The Cramér-Lundberg approximation : a new approach

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Abstract

The well-known Cramér-Lundberg approximation says that for large u, the ultimate ruin probability $\psi(u)$ satisfies

$$\psi(u) \simeq C\, e^{-R\,u},$$

where u is the initial reserve, R is the adjustment coefficient and C is a positive constant.
Our aim in this work is to present a new expression for C in the classical perturbed risk process and to extend this expression in two cases : 1) possibly negative claims and 2) an infinite number of claims on finite time intervals.

Résumé

La célèbre formule de Cramér-Lundberg fournit l'approximation suivante pour la probabilité de ruine sur un horizon infini $\psi(u)$:

$$\psi(u) \simeq C\, e^{-R\,u},$$

où u désigne la réserve initiale, R coefficient d'ajustement et C est une constante positive.
Nous présentons dans cet article une nouvelle expression du paramètre C pour le modèle du risque classique (perturbé) et nous essayons d'étendre cette expression dans deux cas : 1) possibilité de sinistres négatifs et 2) un nombre infini de sinistres sur un intervalle de temps fini.

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1 Introduction

The well-known Cramér-Lundberg approximation says that for large u, the ultimate ruin probability $\psi(u)$ satisfies

$$\psi(u) \simeq C\, e^{-R\,u},$$

where u is the initial reserve, R is the adjustment coefficient and C is a positive constant.

(Dufresne , 1989) studied the classical perturbed risk process and gives an expression for the constant C. Our aim in this work is to present a new expression for C via a change of measure. We show that C is in fact equal (up to the sign) to the ratio between the drifts of the risk process under two measures : the initial measure and the martingale measure.

Unfortunately, this formula is no more valid in case of possible negative claims. We illustrate this point by a counterexample and show how this formula can be modified to take account for negative claims.

We also prove that the expression is still valid when the claims are positive but possibly in an infinite number on finite time intervals (that is the case when the risk process is a Lévy process).

2 The classical perturbed risk model revisited

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a complete filtered probability space satisfying usual conditions (see for instance (Protter , 1990)). All processes considered in the sequel are defined on this space and adapted to the filtration.

Our basic process is the reserve process $(X_t, t \geq 0)$:

$$X_t = u + c\,t + \sigma W_t - S_t. \tag{2.1}$$

In equation 2.1, u, σ and c are positive constants, $(W_t, t \geq 0)$ is a standard Brownian motion and $(S_t, t \geq 0)$ is a compound Poisson process (independent of W) representing the accumulated claims :

$$S_t = \sum_{i=1}^{N_t} Y_i,$$

where N_t is a Poisson process of parameter λ and $(Y_1, Y_2, ...)$ denote the claim values which are independent identically distributed positive random variables with distribution function G.

For a not necessarily positive random variable Z we will denote its moment generating function by $\psi_Z(.)$ and its characteristic function by $\varphi_Z(.)$,provided the first one exists:

$$\psi_Z(w) \doteq E[e^{w\,Z}]; \varphi_Z(w) \doteq E[e^{i\,w\,Z}]. \tag{2.2}$$

It is well-known that (2.1) is equivalent to :

$$\varphi_{X_t}(w) = e^{iuw + iwct - \frac{1}{2}\sigma^2 w^2 t - \lambda t + \lambda t \varphi_Y(-w)} \; \forall t \geq 0. \tag{2.3}$$

We also have :

$$\psi_{X_t}(w) = e^{uw + wct + \frac{1}{2}\sigma^2 w^2 t - \lambda t + \lambda t \psi_Y(-w)}. \tag{2.4}$$

Let τ denote the time of ruin, that is :

$$\tau = inf(t \geq 0 : X_t < 0).$$

The probability of ultimate ruin is, as usual, denoted by

$$\psi(u) = P[\tau < \infty].$$

We make the following assumptions :

Assumption 2.1 (Net profit condition)

$$c > \lambda \, E[Y_1].$$

Assumption 2.2 *We also suppose the existence of the adjustment coefficient(see for instance (*GERBER *, 1979) and (*DUFRESNE *, 1989)), which is denoted by* $R > 0$ *and satisfies :*

$$\lambda + Rc - \frac{\sigma^2}{2} R^2 = \lambda \int_0^\infty e^{Ry} dG(y). \tag{2.5}$$

The Cramér-Lundberg result says that :

$$\psi(u) \simeq C \, e^{-Ru}, \tag{2.6}$$

where C is a positive constant given by (see for example (DUFRESNE , 1989, p137)):

$$C = \frac{1}{R} \frac{c - \lambda \, E[Y_1]}{\frac{\sigma^2}{2} + \lambda \int_0^\infty y \, e^{Ry}(1 - G(y))dy}.$$

The aim of this paper is to find another expression for the constant C in (2.6) valid for (more) general (Lévy) processes and linked to martingale methods. For this purpose, the following lemma is useful.

Let $L_t \doteq e^{-R(X_t - u)}$; $t \geq 0$.

Lemma 2.1 *Under assumptions (2.1) and (2.2):* $(L_t)_{t \geq 0}$ *is a positive martingale with expectation* 1 *.*

Proof
The proof follows trivially from (2.4). Effectively, we get for $s < t$:

$$\begin{aligned} E\left[L_t|\mathcal{F}_s\right] &= E\left[e^{-R(X_t-u)}|\mathcal{F}_s\right], \\ &= e^{-R(X_s-u)}\, E\left[e^{-R(X_t-X_s)}|\mathcal{F}_s\right], \\ &= e^{-R(X_s-u)}\, E\left[e^{-R(X_t-X_s)}\right], \\ &= e^{-R(X_s-u)}\, e^{(t-s)(-Rc+\frac{\sigma^2}{2}R^2-\lambda+\lambda\psi_Y(R))}, \\ &= e^{-R(X_s-u)}, \end{aligned}$$

the last inequality following from (2.5).

■

We can therefore define a new probability measure Q progressively equivalent to P in the following way (see for instance (Schmidli , 1995)):

$$\left(\frac{dQ}{dP}\right)_{\mathcal{F}_t} = L_t.$$

The idea is now to study the process $(X_t)_{t\geq 0}$ under this new probability measure. The following result is from (Schmidli , 1995) and we give a different proof based on the Girsanov theorem and the exponential formula for Lévy processes.

Proposition 2.1 *Under the assumptions (2.1) and (2.2), the process* $(X_t)_{t\geq 0}$ *admits under the probability* Q*, the following representation :*

$$X_t = u + \tilde{c}\,t + \sigma\tilde{W}_t - \tilde{S}_t, \tag{2.7}$$

where

i) $\tilde{c} = c - \sigma^2 R$,

ii) $(\tilde{W}_t, t \geq 0)$ *is a standard Brownian motion ,*

iii) $(\tilde{S}_t, t \geq 0)$ *is a compound Poisson process (independent of* $\tilde{W}$*) :*

$$\tilde{S}_t = \sum_{i=1}^{\tilde{N}_t} \tilde{Y}_i,$$

where $\tilde{N}_t$ *is a Poisson process with parameter* $\tilde{\lambda} = \lambda\,\psi_{X_1}(R)$ *and* $(\tilde{Y}_1, \tilde{Y}_2, ...)$ *denotes the claim amounts which are independent identically distributed positive random variables with distribution function* $d\tilde{G}(x) = \psi_{X_1}(R)^{-1}\, e^{Rx}\, dG(x)$.

Proof
We have to show (from (2.3))that

$$\varphi^Q_{X_t-u}(w) \doteq E_Q[e^{i\,w\,(X_t-u)}] = e^{iw\tilde{c}t-\frac{1}{2}\sigma^2w^2t-\tilde{\lambda}\,t+\tilde{\lambda}\,t\,\varphi_{\tilde{Y}}(-w)},$$

where $E_Q[.]$ denotes the expectation under Q.
But we get by using the fact that under Q the processes $(W_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ still remain independent :

$$\begin{aligned}\varphi^Q_{X_t-u}(w) &= E_Q[e^{i\,w\,(X_t-u)}]\\ &= e^{iwct}\,E[e^{-R(X_t-u)}\,e^{iw\sigma W_t}]E[e^{-R(X_t-u)}\,e^{-iwS_t}].\end{aligned}$$

Let us now define another probability $\bar{Q}$ by

$$\frac{d\bar{Q}}{dP} = e^{-R\sigma W_t-\frac{R^2\sigma^2t}{2}}.$$

Then, from Girsanov theorem, we know that under $\bar{Q}$: $\bar{W}_t = W_t + R\sigma t$ is a standard Brownian motion. We get :

$$\begin{aligned}E[e^{-R(X_t-u)}\,e^{iw\sigma W_t}] &= E[\frac{d\bar{Q}}{dP}\,e^{iw\sigma W_t}]\\ &= E_{\bar{Q}}[e^{iw\sigma W_t}]\\ &= E_{\bar{Q}}[e^{iw\sigma W_t}]\\ &= E_{\bar{Q}}[e^{iw\sigma(\bar{W}_t-R\sigma t)}]\\ &= e^{-iw\sigma^2Rt-\frac{1}{2}w^2\sigma^2t}.\end{aligned}$$

We also have :

$$\begin{aligned}E[e^{-R(X_t-u)}\,e^{-iwS_t}] &= E[e^{R\,S_t}\,e^{t\lambda\int_0^\infty(1-e^{Rx})dG(x)}\,e^{-iwS_t}]\\ &= e^{t\lambda\int_0^\infty(1-e^{Rx})dG(x)}\,E[e^{(R-iw)S_t}].\end{aligned}$$

As, from the exponential formula (see for instance (Bertoin , 1996)), we have :

$$E[e^{(R-iw)S_t}] = e^{-\lambda t\int_0^\infty(1-e^{Rx-iwx})dG(x)}$$

We finally get:

$$\begin{aligned}E[e^{-R(X_t-u)}\,e^{-iwS_t}] &= e^{t\lambda\int_0^\infty(1-e^{Rx})dG(x)}\,E[e^{(R-iw)S_t}]\\ &= e^{-\lambda t\int_0^\infty e^{Rx}(1-e^{-iwx})dG(x)}\\ &= e^{-\tilde{\lambda}\,t+\tilde{\lambda}\,t\,\varphi_{\tilde{Y}}(-w)}.\end{aligned}$$

■

We introduce the following natural definition :

Definition 2.1 *Let $X = (X_t)_{t\geq 0}$ be a process with stationary increments. The drift of X under the probability P is denoted by $dr_P(X)$ and is defined by*

$$dr_P(X) = E_P(X_1 - X_0).$$

Proposition 2.2 *Under assumptions (2.1) and (2.2), the coefficient C in (2.6) is given by :*

$$C = -\frac{dr_P(X)}{dr_Q(X)}. \tag{2.8}$$

Proof
We have to show :

$$\begin{aligned} R\frac{\sigma^2}{2} + R\lambda \int_0^\infty x\, e^{Rx}(1 - G(x))dx &= -\left(\tilde{c} - \tilde{\lambda} \int_0^\infty x\, d\tilde{G}(x)\right) \\ &= -\left(c - R\,\sigma^2 - \lambda \int_0^\infty x\, e^{Rx} dG(x)\right). \end{aligned}$$

From an integration by parts, we get :

$$\begin{aligned} R\frac{\sigma^2}{2} + R\lambda \int_0^\infty x\, e^{Rx}(1 - G(x))dx &= R\frac{\sigma^2}{2} + R\lambda \int_0^\infty \left(\frac{x\, e^{Rx}}{R} - \frac{e^{Rx}}{R^2}\right) dG(x) + \frac{\lambda}{R} \\ &= R\frac{\sigma^2}{2} - \lambda \int_0^\infty x\, e^{Rx} dG(x) + \lambda \int_0^\infty \frac{e^{Rx}}{R} dG(x) - \frac{\lambda}{R}. \end{aligned}$$

And since

$$\lambda \int_0^\infty \frac{e^{Rx}}{R} dG(x) - \frac{\lambda}{R} = c - \frac{\sigma^2}{2} R$$

finally we have :

$$R\frac{\sigma^2}{2} + R\lambda \int_0^\infty x\, e^{Rx}(1 - G(x))dx = -c + R\sigma^2 + \lambda \int_0^\infty x e^{Rx} dG(x).$$

∎

3 The case of possibly negative claims

When the claims can be negative the preceding result may proved to be uncorrect as shown in the following example (see for instance (Gerber , 1979)):
Example : negative claims
The risk process X is then described as follows

$$X_t = u - c\,t + S_t. \tag{3.1}$$

In equation (3.1), $(S_t, t \geq 0)$ is a compound Poisson process :

$$S_t = \sum_{i=1}^{N_t} Y_i,$$

where N_t is a Poisson process with parameter λ and $(Y_1, Y_2, ...)$ denote independent identically distributed positive random variables with distribution function G.
We suppose that $c < \lambda\, E[Y_1]$. Let R be the strictly positive solution of

$$\lambda - Rc = \lambda \int_0^\infty e^{-Ry} dG(y). \tag{3.2}$$

It is easy to see that $(e^{-RX_t}, t \geq 0)$ is a martingale and it is also well-known that $\psi(u) = e^{-Ru}$.
Let us now denote by $L_t \doteq e^{-R(X_t - u)}$; $t \geq 0$. Define the probability Q progressively equivalent to P by

$$\left(\frac{dQ}{dP}\right)_{\mathcal{F}_t} \doteq L_t.$$

Under the probability Q, the process $(S_t, t \geq 0)$ is a compound Poisson process with parameters $\tilde{\lambda} = \lambda \int_0^\infty e^{-Rx}\, dG(x)$ and claim distribution $d\tilde{G}(x) = \frac{e^{-Rx}\, dG(x)}{\int_0^\infty e^{-Rx}\, dG(x)}$.
Then :

$$dr_P(X) = \lambda \int_0^\infty x\, dG(x) - c > 0;$$
$$dr_Q(X) = \lambda \int_0^\infty x e^{-Rx}\, dG(x) - c < 0.$$

Nevertheless, we usually do not have that

$$-\frac{dr_P(X)}{dr_Q(X)} = 1.$$

We show this by the following example : suppose that the claims are exponential random variables with parameter 1, that is $dG(x) = e^{-x}\, dx$. The "net profit condition" implies that $\lambda > c$. Then, we easily find that $R = \frac{\lambda - c}{c}$, $dr_P(X) = \lambda - c$ and $dr_Q(X) = \frac{c^2}{\lambda} - c < 0$. Taking for example $\lambda = 2c$ gives us the counterexample.

■

We come back to the traditionnal risk model :

$$X_t = u + c\,t - S_t. \tag{3.3}$$

but with the possibility of negative claims, that is the distribution function G is defined on the whole real line. We need the following assumptions :

Assumption 3.1 (Net profit condition)

$$c > \lambda\, E[Y_1].$$

Assumption 3.2 *We also will suppose the existence of the adjustment coefficient $R > 0$ defined as the solution*[1] *of the following equation :*

$$\lambda + Rc = \lambda \int_{-\infty}^{\infty} e^{Ry} dG(y).$$

Let $L_t \doteq e^{-R(X_t - u)}$; $t \geq 0$.

Lemma 3.1 *Under assumptions (3.1) and (3.2),$(L_t)_{t\geq 0}$ is a positive martingale with expectation* 1 *.*

Proof
The proof follows trivially from (2.4).

∎

We can again define a probability measure Q progressively equivalent to P :

$$\left(\frac{dQ}{dP}\right)_{\mathcal{F}_t} = L_t.$$

If $\tau(u)$ denotes the ruin time for an initial reserve equal to u, $E \in \mathcal{F}_{\tau(u)}, E \subseteq \tau(u) < \infty$, we get :

$$P(E) = E_Q[L_{\tau(u)}; E]. \tag{3.4}$$

Now it results from 2.8. that $dr_Q(X) < 0$ and the strong law of large number then implies that $Q[\tau(u) < \infty] = 1$. Therefore

$$\begin{aligned} P[\tau(u) < \infty] &= E_Q[L_{\tau(u)}; \tau(u) < \infty] \\ &= E_Q[L_{\tau(u)}] \\ &= E_Q[e^{-R(X_{\tau(u)} - u)}] \\ &= E_Q[e^{-RB(u)}], \end{aligned}$$

where $B(u) \doteq X_{\tau(u)} - u$ denotes the amount of the ruin. Therefore, if the limit (in $Q-$distribution) of $B(u)$ exists when u is going to ∞ and is denoted by $B(\infty)$ we get :

$$C = E_Q[e^{-RB(\infty)}].$$

We therefore generalize the Cramér-Lundberg result in that case. Our goal is to find an expression for the constant C. Let us first expose some preliminary results :

[1] A convexity argument based on the previous assumption then proves the unicity of this strictly positive solution.

Proposition 3.1 *Under assumptions (2.1) and (2.2), the process $(X_t)_{t\geq 0})$ admits under the probability Q the following representation :*

$$X_t = u + c\,t - \tilde{S}_t, \tag{3.5}$$

where $(\tilde{S}_t, t \geq 0)$ is a compound Poisson process :

$$S_t = \sum_{i=1}^{\tilde{N}_t} \tilde{Y}_i,$$

where $\tilde{N}_t$ is a Poisson process with parameter $\tilde{\lambda} = \lambda\,\psi_{\tilde{Y}}(R)$ and $(\tilde{Y}_1, \tilde{Y}_2, ...)$ denotes the claim amounts which are independent identically distributed random variables with distrbution function $d\tilde{G}(x) = \psi_{\tilde{Y}}(R)^{-1}\, e^{Rx}\, dG(x)$.

Proof
This is an easy adaptation of proposition 2.7.

∎

The following proposition is an adaptation of (ASMUSSEN , 1987, Theorem XII.5.3. p 267)

Proposition 3.2 *Let $(U_t)_{t\geq 0}$ be a discrete time non-lattice random walk with $EU_1 < 0$ and $U_0 = 0$. Let $\tau_- = inf_{t\geq 1}\{t : U_t \leq 0\}$,$\tau_+ = inf_{t\geq 1}\{t : U_t > 0\}$ and $M = sup_{t\geq 0}\{U_t\}$ .Then*

$$\psi(u) = P[M > u] \approx C\, e^{-Ru},$$

where C is given by

$$C = \frac{d_P(U)}{d_Q(U)}\;\; \frac{1 - E\left[e^{R\,U_{\tau_-}}\right]}{R\,E\left[U_{\tau_-}\right]},$$

where

i) R is such that $\left(e^{-RU_t}\right)_{t\geq 0}$ is a discrete-time martingale (existence of such an R is assumed);

ii) Q is a probability measure defined by

$$\left(\frac{dQ}{dP}\right)_{\mathcal{F}_t} = e^{-RU_t};$$

iii) $d_P(U) = E_P(U_1); d_Q(U) = E_Q(U_1)$.

Proof
It is easy to see that the limit $B(\infty)$ exists in that case (see for instance (ASMUSSEN , 1987)). Now, by the renewal theorem, the density of $B(\infty)$ on R_+ under Q is $\frac{1-\tilde{G}_+(x)}{\int x d\tilde{G}_+(x)}$. But it comes from (3.4) that if $E = \{U_{\tau_+} \in A, \tau_+ < \infty\}$ then

$$\begin{aligned} G_+(A) = P[E] &= E_Q[e^{-RU_{\tau_+}}; U_{\tau_+} \in A] \\ &= \int_A e^{-Rx} d\tilde{G}_+(x). \end{aligned}$$

This shows that

$$dG_+(x) = e^{-Rx}\, d\tilde{G}_+(x).$$

Similarly we have :

$$dG_-(x) = e^{-Rx}\, d\tilde{G}_-(x).$$

Hence

$$\begin{aligned} C &= E_Q\left[e^{-RB(\infty)}\right] \\ &= \int_0^\infty e^{-Rx} \frac{1-\tilde{G}_+(x)}{\int x d\tilde{G}_+(x)} dx \\ &= \frac{1}{R\int x d\tilde{G}_+(x)} \int_0^\infty (1-e^{-Rx}) d\tilde{G}_+(x) \\ &= \frac{1}{R\int x d\tilde{G}_+(x)} (1 - \int_0^\infty dG_+(x)). \end{aligned}$$

Now it is enough to notice that (see (ASMUSSEN , 1987)) :

$$E_P(\tau_\pm) = \frac{1}{1-\int_0^\infty dG_\mp(x)}$$

and

$$E_Q(\tau_\pm) = \frac{1}{1-\int_{-\infty}^0 d\tilde{G}_{\substack{- \\ +}}(x)}.$$

It follows from Wald's theorem that :

$$E_Q[U_{\tau_-}] = E_Q[U_1]\, E_Q[\tau_-]$$

and

$$E_Q[U_{\tau_+}] = E_Q[U_1]\, E_Q[\tau_+].$$

Therefore :

$$C = \frac{1}{R\int x d\tilde{G}_+(x)} (1 - \int_0^\infty dG_+(x))$$

$$
\begin{aligned}
&= \frac{1}{R\,E_Q(U_{\tau_+})}\,\frac{1}{E_P(\tau_-)} \\
&= \frac{1}{R\,E_Q(U_1)\,E_Q(\tau_+)}\,\frac{E_P(U_1)}{E_P(U_{\tau_-})} \\
&= \frac{E_P(U_1)}{E_Q(U_1)}\,\frac{1-\int_{-\infty}^{0} d\tilde{G}_-(x)}{R\,E_P(U_{\tau_-})} \\
&= \frac{E_P(U_1)}{E_Q(U_1)}\,\frac{1-\int_{-\infty}^{0} e^{Rx}\,dG_-(x)}{R\,E_P(U_{\tau_-})} \\
&= \frac{E_P(U_1)}{E_Q(U_1)}\,\frac{1-E_P\,e^{RU_{\tau_-}}}{R\,E_P(U_{\tau_-})}.
\end{aligned}
$$

■

An expression for C
Let us define $\forall n \geq 1$ the process U^n as follows :

$$Z_i^n \doteq X_{\frac{i+1}{n}} - X_{\frac{i}{n}},\ \ i \in I\!N,$$

$$U_t^n = \sum_{j<i} Z_j^n$$

where $i = \inf\left\{j \in I\!N\ :\ \frac{j}{n} \geq t\right\}$. Let us now denote by $\psi_n(u)$ the ruin probability for the process U^n :

$$\psi_n(u) = P[\exists t > 0 : U_t^n < 0].$$

Then by the preceding proposition,

$$\psi_n(u) \approx C_n\, e^{-Ru},$$

where

i) R is such that $\left(e^{-RU_t^n}\right)_{t\geq 0}$ is a discrete-time martingale (let us note that existence of such an R occurs under assumptions (3.1) and (3.2));

ii) Q_n is a probability measure defined by

$$\left(\frac{dQ_n}{dP}\right)_{\mathcal{F}_t} = e^{-RU_t^n};$$

iii) C_n is given by

$$C_n = \frac{d_P(U)}{d_Q(U)}\ \ \frac{1-E\left[e^{RU^n_{\tau_-^n}}\right]}{R\,E\left[U^n_{\tau_-^n}\right]},$$

where $\tau_-^n = inf_{t\geq 1}\left\{t : U_t^n \leq 0\right\}$

Remark: we obviously have

$$\frac{dr_P(U_n)}{dr_Q(U_n)} = \frac{dr_P(X)}{dr_Q(X)}.$$

■

The following result is now clear:

Theorem 3.1 *Under assumptions (3.1) and (3.2): if for n is going to* ∞

$$\frac{1 - E\left[e^{RU^n_{\underline{\tau^n}}}\right]}{R\,E\left[U^n_{\underline{\tau^n}}\right]} \to A,$$

then

$$C = \frac{dr_P(X)}{dr_Q(X)}\,A.$$

■

Corollary 3.1 *If the claims are positive, then* $A = -1$.

Proof

Let us denote $U^n_{\underline{\tau^n}}$ by Y_n . We obviously have:

$$-\frac{c}{n} \leq Y_n \leq 0.$$

It results then from classical results on the moment generating function (see for instance (Rohatgi , 1984)[p.512] that $E\left[e^{RY_n}\right]$ can be expanded in a Taylor series about zero, which gives :

$$\frac{E\left[e^{RY_n}\right] - 1}{R\,E\left[Y_n\right]} = \frac{R\,E\left[Y_n\right] + \frac{1}{2}\,R^2\,E\left[Y_n\right]^2 + \ldots}{R\,E\left[Y_n\right]}.$$

The result follows now easily from the following fact

$$E|Y_n^{k+1}| \leq \frac{c^k}{n^k}\,E(|Y_n|).$$

■

4 The Cramér-Lundberg approximation in the Lévy case

We suppose in this section that the risk process $(X_t, t \geq 0)$ is a Lévy process i.e. an adapted càdlàg continuous in probability process with stationary independent increments. It is then well-known that X admits the following representation (see for instance (Jacod and Shiryaev , 1987)):

$$X_t = u + c\,t + \sigma\, W_t + \int_0^t \int \phi(x)(\mu - \nu)(ds, dx) + \int_0^t \int (x - \phi(x))\mu(ds, dx), \quad (4.1)$$

where

i) u and c are positive constants,

ii) $(W_t, t \geq 0)$ is a standard Brownian motion,

iii) the truncation function $\phi(x) = x\, 1_{|x| \leq 1}$,

iv) μ is the jump measure of X :

$$\mu(\omega, dt, dx) = \sum_{s \geq 0} 1_{\Delta X_s(\omega) \neq 0}\; \epsilon_{(s, \Delta X_s(\omega))}(dt, dx),$$

v) $\nu(ds, dx) = ds\; F(dx)$ where $F(.)$ is the Lévy measure of X (ν is the compensator of the measure μ).

It is then possible to show that :

$$\varphi_{X_t}(w) = e^{iuw + iwct - \frac{1}{2}\sigma^2 w^2 t - t\,\int_{\Re}\left(1 - e^{iwx} + iwx 1_{|x|<1}\right)F(dx)}\;\forall t \geq 0 \quad (4.2)$$

and, if $\int_{\Re} \|1 - e^{wx}\| < \infty$ (see the exponential formula of (Bertoin , 1996))

$$\psi_{X_t}(w) = e^{uw + wct + \frac{1}{2}\sigma^2 w^2 t - t\,\int_{\Re}(1 - e^{wx})F(dx)} \quad (4.3)$$

We will make the following assumptions :

Assumption 4.1 (Net profit condition)

$$c > - \int_{\Re} x F(dx).$$

Assumption 4.2 *There exists an $R > 0$ such that*

$$R\,c - R^2 \frac{\sigma^2}{2} + \int (1 - e^{-Rx})\, F(dx) = 0.$$

When such an R exists it is called the *adjustment coefficient*. We now introduce some notations and definitions (see (BERTOIN , 1996)):
Let X be a Lévy process and S be the supremum process, that is

$$S_t = sup_{0\leq s\leq t}\{0 \vee X_s\}.$$

Moreover, we define

i) $L = (L(t), t \geq 0)$ is the local time of $S - X$ at 0 and $L^{-1}(.) = inf\{s \geq 0 : L(s) > .\}$ its right-continuous inverse, *i.e.* the ascending ladder time process;

ii) $H = (H(t), t \geq 0)$ where
$H(t) \doteq S_{L^{-1}(t)}$ if $L^{-1}(t) < \infty$, $H(t) = \infty$ otherwise.
The pair (L^{-1}, H) is known as the *ladder process*;

iii) the *renewal function*

$$\mathcal{V}_P(x) = \int_0^\infty P(H(t) \leq x)dt = E\int_0^\infty 1_{S_t\leq x} dL(t), x \in [0, \infty).$$

We then have the following generalization of (2.6) :

Theorem 4.1 *Under assumptions (4.1) and (4.2): there exists a constant C such that :*

$$\psi(u) \simeq C\, e^{-Ru}. \tag{4.4}$$

Proof

i) Since the process X drifts to $-\infty$, it is easy to see from (BERTOIN , 1996, proposition VI.17) that there exists a constant k_1 such that

$$\psi(u) = 1 - \frac{1}{k_1}\,\mathcal{V}_P(u).$$

ii) Define the probability Q by

$$\left(\frac{dQ}{dP}\right)_{\mathcal{F}_t} = e^{-R(X_t-u)}.$$

It exists then a constant k_2 such that :

$$\mathcal{V}_Q(dx) = k_2\, e^{Rx}\mathcal{V}_P(dx).$$

iii) Then, we get:

$$\begin{aligned}\psi(u) &= 1 - \frac{1}{k_1}\,\mathcal{V}_P(u) \\ &= 1 - \frac{1}{k_1 k_2}\int_0^u e^{-Ry}\mathcal{V}_Q(dy).\end{aligned}$$

iv) But moreover

$$\frac{1}{k_1 k_2} \int_0^{\infty} e^{-Ry} \mathcal{V}_Q(dy) = 1,$$

so that we get with the use of renewal theorem for Lévy processes (see for example (Bertoin , 1996, p. 38)) the following result:

$$\begin{aligned} \psi(u) &= \frac{1}{k_1 k_2} \int_u^{\infty} e^{-Ry} \mathcal{V}_Q(dy) \\ &\approx \frac{1}{k_1 k_2\, E_Q(H_1)} \int_u^{\infty} e^{-Ry} dy \\ &= \frac{e^{-Ru}}{R\, k_1 k_2\, E_Q(H_1)}. \end{aligned}$$

■

This result proves that the value of the constant C is given by:

$$C = \frac{1}{R\, k_1 k_2\, E_Q(H_1)}.$$

Our aim is now to prove that the formula (2.8) is still valid in the case of **negative jumps**. A particular case is for example the gamma process studied by (Dufresne et al., 1994). We now follow the steps of the preceding sections.
Let us denote $L_t = e^{-R(X_t - u)}\,; t \geq 0$.
A simple application of the exponential formula leads to the following lemma :

Lemma 4.1 *Under assumptions (4.1) and (4.2), $(L_t)_{t \geq 0}$ is a positive martingale with expectation* 1 .

■

We now consider the probability measure Q progressively equivalent to P defined in the following way :

$$\left(\frac{dQ}{dP}\right)_{\mathcal{F}_t} = L_t.$$

Proposition 4.1 *Under assumptions (4.1) and (4.2), X is, under the probability Q, always a Lévy process and admits the following representation :*

$$X_t = u + \tilde{c}\, t + \sigma\, W_t + \int_0^t \int \phi(x)(\mu - \tilde{\nu})(ds, dx) + \int_0^t \int (x - \phi(x))\mu(ds, dx). \quad (4.5)$$

where

i) $\tilde{c} = c - \sigma^2 R$,

ii) $(W_t, t \geq 0)$ *is a standard Brownian motion,*

iii) μ *is the jump measure of* X :

$$\mu(\omega, dt, dx) = \sum_{s \geq 0} 1_{\Delta X_s(\omega) \neq 0}\ \epsilon_{(s, \Delta X_s(\omega))}(dt, dx),$$

iv) $\tilde{\nu}(ds, dx) = ds\, \tilde{F}(dx)$ *where* $\tilde{F}(dx) \doteq e^{-Rx}\, F(dx)$ *is the Lévy measure of* X *under* Q.

■

Theorem 4.2 *Under assumptions (4.1) and (4.2), and if the jumps of the process are negative (that is* $F(x) - F(0) = 0$, $\forall x > 0$*) then the coefficient* C *in (4.4) is given by :*

$$C = -\frac{dr_P(X)}{dr_Q(X)}. \tag{4.6}$$

Proof
Let us consider the following approximations $(X^\epsilon, \epsilon > 0)$ of the Lévy process X :

$$X_t^\epsilon = u + c\,t + \sigma\, W_t + \int_0^t \int x 1_{x < -\epsilon} \mu(ds, dx). \tag{4.7}$$

The processes $(X^\epsilon, \epsilon > 0)$ are all classical perturbed Poisson processes like in section 2. Those are Lévy processes with Lévy measure $F^\epsilon(x) = F(x) 1_{x < -\epsilon}$. Therefore

$$\psi^\epsilon(u) \simeq -\frac{d_P(X^\epsilon)}{d_Q(X^\epsilon)} e^{-R^\epsilon u}.$$

But of course we get the following convergences when $\epsilon \downarrow 0$:

i) $\psi^\epsilon(u) \uparrow \psi(u)$,

ii) $R^\epsilon \to R$,

iii) $C^\epsilon \to -\frac{dr_P(X)}{dr_Q(X)}$.

The proposition 4.1. gives us the following expression for C :

Proposition 4.2 *Under assumptions (4.1) and (4.2):*

$$\begin{aligned} C &= -\frac{dr_P(X)}{dr_Q(X)} \\ &= -\frac{c + \int_{-\infty}^0 x F(dx)}{c + \sigma^2\, R - \int_{-\infty}^0 x e^{-Rx} F(dx)} \end{aligned}$$

5 Example

We give here an example extending the *Gamma processes* studied by (Dufresne et al., 1994) and which can be considered as *Perturbed Gamma processes*. Let us then consider the following Lévy measure F :

$$F(dx) = a(-x)^{\alpha-1}\, e^{bx} 1_{x<0} dx,$$

with $-1 < \alpha < \infty$ and $a, b > 0$. For $\alpha > 0$, F is integrable and the risk process is then a classical (perturbed) process.
We have the following expression for C :

$$\begin{aligned}
C &= -\frac{c + \int_{-\infty}^{0} xF(dx)}{c - \sigma^2 R + \int_{-\infty}^{0} xF(dx)} \\
&= \frac{c + \int_{-\infty}^{0} xa(-x)^{\alpha-1}\, e^{bx}\, dx}{c - \sigma^2 R + \int_{-\infty}^{0} xe^{-Rx}\, a(-x)^{\alpha-1}\, e^{bx}\, dx} \\
&= \frac{c - a\, \int_0^{\infty} x^{\alpha}\, e^{-bx}\, dx}{c - \sigma^2 R - a\, \int_0^{\infty} x^{\alpha}\, e^{(R-b)x}\, dx};
\end{aligned}$$

where R is the adjustment coefficient defined as the positive root of :

$$\begin{aligned}
0 &= R\,c - R^2\frac{\sigma^2}{2} + \int_{-\infty}^{0} (1 - e^{-R\,x})\, a(-x)^{\alpha-1}\, e^{bx}\, dx \\
&= R\,c - R^2\frac{\sigma^2}{2} + a\, \int_0^{\infty} (1 - e^{R\,x})\, x^{\alpha-1}\, e^{-bx}\, dx.
\end{aligned}$$

The *net profit condition* becomes in that case :

$$c > \frac{a\,\Gamma(\alpha+1)}{b^{\alpha+1}}.$$

If $\alpha > 0$, we get the classical model (that is, the cumulated claims process follows a compound Poisson process). Therefore we only consider α values less than 0. A condition necessary and sufficient for the existence of R is given by :

$$bc - \frac{b^2\sigma^2}{2} + \frac{a\,\Gamma(\alpha+1)}{\alpha\, b^{\alpha}} < 0.$$

5.1 Example

In table 1, we give examples of R and C values numerically computed (with Matlab) for different values of α and We take a relative safety loading[2] equal to 0.15 . Other parameters are fixed as follows : $b = 0.8; a = 5; \sigma = 1$.

[2]Voir (Grandell , 1991).

Table 1

α	c	R	C
-0.8	27.6021	0.8253	0.0412
-0.6	13.9452	0.4902	0.6665
-0.4	9.7896	0.3005	0.7972
-0.2	8.0027	0.2246	0.8363
0	7.1875	0.1834	0.8530

Table 2 gives **approximate** ruin probabilities computed via the Cramér-Lundberg formula for different values of the initial reserve.

Table 2 : approximate ruin probabilities

u	$\alpha=-0.8$	$\alpha=-0.6$	$\alpha=-0.4$	$\alpha=-0.2$	$\alpha=0$
1	0.0180	0.4082	0.5903	0.6681	0.7101
2	0.0079	0.2500	0.4371	0.5337	0.5911
3	0.0035	0.1532	0.3236	0.4263	0.4920
4	0.0015	0.0938	0.2396	0.3406	0.4096
5	0.0007	0.0575	0.1774	0.2721	0.3410
6	0.0003	0.0352	0.1314	0.2173	0.2838
7	0.0001	0.0216	0.0973	0.1736	0.2363
8	0.0001	0.0132	0.0720	0.1387	0.1967
9	0.0000	0.0081	0.0533	0.1108	0.1637
10	0.0000	0.0050	0.0395	0.0885	0.1363
11	0.0000	0.0030	0.0292	0.0707	0.1134
12	0.0000	0.0019	0.0217	0.0565	0.0944
13	0.0000	0.0011	0.0160	0.0451	0.0786
14	0.0000	0.0007	0.0119	0.0360	0.0654
15	0.0000	0.0004	0.0088	0.0288	0.0545
16	0.0000	0.0003	0.0065	0.0230	0.0453
17	0.0000	0.0002	0.0048	0.0184	0.0377
18	0.0000	0.0001	0.0036	0.0147	0.0314
19	0.0000	0.0001	0.0026	0.0117	0.0262
20	0.0000	0.0000	0.0020	0.0094	0.0218

Conclusion

The following result is proved in this paper :
If X is a Lévy process with negative jumps, then for large u,

$$\psi(u) \simeq C\, e^{-R\,u},$$

with the following (new) expression for C :

$$C = -\frac{dr_P(X)}{dr_Q(X)}$$

where P denotes the initial probability measure and Q is another probability measure progressively equivalent to P.
If we allow for positive jumps (that is: negative claims), the result fails and a counterexample is showed.
It could be interesting to extend this formula to more general processes.

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References

ASMUSSEN , S. (1987). *Applied probability and queues.* Wiley , New-York.

BERTOIN , J. (1996). *Lévy Processes.* Cambridge University Press.

DUFRESNE , F. (1989). Probabilité et sévérité de la ruine. Thèse présentée à l'Ecole des Hautes Etudes Commerciales.

DUFRESNE , F., GERBER , H. U., and SHIU , E. (1994). Risk theory with the gamma process. *Astin Bulletin*, 21:177–192.

GERBER , H. (1979). *An Introduction to Mathematical Risk Theory.* Huebner Foundation Monographs, Philadelphia.

GRANDELL , J. (1991). *Aspects of Risk Theory.* Springer-Verlag, New-york.

JACOD , J. and SHIRYAEV , A. (1987). *Limit Theorems for Stochastic Processes.* Springer-Verlag, Heidelberg.

KARATZAS , I. and S.SHREVE (1988). *Brownian Motion and Stochastic Calculus.* Springer, Berlin.

PROTTER , P. (1990). *Stochastic Integration and Differential Equations.* Springer-Verlag, Berlin.

ROHATGI , V. K. (1984). *Statistical Inference.* Wiley, New-York.

SCHMIDLI , H. (1995). Cramér-lundberg approximations for ruin probabilities of risk processes perturbed by diffusion. *Insurance : mathematics and economics*, 16:135–149.