigma_0^{-1}(\mu_1 - \mu_0))^{1/2}$.

In general the procedure for operating the PPCUSUM and FPCUSUM charts consist of three steps. Take the PPCUSUM as an example.

Step 1: Estimate μ_0 and Σ_0 from past data. Calculate the eigenvalues and eigenvectors $f_1 \geq f_2 \geq \cdots \geq f_p$; $e_1, \cdots, e_p$, and let

$$\Sigma_0^{-1/2} = \sum_{j=1}^{p} f_j^{-1/2} e_j^\tau e_j.$$

Step 2: Specify the on-target ARL and SRL, the off-target shift of interest and the parameter k. Using Tables 4.2 and 4.3 to choose the parameter h.

Step 3: Transform the $i-$th sample mean $\bar{x}_i$ by

$$\bar{y}_i = \sqrt{n}\Sigma_0^{-1/2}(\bar{x}_i - \mu_0);$$

calculate C_i and t_i for each i. Then, plot C_i against i and check whether the plotted point falls outside the decision interval $[0, h]$. If the plotted point falls outside the decision interval, then an off-target signal is given. Use the proposal in Section 2 to estimate the direction of the shift and to plot a CUSUM chart for the projected samples in that direction.

For the convenience of using this procedure, the simulated on-target ARL and SRL values of the PPCUSUM and FPCUSUM charts of various control limits are presented in Tables 4.2 and 4.3. These values are the sample mean and sample standard deviation of the simulated run lengths. In these simulations, the number of replications is 6000 for Table 4.2 and at least 3000 for Tables 4.3, respectively. The relative standard error of the estimated ARL can be estimated by $SRL/(ARL \cdot u^{1/2})$, where u is the number of the replications. Thus the estimated relative standard errors of these simulations are smaller than 0.03.

5. EXAMPLES

We use the following examples to illustrate how the PPCUSUM and FPCUSUM charts are applied in the practical setting.

In a manufacturing process, a raw material can have a large effect on the quality of finished good properties. Abnormalities in some characteristics of the raw material may cause production upsets. So it is important to use control charts in raw material shipments. Prusinoski (1979) and Sultan (1986) applied Hotelling's T^2-charts to this case. In the following example we consider this case again by using the PPCUSUM, FPCUSUM, MC1 and MEWMA1 charts (see Appendix 1).

Example 5.1: Two quality characteristics of steel sections are monitored: x_1 = Brinell hardness and x_2 = tensile strength, which are measured in Kg/mm^2 (see Sultan,1986; Chan and Li, 1992). There are 30 samples of size 1 (that is, $n = 1$) with the first 25 from the original shipment and the next 5 from a new shipment.

Case 1: To illustrate the situation that the process mean is off-target at the beginning of the process, we take the first 25 observations as the on-target ones and use the PPCUSUM, FPCUSUM, MC1 and MEWMA1 charts to test 5 observations from the new shipment. Take the PPCUSUM chart as an example, the procedure is as follows.

Step 1: Estimate the target values by the sample mean and covariance matrix based on 25 observations from the original shipment:

$$\mu_0 = \begin{pmatrix} 177 \\ 53 \end{pmatrix} \text{ and } \Sigma_0 = \begin{pmatrix} 324 & 65 \\ 65 & 20 \end{pmatrix}.$$

Step 2: Let the interesting off-target shift, say μ_1, be of $d(\mu_1, \mu_0) = 1$. Then choose the reference value $k = 0.5$. Assume that the on-target ARL is about 132. By Table 4.1 we choose the control limit $h = 5$ in the PPCUSUM chart.

Step 3: Transform the $i-$th sample mean, calculate C_i and plot it against i in the PPCUSUM chart (the figure is omitted) and check whether it falls outside the decision intervals. It turns out that the PPCUSUM chart gives an off-target signal at the 4$-$th sample from the new shipment.

Similarly, let $h = 5.3$, $k = 0.5$ and $\lambda = 0.55$ in the FPCUSUM chart; $h = 4.75$ and $k = 0.5$ in the MC1 chart; $h = \sqrt{8.79}$ and $r = 0.1$ in the MEWMA1 chart. Calculate and plot the statistics $CM_i, MC1_i, NZ_i$ against i in the FPCUSUM, MC1, MEWMA1 charts, respectively. Then the FPCUSUM chart gives an off-target signal at the 3$-$rd sample. The MEWMA1 chart gives an off-target signal at 4$-$th sample. No off-target signal is generated from the MC1 chart.

Note that if we use the four charts to make retrospective test for the samples from the original shipment, then we find that the first sample is off-target and the last 24 are on-target. In Case 2 below we apply the above four charts to the data set composed of the last 24 samples from the original shipment and 5 samples from the new shipment to illustrate the situation that the process mean is off-target after the process mean has been on-target for a long time.

Case 2: We assume that μ_0 and Σ_0 are known and equal to the estimated values in Case 1. Then, the PPCUSUM gives an off-target signal at the $4-$th sample from the new shipment. The FPCUSUM gives an off-target signal at the $3-$rd sample from the new shipment while no off-target signal is presented by the MC1 and MEWMA1 charts.

Example 5.2: This example involves twenty three-dimensional samples of size 5, collected by Professor Weiming Zhang, from a Chinese mechanical manufacturing plant. In that plant the mouth bushes of joint pipes are lathed. It is known that the quality of each bush is determined mainly by three characteristics. To monitor these quality characteristics, a sample of size 5 is sampled once every two hours. Twenty samples, obtained in the two days running, are presented in Table 5.1. See Zhang (1996) for a detailed description. In the following we use the PPCUSUM with parameters $k = 0.5$, $h = 7$ to detect any mean shift in these samples. Here, according to Table 4.2, the on-target ARL and SRL are 300 and 275.

We first calculate the sample means and sample covariances of these samples. The on-target samples are obtained by the following heuristic way: first take the averages of the first eight sample means and sample covariances

$$\hat{\mu}_0 = \begin{pmatrix} 0.798 \\ 0.177 \\ 0.966 \end{pmatrix} \text{ and } \hat{\Sigma}_0 = \begin{pmatrix} 0.0006 & -0.0001 & 0.0003 \\ -0.0001 & 0.0008 & 0.0001 \\ 0.0003 & 0.0001 & 0.0007 \end{pmatrix}$$

as the nominal values of the process mean and covariance, and then use the above PPCUSUM to make a retrospective test for the first eighty samples. As a result, $C_i, i = 1, \cdots, 8$ are, respectively, 0.72, 0.79, 0.77, 2.49, 2.18, 1.20, 1.53, 1.06,which are far below the control limit $h = 6.5$. This implies that the first eighty samples are on-target with respect to $\hat{\mu}_0$. Similarly, we use the chart for covariance (see, for example, Zhang, 1996) to show that the first eighty samples are on-target with respect to $\hat{\Sigma}_0$. Therefore we believe that the first eighty samples are from a stable production process and it is reasonable to take $(\hat{\mu}_0, \hat{\Sigma}_0)$ as the nominal values. Finally, use the PPCUSUM to check the last 12 samples. For the convenience of illustration, we assume that the process covariance does not changed. The sequence of $\{C_i\}$ are plotted in Figure 5.1. The PPCUSUM give an off-target signal at $i = 2$ (that is, the tenth sample of size 5 in Table 5.1). In addition, looking at the changes of slopes in the curve of Figure 5.1, we find that the curve is increasing and also find two big changes at $i = 7$ and $i = 10$ (that is, the fifteenth and eighteenth samples of size 5 in Table 5.1). This suggests that the process mean has shifted at least three times.

The last example is used to show how to estimate the shift time.

Example 5.3. In this example, we apply the PPCUSUM, FPPCUSUM charts defined in Example 5.1 to a data set given by Crosier (1988). This data set is composed of the samples of size 1 from the normal distribution with unit variances and a correlation coefficient of 0.5. The process mean is on target for the first five samples and then shifts to $(1, 2)^\tau$ for the last five samples. In this example μ_0 is zero and Σ_0 is

$$\begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}.$$

The PPCUSUM, FPCUSUM charts give the off-target signal at the 10−th sample.

We project the sphered samples in the optimal estimated direction of the shift, $a^\tau = (-0.039023, 0.9992383)$. The univariate CUSUM sequences are 1.071, 2.044, 3.073, 3.488, 2.339, 3.162, 5.832, 7.587, 8.833, and 11.769. Then, we plot these projected CUSUM sequences in the univariate CUSUM chart (see Figure 5.2). It is easy to find from Figure 5.2 that the curve is increasing after 5−th sample. So we can use 6 to estimate the shift time.

6. PERFORMANCES OF THE NEW CHARTS

At first, we see that the method of Pignatiello and Runger (1990, P.185) can be applied directly to prove the first property.

Property 1. Both the PPCUSUM and FPCUSUM charts are directionally invariant.

That is, the ARL and SRL of the PPCUSUM and FPCUSUM charts depend on the shift μ only through the distance $d = d(\mu, \mu_0) = \{(\mu - \mu_0)^\tau \Sigma_0^{-1}(\mu - \mu_0)\}^{1/2}$.

This property makes the calculation and comparison of the ARL and SRL easier because we need only to consider the shifts of the form $\mu - \mu_0 = (\delta, \cdots, \delta)$ and $\Sigma_0 = I_p$ (the p by p unit matrix).

It is easy to show the following,

Property 2. When $p = 1$ the PPCUSUM chart is reduced to the two-sided CUSUM chart. This implies that the PPCUSUM chart is a natural multivariate extension of the univariate CUSUM chart.

Pignetiello and Runger (1990) suggested a multiple CUSUM chart: using univariate CUSUM charts along several uniformly selected directions, say a_j $1 \leq j \leq l$, simultaneously. In the language of repeated significance tests, this chart is equivalent to using $C_i^{a_j}$ $1 \leq j \leq l$ to test the hypothesis $\mu = \mu_0$ simultaneously when the first i samples are obtained. If we use the same control limit, say h, for all these charts, then an off-target signal appears in one of these charts if and only if $\max_{1\leq j\leq l} C_i^{a_j} > h$. Note that if l tends to infinity, then $\max_{1\leq j\leq l} C_i^{a_j}$ converges to $\max_{||a||=1} C_i^{a}$, which is just the statistic used in the PPCUSUM. In this sense we have the next property:

Property 3. The PPCUSUM chart is the limit of the Pignatiello-Runger multiple CUSUM chart when the number of uniformly selected directions are large enough.

Note that in higher dimension, a very large l is required and in this case the computation of $\max_{1\leq j\leq l} C_i^{a_j}$ is harder than that of C_i.

In this section, the ARL, SRL, ADRL and SDRL performances of the PPCUSUM, FPCUSUM, MEWMA1, MEWMA2 and MC1 charts are compared in Tables 6.1 $\sim$ 6.4 for $p = 2, 5$ via the Monte Carlo simulation. The χ^2-chart, Crosier's MCUSUM chart and the multiple CUSUM chart have not been involved in this study because they have been compared with the MEWMA chart in Pignatiello and Runger (1990), and it has been shown that the MEWMA chart performs slightly better.

In Pignatiello and Runger (1990) and Lowry et al. (1992), the ARL performances of the MC1, MEWMA1 and MEWMA2 charts are compared only under initial off-target conditions. This is in contrast to how quality control charts are applied in practice: a shift often occurs after the process mean has been on-target for some time. Thus it is necessary to make a comparison of the PPCUSUM, FPCUSUM, MC1, MEWMA1 and MEWMA2 charts in terms of the average delayed run length (ADRL) and the standard deviation of the delayed run lengths (SDRL) under the following model: for a fixed integer v the process mean is on-target before the first $v-1$ samples, and off-target after the v-th sample (including the v-th sample). Mathematically, the ADRL and SDRL under the above model are just the conditional expectation and conditional standard deviation, $E((RL-v+1)|RL>v)$ and $(Var((RL-v+1)|RL>v))^{1/2}$. By the simulation result, we choose v so that these quantities are near their limits, the steady-state ADRL and SDRL. Here, we find that $v=15$ is a reasonable choice. In this study, we adopt Yashchin's approach to assume that there is no false alarm before the shift (see, for example, Yashchin, 1985; Reynolds et al., 1990). Similar to Taylor (1968), if the control chart signals during the first $v-1$ samples, then this sequence of samples is discarded and the count of samples is reset to zero. Each simulated value of the delayed run length is obtained by subtracting $v-1$ from a simulated run length.

For the off-target case, in Tables 6.1 and 6.3 we used the assumption that the shift has occurred prior to the application of charts, while in Tables 6.2 and 6.4 we used the assumption that the shift occurs after $v-1(=14)$ samples. As before ARL and SRL (or ADRL and SDRL) in these tables are, respectively, the average and the standard deviation of at least 6000 simulated run lengths. The relative standard errors of the estimated ARL and ADRL are smaller than 0.01. In addition, the following remarks are used in these tables.

1. Invoking the results at the beginning of Section 4, we let the parameter $\lambda=0.6$ in the FPCUSUM chart and let the reference values of all these charts be 0.5. For the parameter r of the MEWMA1 and MEWMA2 chart, $r=0.1$ is recommended by Lowry et al. (1992). Our simulation study shows for the MEWMA1 with $r=0.1$ the on-target SRL is larger than the on-target ARL (see Tables 6.1 and 6.3). For example, when $p=2$, $h=\sqrt{7.88}$ and $r=0.1$, the on-target ARL and SRL are 132 and 135. As pointed out by Chan and Zhang (1996a), it will result a higher chance that the run length takes a small value.

2. Note that the PPCUSUM1, FPCUSUM, MEWMA1 and MEWMA2 and MC1

charts are directionally invariant (see Pignatiello and Runger,1990; Lowry, et al., 1992). Hence, without loss of generality, we assume that $\Sigma_0 = I_p$ (the p by p identity matrix) and $\mu - \mu_0$ is of the form $(\delta, \cdots, \delta)$. For $p = 2, 5$, we let $d = d(\mu, \mu_0) = 0.5, 1.0,\ 1.5, 2.0,\ 2.5, 3.0$.

3. To compare the different control procedures fairly, we need to calibrate each procedure so that the on-target ARL is approximately the same for all the procedures (see, Pignatiello and Runger, 1990).

When $p = 2$, it is clear from Tables 6.1 that: **(i)** For the initial shift μ_1 with $0.5 \leq d(\mu_1, \mu_0) \leq 2$, the PPCUSUM and FPCUSUM (with $\lambda = 0.55$) produce slightly larger ARL than the MC1 chart, while for the initial shift μ_1 with $d = d(\mu_1, \mu_0) > 2$, the FPCUSUM gives a slightly shorter ARL than the MC1. **(ii)** The MEWMA1 chart with $r = 0.1$ is slightly quicker than the PPCUSUM and FPCUSUM charts in detecting initial shifts in terms of ARL. However, the on-target SRL of that MEWMA1 chart is larger than those of the PPCUSUM and FPCUSUM charts. And if we let $r = 0.4$ so that the on-target $SRL = 128$ is smaller than the on-target $ARL = 132$, then the PPCUSUM and FPCUSUM charts perform better than the MEWMA1 chart. **(iii)** The MEWMA2 chart with $r = 0.1$ performs slightly better than the PPCUSUM and FPCUSUM charts when $0.5 \leq d(\mu, \mu_0) \leq 1.0$, while the latter two perform better when $1.5 \leq d(\mu, \mu_0) \leq 3.0$. **(iv)** Table 6.2 indicates that when $p = 2$ the PPCUSUM and FPCUSUM charts have better ADRL and SDRL performances than the competing charts.

For $p = 5$ a similar conclusion can be drawn from Tables 6.3 and 6.4.

Generally speaking, we conclude that the PPCUSUM and FPCUSUM charts perform roughly the same as the MC1, MEWMA1 and MEWMA2 charts in detecting mean shifts. We recommend using the MCUSUM, MC1, MEWMA1 and MEWMA2 charts for detecting initial off-target shifts and using the PPCUSUM and FPCUSUM charts for detecting delayed off-target shifts.

7. Inertia Problems

The "inertia" problem is originally studied by Yashichin (1987) for the univariate EWMA chart. Lowry et al. (1992) point out that the MC1, MEWMA1, MEWMA2 and MCUSUM charts can have "inertia" in reacting to shifts in the process mean relative to the univariate CUSUM chart. They presented a worst-case inertia existing in the MEWMA1 and MEWMA2 charts. In contrast, it is easy to see that the PPCUSUM and FPCUSUM charts can prevent this kind of inertia. In the same paper Lowry et al. propose that the χ^2-chart should always used in conjunction with the MEWMA chart to help to prevent such delays. However, as noted by them in that paper, the use of the χ^2-chart does not completely solve the inertia problem. For when if inertia has built up, then a sequence of relatively large shifts from the target value that does not trigger the χ^2- chart may still not result in off-target signal from the MCUSUM, MC1, MEWMA1 and MEWMA2 charts. As expected, in this case the simulation results displayed in Tables 7.1$\sim$ 7.2 do show that the PPCUSUM and FPCUSUM charts can be more effective than the MC1, MEWMA1, MEWMA2, the combined χ^2-MEWMA1 and the combined χ^2-MEWMA2 charts against the moderate inertia (see Appendix 1 for the definitions of these charts and note that the rough rule of thumb of Lowry et al. (1992) is used here in designing χ^2-MEWMA1 and χ^2-MEWMA2 charts).

In Tables 7.1 and 7.2, we adopt a special model, called **Model 1**. In this model, we let the sample size be 1. Let $y_i = 0$, for $1 \leq i \leq 17$, $y_{18} = (-2.8, -0.5)^\tau$ and $y_{19} = (-1.5, -1.5)^\tau$ for Table 7.1, and let $y_i = 0$, $1 \leq i \leq 17$, and $y_{18} = (-2.8, -0.5, -0.5, -0.5, -0.5)^\tau$ and $y_{19} = (-1.5, -1.5, -1.5, -1.5, -1.5)^\tau$ for Table 7.2. Assume that $\mu_0 = 0$. Let $y_i, 20 \leq i < \infty$ be samples from $N(\mu, I_2)$ for Table 7.1 and $N(\mu, I_5)$ for Table 7.2.

In Tables 7.3 and 7.4, an alternative normal model is used. This model is used to describe a systematic effect produced by two machines such as spindles, operators or vendors used alternatively (see, Nelson, 1985). In this model, let y_i, $1 \leq i < \infty$ be alternatively sampled from $N(\mu_1, I_p)$ and $N(\mu_2, I_p)$ and assume that μ_1 is of the form $(\delta, \cdots, \delta)$ and $\mu_2 = -\mu_1$. For the alternative model, Tables 7.3 and 7.4 show that the PP-CUSUM and FPCUSUM charts perform better than the MC1, MEWMA1 and MEWMA2 charts. When $p = 2$ the FPCUSUM chart is superior to the combined χ^2-MEWMA1 and χ^2-MEWMA2, and when $p = 5$ the latter are better.

Remark: The shifts of the form $(\delta, \cdots, \delta)^\tau$ are considered in Tables 7.1 and 7.4. Here, $\delta = d/\sqrt{p}$ and $d = 0.5, 1.0, 1.5, 2.0, 2.5$ and 3.0. Each pair of ARL and SRL values in Tables 7.1 and 7.4 is obtained from 6000 replications.

8. CONCLUSIONS

Two new charts, namely PPCUSUM and FPCUSUM, are developed for detecting mean shifts in a multivariate normal distribution. It is shown that these charts are directionally invariant. The PPCUSUM chart is a natural multivariate extension of the univariate CUSUM chart because it is reduced to the two-sided CUSUM chart when the dimension p is equal to 1. Simulations indicate that the new charts have slightly superior SRL, ADRL and SDRL performances than the MC1 and MEWMA1 charts, while performing a little worse in terms of ARL. The main advantage of the new charts is that they can be more effective in avoiding the inertia problem than the MCUSUM, MC1, MEWMA1, MEWMA2, the combined χ^2-MEWMA1 and the combined χ^2-MEWMA2 charts. However, there is a drawback of using the new charts: they require more computation than the competing charts. On the whole, the new charts seem promising when on-line computers are used during production. For the very high dimensional set, it is helpful to use the dimension reduction technique to reduce the dimension before applying our method.

For the convenience of application, the on-target ARL and SRL of these charts under various control limits are simulated. Using the least squares fitting, we obtain the following approximate formula for the on-target $\log(ARL)$:

$$\log(ARL) = c_o + c_1 h$$

where

$$c_o = \begin{cases} -0.2247889, & when \quad p=2, \lambda=0.55, k=0.5; \\ 0.6699669, & when \quad p=3, \lambda=0.6, k=0.5; \\ -0.186100, & when \quad p=4, \lambda=0.6, k=0.5; \\ -0.5101398, & when \quad p=5, \lambda=0.6, k=0.5; \end{cases}$$

$$c_1 = \begin{cases} 0.9693552, & when \quad p=2, \lambda=0.55, k=0.5; \\ 0.7091466, & when \quad p=3, \lambda=0.6, k=0.5; \\ 0.7372557, & when \quad p=4, \lambda=0.6, k=0.5; \\ 0.7002355, & when \quad p=5, \lambda=0.6, k=0.5. \end{cases}$$

The standard deviation of fitting is less than 0.1. Their ADRL and SDRL values are also available from the authors.

Appendix 1: The brief definitions of the MC1 chart and MEWMA chart

MC1 chart: Let $l_0 = 0, l_1 = 1$ and for $i = 1, 2, \cdots$,

$$D_i = \sum_{i-l_i+1}^{i} y_j, MC_i = \max\{0, ||D_i|| - kl_i\},$$
$$l_i = \begin{cases} l_{i-1} + 1, & MC_{i-1} > 0, \\ 1, & \text{otherwise} . \end{cases} .$$

An off-target signal is given as soon as $MC_i > h$, where $k > 0$ and $h > 0$.

MEWMA1 chart (using the exact covariance of the MEWMA statistic): Let $Z_0 = 0$ and

$$Z_i = ry_i + (1-r)Z_{i-1}, \quad 1 \le i < \infty,$$

where $0 < r < 1$ is a constant. An off-target signal is given as soon as

$$NZ_i = ||Z_i||\sqrt{\frac{2-r}{r(1-(1-r)^{2i})}} > h.$$

MEWMA2 chart (using the asymptotic covariance of the MEWMA statistic): replace NZ_i in MEWMA1 chart by

$$NZ_i = ||Z_i||\sqrt{\frac{2-r}{r}}.$$

The combined χ^2-MEWMA1 chart: Let h_1^2 be the upper control limit of χ^2- chart. Let Z_i and NZ_i be those used in the MEWMA1 chart. Then, the combined χ^2-MEWMA1 chart gives an off-target signal as soon as $||Z_i|| > h_1$ or $NZ_i > h$.

Similarly, we can define the combined χ^2-MEWMA2 chart.

Appendix 2: Proof of Property 2

Let $p = 1$. Then $\bar{y}_j, 1 \le j \le \infty$ defined in Section 2 are univariate sample means. Let h and k be the upper control limit and the reference value in the PPCUSUM chart respectively. Note that the PPCUSUM chart gives an off-target signal as soon as $C_i > h$. Denote by S_i^+ and S_i^- the upper one-sided and lower one-sided CUSUM values for the period i where the reference value is k. Using S_i^+ and S_i^- we define a two-sided CUSUM chart which gives an off-target signal as soon as $\max\{S_i^+, S_i^-\} > h$ (see Page, 1954). To complete the proof, it suffices to prove that

$$\max\{S_i^+, S_i^-\} = C_i$$

for $i = 1, 2, \cdots$.

To this end, we first show by induction that

$$S_i^+ = \max\{0, y_i - k, \cdots, y_1 + \cdots + y_i - ik\},$$
$$S_i^- = \max\{0, -y_i - k, \cdots, -y_1 + \cdots - y_i - ik\}$$

for $i = 1, 2, \cdots$. Then, by definition, we have

$$\begin{aligned}
&\max\{S_i^+, S_i^-\} \\
&= \max\{0, \max\{y_i - k, -y_i - k\}, \cdots, \max\{y_1 + \cdots + y_i - ik, -y_1 - \cdots - y_i - ik\}\} \\
&= \max\{0, |y_i| - k, \cdots, |y_1 + \cdots + y_i| - k\} \\
&= C_i
\end{aligned}$$

for $i = 1, 2, \cdots$.

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Table 4.1. ARL and SRL comparison for the PPCUSUM charts with $k = 0.5$ and different λ for $p = 2, 5$

	p=2		p=5		
	h=6.0	h=6.5	h=10.3	h=8.6	h=8.6
	$\lambda = 0.45$	$\lambda = 0.55$	$\lambda = 0.3$	$\lambda = 0.4$	$\lambda = 0.6$
d	ARL,SRL	ARL,SRL	ARL,SRL	ARL,SRL	ARL,SRL
0.0	282,266	285,267	166,169	162,156	162,154
0.5	43.4,34.9	38.6,30.4	53.8,45.2	40.1,30.0	37.4,27.7
1.0	11.8,6.30	11.1,5.74	15.2,8.20	13.0,6.50	12.63,5.69
1.5	6.28,2.88	6.11,2.54	7.93,4.13	7.20,3.12	7.38,2.50
2.0	3.99,1.88	4.09,1.61	4.67,2.77	4.67,2.18	5.22,1.57
2.5	2.73,1.35	2.97,1.21	2.97,1.91	3.20,1.62	3.99,1.14
3.0	2.01,1.00	2.22,0.94	1.98,1.25	2.24,1.20	3.17,0.95

Table 4.2. The on-target ARL and SRL of the PPCUSUM charts with $k = 0.5$ for $p = 2, 3, 4$ and various control limits h

p=2			p=3			p=4		
h	ARL	SRL	h	ARL	SRL	h	ARL	SRL
4.75	107	98.6	5.75	108	103	7.00	143	133
5.00	133	124	6.00	132	125	7.50	212	195
5.50	206	196	6.25	163	153	7.75	260	246
5.75	261	249	6.50	195	187	8.00	305	285
6.00	326	311	6.75	245	232	8.25	378	340
6.25	406	359	7.00	300	275	8.50	446	392
6.50	485	421	7.25	365	337	8.75	524	437
7.00	689	504	7.50	450	391	9.00	628	486

Table 4.3. On-target ARL and SRL of the FPCUSUM charts for $p = 2, 3, 4, 5$

$p = 2, \lambda = 0.55, k = 0.5$						$p = 3, \lambda = 0.6, k = 0.5$						
h	5.3	5.5	6.0	6.5	7.0	h	6.0	6.5	7.0	7.5	8.0	9.0
ARL	131	163	285	455	669	ARL	124	190	300	442	629	10008
SRL	126	157	267	401	490	SRL	118	185	282	387	483	531
$p = 4, \lambda = 0.6, k = 0.5$						$p = 5, \lambda = 0.6, k = 0.5$						
h	7.0	7.5	8.0	8.5	9.0	h	7.5	8.0	8.5	9.0	9.5	10
ARL	143	210	307	440	624	ARL	113	162	229	344	472	636
SRL	133	198	287	392	484	SRL	106	154	216	321	408	481

Table 5.1. The mouth bush data used in Example 5.1

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
.81	.85	.76	.79	.82	.82	.79	.74	.77	.80
.17	.18	.20	.19	.20	.19	.20	.17	.18	.18
1.0	.98	.95	.98	.96	1.02	.99	.87	.96	.95
x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}	x_{17}	x_{18}	x_{19}	x_{20}
.83	.81	.79	.80	.81	.79	.75	.76	.78	.80
.15	.14	.17	.16	.20	.13	.17	.14	.15	.19
.99	.97	.95	.94	.98	.94	1.0	.96	.92	.99
x_{21}	x_{22}	x_{23}	x_{24}	x_{25}	x_{26}	x_{27}	x_{28}	x_{29}	x_{30}
.82	.80	.84	.78	.79	.81	.80	.84	.79	.76
.12	.14	.16	.20	.21	.13	.17	.21	.22	.19
.94	.97	.98	.95	.97	.98	.95	.96	.94	.95
x_{31}	x_{32}	x_{33}	x_{34}	x_{35}	x_{36}	x_{37}	x_{38}	x_{39}	x_{40}
.80	.82	.83	.78	.81	.81	.78	.79	.77	.83
.14	.16	.22	.23	.20	.14	.17	.21	.22	.19
.98	.97	.98	.95	.96	.96	.98	.99	.97	.99
x_{41}	x_{42}	x_{43}	x_{44}	x_{45}	x_{46}	x_{47}	x_{48}	x_{49}	x_{50}
.76	.77	.75	.74	.75	.76	.76	.75	.76	.82
.12	.12	.14	.17	.18	.13	.20	.10	.10	.18
.96	1.02	1.0	.98	.97	.94	.90	1.0	.88	.96

Table 5.1. (continued)

x_{51}	x_{52}	x_{53}	x_{54}	x_{55}	x_{56}	x_{57}	x_{58}	x_{59}	x_{60}
.82	.78	.78	.77	.80	.81	.83	.77	.74	.76
.14	.21	.14	.15	.20	.15	.17	.16	.14	.26
.96	.90	.91	.93	.98	.94	.96	.95	.94	.99
x_{61}	x_{62}	x_{63}	x_{64}	x_{65}	x_{66}	x_{67}	x_{68}	x_{69}	x_{70}
.83	.86	.79	.76	.80	.75	.75	.75	.70	.75
.16	.18	.17	.16	.19	.12	.10	.12	.14	.14
.93	.95	.96	.98	.97	.99	.95	.95	.95	.97
x_{71}	x_{72}	x_{73}	x_{74}	x_{75}	x_{76}	x_{77}	x_{78}	x_{79}	x_{80}
.75	.72	.80	.72	.75	.81	.70	.72	.71	.77
.13	.18	.15	.13	.13	.13	.17	.16	.16	.14
.97	.98	.99	.99	.97	.97	.96	.97	.98	.97
x_{81}	x_{82}	x_{83}	x_{84}	x_{85}	x_{86}	x_{87}	x_{88}	x_{89}	x_{90}
.63	.70	.70	.70	.70	.80	.75	.75	.67	.70
.13	.15	.13	.15	.15	.17	.16	.15	.17	.16
.97	.97	.98	.98	.96	.97	.97	.97	.97	.99
x_{91}	x_{92}	x_{93}	x_{94}	x_{95}	x_{96}	x_{97}	x_{98}	x_{99}	x_{100}
.73	.73	.75	.72	.80	.74	.73	.90	.70	.75
.15	.15	.18	.17	.16	.16	.15	.13	.15	.16
.90	.90	.87	.90	.86	.92	.90	.90	.88	.89

Table 6.1. ARL and SRL comparison when $p = 2$

d	0.0	0.5	1.0	1.5	2.0	2.5	3.0
PPCUSUM with $h = 5, k = 0.5$							
ARL	132	27.4	9.33	5.41	3.82	3.00	2.51
SRL	124	21.3	4.71	2.10	1.22	0.84	0.64
FPCUSUM with $h = 5.3, \lambda = 0.5, k = 0.5$							
ARL	131	29.8	9.58	5.29	3.48	2.48	1.98
SRL	126	24.0	5.47	2.46	1.62	1.16	0.87
MC1 with $h = 4.33, k = 0.5$							
ARL	131	26.2	8.57	4.80	3.40	2.70	2.27
SRL	126	21.3	4.83	2.01	1.18	0.82	0.60
MEWMA1 with $h = \sqrt{7.88}, r = 0.1$							
ARL	132	20.8	6.96	3.67	2.40	1.75	1.41
SRL	135	17.5	4.61	2.21	1.30	0.86	0.60
MEWMA1 with $h = 3.077, r = 0.4$							
ARL	132	39.2	10.6	4.76	2.86	2.03	1.58
SRL	128	38.4	8.73	3.21	1.62	1.02	0.71
MEWMA2 with $h = 2.763, r = 0.1$							
ARL	132	23.7	9.18	5.64	4.11	3.29	2.74
SRL	123	16.3	4.18	1.94	1.18	0.83	0.65

Table 6.2. ADRL and SDRL comparison when $p = 2, v = 15$

d	0.5	1.0	1.5	2.0	2.5	3.0
PPCUSUM with $h = 5, k = 0.5$						
ADRL	26.1	8.45	4.84	3.44	2.74	2.26
SDRL	22.7	5.16	2.25	1.37	0.95	0.72
FPCUSUM with $h = 5.3, \lambda = 0.5, k = 0.5$						
ADRL	27.8	8.79	4.89	3.33	2.48	1.98
SDRL	23.9	5.22	2.40	1.47	1.07	0.79
MC1 with $h = 4.33, k = 0.5$						
ADRL	26.5	8.81	5.22	3.78	3.00	2.56
SDRL	22.2	5.00	2.48	1.59	1.18	0.92
MEWMA1 with $h = \sqrt{7.88}, r = 0.1$						
ADRL	24.4	9.29	5.65	4.12	3.30	2.75
SDRL	17.0	4.25	4.12	1.20	0.85	0.65
MEWMA1 with $h = 3.077, r = 0.4$						
ADRL	40.8	11.1	5.18	3.23	2.42	1.93
SDRL	38.5	8.65	3.12	1.52	0.95	0.69
MEWMA2 with $h = 2.763, r = 0.1$						
ADRL	23.6	9.15	5.64	4.10	3.31	2.75
SDRL	16.2	4.08	1.97	1.18	0.85	0.64

Table 6.3. ARL and SRL comparison when $p = 5$

d	0.0	0.5	1.0	1.5	2.0	2.5	3.0
PPCUSUM with $h = 8, k = 0.5$							
ARL	163	37.5	12.6	7.44	5.36	4.22	3.51
SRL	154	28.1	5.71	2.41	1.44	0.98	0.74
FPCUSUM with $h = 8, \lambda = 0.6, k = 0.5$							
ARL	162	37.4	12.6	7.38	5.22	3.99	3.17
SRL	154	27.7	5.69	2.50	1.56	1.14	0.95
MC1 with $h = 6.55, k = 0.5$							
ARL	163	35.4	10.5	6.15	4.42	3.51	2.93
SRL	159	30.2	5.53	2.25	1.30	0.90	0.68
MEWMA1 with $h = \sqrt{14.3}, r = 0.1$							
ARL	161	31.4	9.62	4.95	3.13	2.26	1.75
SRL	164	26.5	6.21	2.79	1.63	1.09	0.78
MEWMA1 with $h = \sqrt{15.97}, r = 0.4$							
ARL	161	65.3	17.6	6.99	3.88	2.62	1.97
SRL	156	63.4	15.4	4.98	2.27	1.33	0.90
MEWMA2 with $h = 3.75, r = 0.1$							
ARL	163	35.0	12.4	7.40	5.34	4.22	3.52
SRL	152	24.0	5.55	2.41	1.42	0.96	0.74

Table 6.4. ADRL and SDRL comparison when $p = 5, v = 15$

d	0.5	1.0	1.5	2.0	2.5	3.0
PPCUSUM with $h = 8, k = 0.5$						
ADRL	33.7	11.1	6.60	4.71	3.69	3.06
SDRL	27.9	5.80	2.70	1.65	1.18	0.94
FPCUSUM with $h = 8, \lambda = 0.6, k = 0.5$						
ADRL	33.5	11.1	6.54	4.66	3.60	2.91
SDRL	27.8	5.80	2.63	1.66	1.16	0.92
MC1 with $h = 6.55, k = 0.5$						
ADRL	37.0	12.8	7.69	5.55	4.37	3.67
SDRL	30.1	7.16	3.59	2.35	1.74	1.41
MEWMA1 with $h = \sqrt{14.3}, r = 0.1$						
ADRL	35.9	12.6	7.39	5.32	4.20	3.50
SDRL	26.0	5.64	2.42	1.43	0.96	0.73
MEWMA1 with $h = \sqrt{15.97}, r = 0.4$						
ADRL	66.3	18.1	7.41	4.33	3.06	2.41
SDRL	63.8	15.2	4.81	2.15	1.21	0.82
MEWMA2 with $h = 3.75, r = 0.1$						
ADRL	34.5	12.4	7.37	5.33	4.24	3.53
SDRL	24.6	5.46	2.42	1.41	0.98	0.72

Table 7.1. ARL and SRL comparison under **model 1** for the following charts of the same on-target ARL approximately and $p = 2, \delta = d/\sqrt{p}$

d	0.5	1.0	1.5	2.0	2.5	3.0
PPCUSUM with $h = 5, k = 0.5$						
ARL-19	26.7	9.26	5.38	3.83	2.98	2.51
SRL	21.9	4.85	2.11	1.24	0.83	0.64
FPCUSUM with $h = 5.3, \lambda = 0.5, k = 0.5$						
ARL-19	28.7	9.43	5.29	3.48	2.50	1.88
SRL	23.5	5.35	2.48	1.55	1.16	0.87
MC1 with $h = 4.33, k = 0.5$						
ARL-19	27.8	11.8	7.79	5.90	4.77	4.02
SRL	23.2	5.43	2.47	1.50	1.06	0.80
MEWMA1 with $h = \sqrt{7.88}, r = 0.1$						
ARL-19	28.8	12.3	7.93	5.86	4.72	3.99
SRL	17.1	4.52	2.23	1.36	0.98	0.75
χ^2-MEWMA1 with $h = 2.823, h_1 = 4, r = 0.1$						
ARL-19	29.1	12.3	7.16	5.58	4.18	3.14
SRL	17.6	4.67	2.43	1.75	1.57	1.44
MEWMA2 with $h = 2.763, r = 0.1$						
ARL-19	27.9	12.2	7.85	5.83	4.68	3.95
SRL	16.6	4.42	2.17	1.34	0.95	0.75
χ^2-MEWMA2 with $h = 2.78, h_1 = 4, r = 0.1$						
ARL-19	28.1	12.1	7.68	5.52	4.15	3.13
SRL	16.9	4.60	2.41	1.73	1.54	1.42

Table 7.2. ARL and SRL comparison under **model 1** for the following charts of the same on-target ARL approximately and $p = 5, \delta = d/\sqrt{p}$

d	0.5	1.0	1.5	2.0	2.5	3.0
PPCUSUM with $h = 8, k = 0.5$						
ARL-19	36.2	12.5	7.45	5.36	4.21	3.50
SRL	26.9	5.59	2.40	1.44	0.98	0.73
FPCUSUM with $h = 8, \lambda = 0.6, k = 0.5$						
ARL-19	36.8	12.5	7.40	5.19	4.00	3.16
SRL	27.7	5.62	2.49	1.55	1.14	0.95
MC1 with $h = 6.55, k = 0.5$						
ARL-19	43.2	17.4	10.9	7.97	6.29	5.23
SRL	31.7	6.86	3.10	1.81	1.25	0.92
MEWMA1 with $h = \sqrt{14.3}, r = 0.1$						
ARL-19	40.8	16.1	10.0	7.39	5.90	4.94
SRL	26.2	5.79	2.67	1.59	1.11	0.85
χ^2-MEWMA1 with $h = 3.8, h_1 = 4.83, r = 0.1$						
ARL-19	41.2	16.1	9.95	7.21	5.54	4.26
SRL	26.7	5.97	3.84	1.96	1.67	1.61
MEWMA2 with $h = 3.75, r = 0.1$						
ARL-19	39.5	15.9	9.95	7.34	5.87	4.93
SRL	25.1	5.69	2.64	1.58	1.11	0.84
χ^2-MEWMA2 with $h = 3.77, h_1 = 4.8, r = 0.1$						
ARL-19	40.0	15.9	9.85	7.14	5.48	4.22
SRL	25.7	5.85	2.83	1.95	1.71	1.63

Table 7.3. ARL and SRL comparison under the alternating normal model for the following charts of the same on-target ARL approximately and $p = 2, \delta = d/\sqrt{p}$

d	0.5	1.0	1.5	2.0	2.5	3.0
PPCUSUM with $h = 5, k = 0.5$						
ARL	120	97.8	67.7	43.0	22.8	10.7
SRL	113	94.9	64.5	44.8	28.0	10.1
FPCUSUM with $h = 5.3, \lambda = 0.5, k = 0.5$						
ARL	105	58.1	25.1	10.1	4.48	2.31
SRL	100	57.3	25.0	11.1	4.75	2.14
MC1 with $h = 4.33, k = 0.5$						
ARL	124	108	84.9	62.0	43.4	28.5
SRL	119	106	83.1	59.4	41.8	28.0
MEWMA1 with $h = \sqrt{7.88}, r = 0.1$						
ARL	125	104	79.3	55.0	34.1	18.9
SRL	130	113	91.5	74.7	56.8	40.9
χ^2-MEWMA1 with $h = 2.823, h_1 = 4, r = 0.1$						
ARL	119	85.4	44.5	18.0	6.71	2.85
SRL	123	91.2	50.4	22.7	9.01	3.68
MEWMA2 with $h = 2.763, r = 0.1$						
ARL	126	111	95.9	80.0	63.9	55.4
SRL	116	104	88.9	72.3	58.3	51.9
χ^2-MEWMA2 with $h = 2.78, h_1 = 4, r = 0.1$						
ARL	121	91.5	52.3	24.7	10.7	5.00
SRL	113	84.6	49.1	22.9	10.0	4.44

Table 7.4. ARL and SRL comparison under the alternating normal model for the following charts of the same on-target ARL approximately and $p = 5, \delta = d/\sqrt{p}$

d	0.5	1.0	1.5	2.0	2.5	3.0
PPCUSUM with $h = 8, k = 0.5$						
ARL	158	145	126	105	85.9	63.2
SRL	149	132	116	97.2	82.8	65.9
FPCUSUM with $h = 8, \lambda = 0.6, k = 0.5$						
ARL	158	140	124	91.1	57.6	28.0
SRL	151	131	116	86.9	58.9	31.7
MC1 with $h = 6.55, k = 0.5$						
ARL	161	152	140	122	106	91.4
SRL	160	146	138	119	104	88.4
MEWMA1 with $h = \sqrt{14.3}, r = 0.1$						
ARL	156	144	126	102	77.4	51.9
SRL	161	153	138	119	102	84.6
χ^2-MEWMA1 with $h = 3.8, h_1 = 4.83, r = 0.1$						
ARL	153	126	83.8	41.8	17.2	6.88
SRL	158	130	87.8	47.5	21.0	8.75
MEWMA2 with $h = 3.75, r = 0.1$						
ARL	159	149	137	122	108	32.7
SRL	151	142	128	114	101	85.2
χ^2-MEWMA2 with $h = 3.77, h_1 = 4.8, r = 0.1$						
ARL	154	131	89.0	47.5	21.7	9.77
SRL	146	121	81.4	44.4	20.1	9.08

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