Assessing Elliptical Symmetry via a Computer-assisted Test Procedure

Li-Xing Zhu*

Catholic University of Louvain and Chinese Academy of Sciences

May 23, 1997

Abstract

The paper presents a conditional test procedure which is a computer-assisted test one (CAT) for assessing elliptical symmetry of a multivariate variable. The resulting CAT is strictly distribution free if the underlying null distribution of variable is elliptical symmetry. Furthermore, the CAT is strictly valid if the symmetric center is known and are asymptotic valid if the center is an unknown point. The equivalence, in the large sample sense, between the CAT and its unconditional counterpart is established. The power behavior of the CAT and its unconditional counterpart under local alternative is investigated. Some simulations are conducted to demonstrate how the CAT works.

Key words and phrases: Empirical characteristic function, empirical process, computer-assisted test, elliptical symmetry, validity of test.

1991 MSC. Primary: 62G10, Secondary: 62G20, 62G09.

Short Title: Computer-assisted Test.

*Supported partly by the NNSF of China

1 Introduction

Testing for symmetry of a random variable has received considerable attention. In univariate cases, many statistics have been proposed. For example, Butler (1969), Rothman & Woodroofe (1972), Doksum, Fenstad & Aaberge (1977), Antille, Kersting & Zucchini (1982), Shorack & Wellner (1986, section 22), Aki (1987), Csörgö & Heathcote (1987) and Schuster & Barker (1987). There are two different but related issues in multivariate cases, testing for reflected symmetry and for elliptical symmetry. The one which we will touch upon in this paper is the later. There are many references concerning this issue in literature. For example, among others, Kariya & Eaton (1977), Beran (1979), Blough (1989), Baringhaus (1991), Baringhaus & Henze (1991), Fang, Zhu & Bentler (1993) and Zhu, Fang & Zhang (1995). Ghosh & Ruymgaart (1992) extended the statistic proposed by Feuerverger & Mureika (1972) to multivariate case, which bases on integrated empirical characteristic function.

Since little is known about the sampling and limiting null distributions of the test statistics (see, Ghosh & Ruymgaart, 1992, pp.439), even in the simplest case where both symmetric center and shape matrix are completely known except for some special cases. Hence approximations for them are necessary. Some procedures including Bootstrap procedure have been suggested for practical use of the tests.

As an alternative, we will in the present paper develop a computer-assisted test procedure (CAT) for the elliptical symmetry of a multivariate random variable. We mainly present the following three conclusions: a) The CAT is strict (asymptotic) valid under elliptical symmetry about known (unknown) center and shape matrix; b) The CAT is conditionally distribution-free under the null hypothesis, and c) the CAT is asymptotically equivalent to its unconditional counterpart.

Section 2 will contain the some tests which handle separately the cases of known and unknown center and shape matrix. The CAT procedure will be defined in section 3 and the validity and asymptotical validity of the tests will be presented in the same section. Power study under local alternative will be put in section 4. Section 5 will contain some simulation experiments. Section 6 will present some concluding remarks. All proofs of the theorems presented in this paper are postponed to last section, section 7.

2 Test Statistics

We consider the covariance matrix of random d-variate vector x is positive definite throughout this paper. Hence it will not be mentioned further.

A random d-variate vector x is said to be spherically symmetric if

$$Hx \text{ have the same distribution for every orthogonal } d \text{ by } d \text{ matrix } H. \tag{2.1}$$

A random d-variate vector x is said to be elliptically symmetric with a center $\mu \in R^d$ and a shape matrix Σ (the covariance matrix in our case) if for some $\mu \in R^d$ such that $A(x-\mu)$ is spherically symmetric, where $A = \Sigma^{-1/2}$.

Consider both μ and Σ are known center and shape matrix first. Assume no loss of generality that $\mu = 0$ and $\Sigma = I_d$ an identical matrix. For this case Guosh & Ruymgaart (1992) extended the statistic proposed by Feuerverger & Mureika (1972) to multivariate version and the test statistic is defined as

$$\int\int (\sqrt{n} P_n\{\sin(ta'A(x-\mu))\})^2 w(t)\,dt\,d\,\nu(a), \tag{2.2}$$

where P_n is empirical measure based on the sample points $X_n = (x_1, \cdots, x_n)$ which are iid copies of x, $w(\cdot)$ is a weight function, ν is the uniform distribution on S^d and $P_n(f(x))$ stands for $(1/n)\sum_{j=1}^n f(x_j)$ for each function $f(\cdot)$. A natural extension to unknown center and shape matrix cases is to replace these two by their estimators. In order to describe the construction of the computer-assisted tests defined in next section, we define some empirical processes and associated test statistics. Let

$$V_{n1} = \{V_{n1}(X_n, t, a) = \sqrt{n} P_n\{\sin(ta'A(x-\mu))\} : (t,a) \in I \times S^d\}. \tag{2.3}$$

Based on (2.3) the test statistic in (2.2) can be written as

$$E_{n1} = \int_{S^d}\int_I \{V_{n1}(X_n, t, a)\}^2 dw(t) d\nu(a) \tag{2.4}$$

where I is a working region. For the case where the center is known but the shape matrix is unknown, we then need to estimate Σ by a consistent estimator. Here we use the sample covariance matrix to estimate it, say $\hat{\Sigma}$. and then $\hat{A} = \hat{\Sigma}^{-1/2}$. The estimatedly empirical process is:

$$V_{n2} = \{V_{n2}(X_n, \hat{A}, t, a) = \sqrt{n} P_n\{\sin(ta'\hat{A}(x-\mu))\} : (t,a) \in I \times S^d\}. \tag{2.5}$$

The associated test statistic is

$$E_{n2} = \int_{S^d} \int_I \{V_{n2}(X_n, \hat{A}, t, a)\}^2 dw(t) d\nu(a). \tag{2.6}$$

Moreover, when the center μ is also unknown, define an empirical process by

$$V_{n3} = \{V_{n3}(X_n, \bar{x}, \hat{A}, t, a) = \sqrt{n} P_n\{\sin(ta'\hat{A}(x-\bar{x}))\} : (t,a) \in I \times S^d\}, \tag{2.7}$$

the associated test statistic is defined by

$$E_{n3} = \int_{S^d} \int_I \{V_{n3}(X_n, \bar{x}, \hat{A}, t, a)\}^2 dw(t) d\nu(a). \tag{2.8}$$

The following theorem states the limit behavior of the test statistics defined above.

Theorem 2.1. *Assume that $P\{x = \mu\} = 0$ and $E||x-\mu||^4 < \infty$. Then*

1). the processes V_{n1} and V_{n2} have a common limiting Gaussian process $V_1 = \{V_1(t,a) : t \in I, a \in S^d\}$ with the covariance kernel:

$$E\{\sin(ta'A(x-\mu))\sin(sb'A(x-\mu))\} : for \quad t, s \in I, a, b \in S^d, \tag{2.9}$$

2). Let $k(t,a,x_j) = \sin(ta'A(x-\mu)) - ta'A(x-\mu)E(\cos(ta'A(x-\mu)))$. The process V_{n3} converges in distribution to a centered Gaussian process $V_2 = \{V_2(t,a) : t \in I, a \in S^d\}$ with the covariance kernel: for $t, s \in I, a, b \in S^d$,

$$E\{k(t,a,x)k(s,b,x)\}. \tag{2.10}$$

The convergence of the statistics is a direct consequence of Theorem 2.1.

Corollary 2.2. *The test statistics E_{n1} and E_{n2} associated with V_{n1} converge in distribution to the same quadratic functional of V_1 , $\int_{S^d}\int_I V_1^2 d\,w(t)\,d\,\nu(\mu)$ say and*

E_{n3} associated with V_{n2} converges in distribution to the quadratic functional of V_2, $\int_{S^d}\int_I V_2^2 d\,w(t)\,d\,\nu(\mu)$ say.

As we know, neither the sampling and limiting null distributions are available for determining the critical value of the tests. It is necessary to consider an approximation for them so that the critical values can be determined in practical use of these tests. For this purpose, we propose a computer-assisted procedure in next section.

3 A Computer-Assisted Test Procedure

Consider a computer-assisted procedure for the test statistic (2.4) first. It is known that x is spherically symmetric if and only if

$$x = u||x||, \tag{3.1}$$

where u is uniformly distributed onS^d and independent of $||x||$. Based on this property, we then construct a computer-assisted test. For any integer n, let $U_n = (u_1, \cdots, u_n)$ where u_i are uniformly distributed on S^d and independent of each other. Then accordingly as the empirical process defined in (2.3), we define a computer-assisted empirical processes. For fixed $X_n = (x_1, \cdots, x_n)$, let

$$V_{n1}(U_n) = \{V_{n1}(X_n, U_n, t, a) = \sqrt{n}P_n\{\sin(ta'u||A(x-\mu)||)\} : (t,a) \in I \times S^d\}. \tag{3.2}$$

The CAT statistics can be constructed as

$$E_{n1}(U_n) = \int_{S^d}\int_I \{V_{n1}(X_n, U_n, t, a)\}^2 dw(t) d\nu(a). \tag{3.3}$$

For the case where the center is known but the shape matrix is unknown, the computer-assisted empirical process is:

$$V_{n2}(U_n) = \{V_{n2}(X_n, U_n, \hat{A}, t, a) = \sqrt{n}P_n\{\sin(ta'u||\hat{A}(x-\mu)||)\} : (t,a) \in I \times S^d\} \tag{3.4}$$

the associated CAT statistic is as follows

$$E_{n2}(U_n) = \int_{S^d}\int_I \{V_{n2}(X_n, U_n, \hat{A}, t, a)\}^2 dw(t) d\nu(a). \tag{3.5}$$

As we see, V_{n2} is a natural extension of V_{n1}. However, for the case where the center μ is also unknown, we will define a computer-assisted empirical process which has a different version from the above. Let

$$V_{n3}(U_n) = \{V_{n3}(X_n, U_n, \bar{x}, \hat{A}, t, a) : (t, a) \in I \times S^d\}, \tag{3.6}$$

where

$$\begin{aligned} & V_{n3}(X_n, U_n, \bar{x}, \hat{A}, t, a) \\ = \quad & \sqrt{n} P_n\{\sin(ta'u||\hat{A}(x - \bar{x})||)\} \\ & -\sqrt{n} P_n\{\cos(ta'u||\hat{A}(x - \bar{x})||) \sin(ta' P_n(u||\hat{A}(x - \bar{x})||)). \end{aligned} \tag{3.7}$$

The associated CAT statistic is defined as

$$E_{n3}(U_n) = \int_{S^d} \int_I \{V_{n3}(X_n, U_n, \bar{x}, \hat{A}, t, a)\}^2 dw(t) d\nu(a). \tag{3.8}$$

The following theorem states the exact validity of the tests in (3.3).

Theorem 3.1 *Assume that $x_1, \cdots, x_n$ are iid d-variate variables which are spherically symmetric. Let $U_n^{(1)}, \cdots, U_n^{(m)}$ be the independent copies of U_n. Then for any $0 < \alpha < 1$,*

$$\begin{aligned} P_{n,m}^{(1)}(\alpha) &= P\{E_{n1} > m - [m\alpha] \text{ of } E_{n1}(U_n^{(j)})'s\} \\ &\leq \frac{[m\alpha] + 1}{m + 1} \end{aligned} \tag{3.9}$$

where [z] stands for the largest integer part of z.

Remark 3.1. *Inequality (3.8) is strict only if E_{ni} fails to resolve certain X_n, which can happen because of discreteness of X_n. But if m is reasonably large, and x satisfies some regular condition such as continuity, (3.8) will be close. In fact under some conditions on distribution of x, it can be showed that*

$$\lim_{n \to \infty} \lim_{m \to \infty} P_{n,m}^{(i)}(\alpha) = \alpha. \tag{3.10}$$

for any $0 < \alpha < 1$, which is a consequence of Theorem 3.2 below.

Theorem 3.2. *Assume, in addition to the conditions of Theorem 3.1, that* $P\{x = \mu\} = 0$. *Then the empirical computer-assisted processes* $\{V_{n1}(U_n, a, t) : t \in I, a \in S^d\}$ *given* X_n *in (3.2) converges in distribution to the Gaussian process* $V_1 = \{V_1(a, t) : t \in I, a \in S^d\}$ *in Theorem 2.1 for almost all series* $\{x_1, \cdots, x_n, \cdots\}$, *which is, in turn, the limit of the empirical process* $\{V_n(a, t) : t \in I, a \in S^d\}$ *in (2.4). This leads up to that* $E_{n1}(U_n)$ *given* X_n *in (3.3) and* E_{n1} *in (2.3) have almost surely the same limit,* $E_1 = \int(V_1(a, t))^2 dw(t)d\nu(a)$.

Following this conclusion, the convergence of the associated quantiles is established.

Denote by $\lambda_n^{(1)}(\alpha, X_n)$, $\lambda_n^{(1)}(\alpha)$ and $\lambda^{(1)}(\alpha)$ the $1-\alpha$ quantiles of the distributions of $E_{n1}(U_n)$ given X_n, E_{n1} and E_1 respectively.

Corollary 3.3. *Under the conditions in Theorem 3.2, for almost all series* $\{x_1, \cdots, x_n, \cdots\}$,

$$\lambda_n^{(1)}(\alpha, X_n) \longrightarrow \lambda^{(1)}(\alpha) \qquad \textit{in Probab.} \tag{3.11}$$

$$\lambda_n^{(1)}(\alpha) \longrightarrow \lambda^{(1)}(\alpha) \qquad \textit{in Probab.} \tag{3.12}$$

as $n \to \infty$.

For the symmetry testing with an unknown center μ and an unknown shape matrix Σ, the strict validity of the CAT does no longer hold but asymptotic validity.

Theorem 3.4. *Assume that* $\{x_1, \cdots, x_n\}$ *are iid d-variate variables which are elliptically symmetric with an unknown shape matrix. Let* $U_n^{(1)}, \cdots, U_n^{(m)}$ *be the independent copies of* U_n. *Then for any* $0 < \alpha < 1$,

$$\lim_{n\to\infty} \lim_{m\to\infty} P\{E_{n2} > m - [m\alpha] \text{ of } E_{n2}(U_n^{(j)})'s\} \leq \alpha \tag{3.13}$$

and similarly for E_{n3} *in the case where the variables are elliptically symmetric with unknown center and shape matrix. Furthermore, the assertions of Theorem 3.2 and of Corollary 3.3 continue to hold for processes* V_{ni} $i = 2, 3$ *and the Gaussian processes* V_i *for* $i = 1, 2$ *respectively.*

Remark 3.2. *For making the use of above tests, it needs to choose working region* A *and weight function* $w(\cdot)$. *Heathcote, Rachev and Cheng (1995) have discussed it* *for the setup of testing reflected (diagonal) symmetry. In our simulations in section 5 below, $A = [-2, 2]^d$ and $w(\cdot)$ is a uniform distribution on A.*

4 Power Study

We here investigate the behavior of the tests and the associated CAT for local alternative. For convenience, let $\sin^{(i)}(c)$ be i-th derivative of $\sin(\cdot)$ at the point c.

Suppose that *iid* d- variate variables $x_i = x_{in}$ have the expression $z_i + y_i/n^\alpha, i = 1, \cdots, n$ for some $\alpha > 0$. The center $\mu = \mu_n = Ez + Ey/n^\alpha$ The following theorem reveals the power behavior of the tests in local alternative.

Theorem 4.1. *Assume that the following conditions hold:*

1) *Both distribution of z and of y are continuous. In addition, z is elliptically symmetric with a known center Ez and a shape matrix $\Sigma = A^{-2}$;*

2) *For l, which is the smallest integer, such that*

$$\begin{aligned} \sup_{t\in I, a\in S^d} |B_l(t,a)| &=: \sup_{t\in I, a\in S^d} |E((ta'(y-Ey)^l \sin^{(l)}(ta'(z-Ez)))| \neq 0, \\ E(||y||^{2l}) &< \infty, \qquad and \qquad E(||y||^{2(l-1)}||z||^2) < \infty. \end{aligned} \tag{4.1}$$

Then

$$V_{n1} = \{\sqrt{n} P_n\{\sin(ta'A(z-Ez)) + (1/l!)B_l(t)\} : (t,a) \in I \otimes S^d\} + o_p(1), \tag{4.2}$$

and similarly for V_{n2}. This leads to

$$\begin{aligned} E_{n1} &= \int_I \int_{S^d} \left\{V_{n1}(t,a)\right\}^2 dw(t) d\nu(a) \\ \Longrightarrow &\int_I \int_{S^d} (V_1(t,a) + 1/l! B_l(t,a))^2 dw(t) \nu(a) \\ E_{n2} &= \int_I \int_{S^d} \left\{V_{n2}(t,a)\right\}^2 dw(t) d\nu(a) \\ \Longrightarrow &\int_I \int_{S^d} (V_1(t,a) + 1/l! B_l(t,a))^2 dw(t) \nu(a). \end{aligned} \tag{4.3}$$

where $\{V_1(t) : (t,a) \in I \times S^d\}$ is a Gaussian process defined in Theorem 3.2.

Remark 4.1. *This conclusion means that the tests can detect the local alternative converging to the null as fast as* $n^{1/(2l)}$*-rate. In some cases, this rate can reach a parametric one, that is,* $l = 1$*. For example,* z *has a uniform distribution on* S^d *and* $y = (z_1^2 - 1, \cdots, z_d^2 - 1)$*, we can see easily that, via a little elementary calculation,* $\sup_{(t,a)\in I \bigotimes S^d} |E(ta'y\cos(ta'z))| \neq 0$*. Hence,* l *will be one. On the other hand, when* x *and* y *are independent of each other,* l *is at least* 3*, namely, the tests can detect, at most, the alternative converging to the null with* $n^{1/6}$*-rate. In fact, it is clear that for* $l = 1, 2$

$$\sup_{(t,a)\in I\times S^d} |E((ta'y)^l \sin^{(l)}(ta'z))| = 0.$$

We now study the CAT for symmetry with a known center μ under local alternative. The following theorem indicates that the critical values determined by the computer-assisted procedure, under local alternative, equals approximately the ones under the null. This means that the critical values is not affected, in the sense of large sample, by underlying distribution of the sample with small perturbation for symmetry in certain class of distributions of variable x.

Theorem 4.2. *Under the conditions of Theorem 4.1, the empirical computer-assisted processes* $\{\sqrt{n}P_n\{\sin(ta'u||\hat{A}(z + y/n^{1/(2l)} - Ez - Ey/n^{1/(2l)})||)\} : (t,a) \in I \times S^d\}$ *given* $(z_i, y_i)'s$ *and* $\{\sqrt{n}P_n\{\sin(ta'u||A(z - Ez)||)\} : (t,a) \in I \times S^d\}$ *given* $z_i's$ *have, almost surely, the same limiting Gaussian process* $\{V_1(t,a) : (t,a) \in I \times S^d\}$ *as the empirical process* $\{\sqrt{n}P_n\{\sin(ta'A(z - Ez))\} : (t,a) \in I \times S^d\}$*. Then the quadratic functionals of these processes have, almost surely, a common limiting functional of Gaussian process,* $\int_I \int_{S^d} (V_1(t))^2 dw(t)\, d\nu(a)$*.*

For unknown center case, there exists also, similar to known center case, a shift of non-random function of t, $1/l!B_l(t)$, in limiting process $\{\sqrt{n}P_n\{\sin(t'(z+y/n^{1/(2l)} - \bar{z} - \bar{y}/n^{1/(2l)}))\} : (t,a) \in I \times S^d\}$ under local alternative cases. The following result describes this.

Theorem 4.3. *Under the same conditions as those of Theorem 4.1, Then*

$$\{\sqrt{n}P_n\{\sin(ta'\hat{A}(z + y/n^{1/(2l)} - \bar{z} - \bar{y}/n^{1/(2l)}))\} : (t,a) \in I \times S^d\}$$

$$
\begin{aligned}
&= \{\sqrt{n}P_n(\sin(ta'A(z-Ez))) + \sqrt{n}\sin(ta'P_nA(z-Ez)E(\cos(ta'A(z-Ez))) \\
&\quad +(1/l!)B_l(t,a) : (t,a) \in I \times S^d\} + o_p(1) \\
&= \{V_2(\cdot,t,a) + (1/l!)B_l(t,a) : (t,a) \in I \times S^d\} + o_p(1) \qquad (4.4)
\end{aligned}
$$

and then

$$
\begin{aligned}
&\int_I \int_{S^d} \{\sqrt{n}P_n(sin(ta'\hat{A}(z + y/n^{1/(2l)} - \bar{z} - \bar{y}/n^{1/(2l)})))\}^2 dw(t)\, d\nu(a) \\
&\Longrightarrow \int_I \int_{S^d} (V_2(t,a) + (1/l!)B_l(t,a))^2 dw(t)\, d\nu(a), \qquad (4.5)
\end{aligned}
$$

where $\{V_2(t) : (t,a) \in I \times S^d\}$ *is a Gaussian process defined in (2.10).*

For the empirical computer-assisted process, there is a parallel conclusion to Theorem 4.2 .

Theorem 4.4. *Under the conditions of Theorem 4.1, the empirical computer-assisted processes, given* $x_{in} = (z_i + y_i/n^{1/(2l)})'s$

$$\{ \quad V_{n3}(X_n, U_n, \bar{x}, \hat{A}, t, a) : (t,a) \in I \times S^d\}$$

has almost surely the limiting Gaussian process $\{V_2(t,a) : (t,a) \in I \times S^d\}$ *defined in (2.10). Then the quadratic functional of the process converge in distribution* $\int_I \int_{S^d} (V_2(t))^2 dw(t)\, d\nu(a)$.

5 Simulations

In order to demonstrate how performance of the CAT's is, a small simulation study is conducted. In the simulation results reported in the table below, the sample size $n = 20$. the dimension of random variable, x, $di = 3$. The hypothetical distribution is normal $N(\mu, V)$. For different cases, we consider that 1) μ and V are both known; 2) μ is known, but not for V and 3) μ and V are both unknown. In order to reveal power behavior of the tests in local alternatives, we consider the variable $x = z + b \cdot y$ for $b = 0$, 0.25, 0.5, 0.75, 1.0, and 1.25, where z has a normal distribution $N(\mu, V)$ and $y = ||V^{-1/2}(z-\mu)||^2$ has a chi-squared distribution with degree three of freedom. In simulation, we generate data from $N(0, I_3)$ and, accordingly as different setup, regard the symmetric center (shape matrix) is known or unknown separately. In order to get a critical value when given the data $\{(y_1, z_1), \cdots, (y_n, z_n)\}$, 1000 U_n pseudo-random numbers of size $n = 20$ are generated by Monte Carlo method. The basic experiment is replicated 1000 times for each combination of sample size and the underlying distribution of the variable. The nominal level we are considering is 0.05. The proportion of times that the values of the statistics exceeded the critical values are recorded as the simulated power of the tests. On the other hand, a comparison between the power based on the CAT and the "accurate" power of the test is done. Here the "accurate" power means that regarding the underlying distribution of the variable is completely known and the null distribution being assumed to be the standard normal $N(\mu, V)$, the power of the tests is simulated by Monte Carlo. The replication time of basic experiments is 3000. Why we do so is just to wonder how the CAT works.

Table power of the tests with $n = 20$

	b	0.00	0.25	0.50	0.75	1.00	1.25
simulated	E_{n1}	0.041	0.120	0.281	0.576	0.773	0.871
accurate		0.051	0.090	0.264	0.561	0.757	0.860
simulated	E_{n2}	0.046	0.201	0.332	0.584	0.767	0.860
accurate		0.056	0.115	0.227	0.551	0.711	0.813
simulated	E_{n3}	0.035	0.363	0.402	0.571	0.610	0.643
accurate		0.060	0.167	0.373	0.523	0.551	0.563

Looking at the table, we see that, under the null hypothesis, the simulated levels of E_{n1} and E_{n2} are close to the nominal one. Under the alternative which we consider, E_{n1} and E_{n2} have good performance. The results in the table show that the power of E_{n1} and E_{n2}, especially for E_{n1}, is close to the "accurate" one. This means that E_{n1} (E_{n2}) is acting as a test that is based on known null distribution. Hence E_{n1} (E_{n1}) can be regarded as a distribution free test , not only in theory but also in practice. On the other hand, the performance of E_{n3} seems not to be encouraging although theoretically it is also a conditional distribution free test. The power under alternative is quite higher than the "accurate" one. Hence E_{n3} may be only applicable in large sample cases.

To summary, the CAT has good performance in the case where the symmetric center is known but not in the situation where the symmetric center is unknown when sample size is small.

6 Concluding Remarks

We have developed a computer-assisted test procedure for testing elliptical symmetry of a multivariate variable, which is based on empirical characteristic function. Under some regular conditions on the distribution of the variable, we have investigated the validity of the CAT and the power behavior of the tests and their CAT counterparts under local alternative.

Remark 6.1. As a principle, other test can be defined to do the same testing. When symmetric center is assumed to be known, a CAT is based on its unconditional counterpart without any modification as the one in (3.3) or (3.5). It is worthwhile say, that for unknown center case the CAT will not have the same form as its unconditional counterpart as the one in (3.8). The modification will guarantee an equivalence, in large sample sense, between the test and the associated CAT. On the other hand, suppose we want to have a test whose CAT counterpart is strict valid for the unknown center case, it seems to us that the test would be location-invariant.

Remark 6.2. For the unknown center case, the CAT defined in the present paper does not have a good performance in small-sample case. It will be interested in searching a test with strict validity even in unknown center case. we reserve it in further research.

7 Appendix: Proofs of Theorems

7.1 Proof of Theorems in Section 2

Proof of Theorem 2.1 It is well known that the process V_{n1} converges in distribution to V_1 with the covariance kernel in (2.9). For V_{n2}, applying the triangle identity, we have By the well-known conclusion,

$$\begin{aligned}\sqrt{n}P_n(\sin(ta'\hat{A}(x-\mu))) &= \sqrt{n}P_n(\sin(ta'A(x-\mu)))\cos(ta'(\hat{A}-A)(x-\mu))) \\ &\quad -\sqrt{n}(P_n(\cos(ta'A(x-\mu)))\sin(ta'(\hat{A}-A)(x-\mu))) \\ &=: I_{n1}(t)-I_{n2}(t). \qquad (7.1)\end{aligned}$$

By a well-known conclusion, $\sqrt{n}(\hat{A}A^{-1/2}-I_d)=O_p(1)$, $\max_{1\leq j\leq n}||x_j-\mu||/n^{1/4}\to 0, a.s.$ and $E(A(x-\mu)\cos(ta'A(x-\mu)))=0$ which is implied by the spherical symmetry of $A(x-\mu)$, we then easily derive that, uniformly over t and a,

$$\begin{aligned}I_{n1}(t,a) &= \sqrt{n}P_n(\sin(ta'A(x-\mu)))+O_p(1/\sqrt{n}), \\ I_{n2}(t,a) &= ta'\sqrt{n}(\hat{A}A^{-1/2}-I_d)(P_n(A(x-\mu)\cos(ta'A(x-\mu)))o_p(1) \\ &= o_p(1) \qquad (7.2)\end{aligned}$$

conclusion 1) is proved. For conclusion 2), the argument is similar since we can derive that

$$\begin{aligned}\sqrt{n}P_n(\sin(ta'\hat{A}(x-\bar{x}))) &= \sqrt{n}P_n(\sin(ta'\hat{A}(x-\mu)))\cos(ta'\hat{A}(\bar{x}-\mu))) \\ &\quad -\sqrt{n}(P_n(\cos(ta'\hat{A}(x-\mu)))\sin(ta'\hat{A}(\bar{x}-\mu))) \\ &= \sqrt{n}P_n(\sin(ta'\hat{A}(x-\mu))) \\ &\quad -\sqrt{n}ta'\hat{A}(\bar{x}-\mu))E(\cos(ta'\hat{A}(x-\mu)))+o_p(1) \qquad (7.3)\end{aligned}$$

The proof of Theorem 2.1 is complete.

7.2 Proof of Theorems in Section 3

Proof of Theorem 3.1 Define $||X_n|| = (||x_1||, \cdots, ||x_n||)$. Note that $x = u||x||$. Then given $||X_n||$, E_{n1} and $E_{n1}(U_n^{(j)})$, are $m+1$ *iid* variables, which implies that

$$P\{E_{n1} > m - [m\alpha] \quad \text{of} \quad E_{n1}(U_n^{(j)})'s|||X_n||\} \leq \frac{[m\alpha]+1}{m+1}.$$

The proof is concluded from integrating $||X_n||$.

Proof of Theorem 3.2. In the following, we firstly show that the process $\{V_n(U_n, X_n, t, a) : (t, a) \in I \times S^d\}$ given X_n converges almost surely to the process $\{V_1(t) : (t, a) \in I \times S^d\}$ which is the limit process of $\{V_{n1}(X_n, t, a) : (t, a) \in I \times S^d\}$. The conclusion of Theorem 3.2 will then hold. By the way, the argument used will be applicable for showing Theorem 3.4, Theorem 4.2 and 4.2 below.

For the simplicity of the notation, write x_j for $A(x_j - \mu)$. Define sets $D_1 = \{\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^n ||x_j||^2 = E||x||^2\}$,

$$D_2 = \{\lim_{n\to\infty} \sup_{(t,a),(s,b)} |\frac{1}{n}\sum_{j=1}^n (\sin(ta'x_j)\sin(sb'x_j)) - E((\sin(ta'x) - \sin(sb'x))| = 0\} \tag{7.4}$$

and $D = D_1 \cap D_2$. By the Lipschitz continuity of *sine* function and Glivenko-Cantelli theorem for general class of functions (e.g. Pollard, 1984, Theorem II 24, pp. 25), , it is clear that D is a subset of sample space with probability measure one.

In the following, we always assume without further mentioning that the given $\{x_1, \cdots, x_n, \cdots\} \in D$.

For convergence of the empirical process defined in the theorem, all we need to do is to prove *Fidis convergence* and *Uniform tightness*. The proof of *fidis convergence* is standard, so we only describe an outline. For any integer k, $(t_1, a_1) \cdots (t_k, a_k) \in I \times S^d$. Let

$$V_1^{(k)} = \Big(E((\sin(t_i a_i' x) - \sin(t_l a_l' x)))\Big)_{1\leq i,l\leq k}.$$

it needs to show that

$$V_{n1}^{(k)} = \{V_{n1}(U_n, X_n, t_i, a_i) : i = 1, \cdots, k\} \Longrightarrow N(0, V_1^{(k)}) \tag{7.5}$$

where "$\Longrightarrow$" means the weak convergence. It suffices to show that for any unit k-dimensional vector γ

$$\gamma' V_{n1}^{(k)} \Longrightarrow N(0, \gamma' V_1^{(k)} \gamma) \tag{7.6}$$

Note that the variance of LHS in (7.2), as follows, converges in probability to $\gamma' V_1^{(k)} \gamma$

$$\gamma' \Big(P_n(\sin(t_i a_i' x_j) - \cos(t_l a_l' x_j))) \Big)_{1 \le i,l \le k} \gamma. \tag{7.7}$$

Hence if $\gamma' V_1^{(k)} \gamma = 0$, (7.2) is trivial. Assume $\gamma' V_1^{(k)} \gamma > 0$. Invoking the boundedness of the *sine* function and the Lindeberg condition,

$$\gamma' V_{n2}^{(k)} / \sqrt{\gamma' V_2^{(k)} \gamma} \longrightarrow N(0,1)$$

The *fidis convergence* hold via combining (7.3).

We now turn to prove *Uniform tightness* of the process. All we need to do is to show that for any $\eta > 0$ and $\epsilon > 0$, there exists an $\delta > 0$ for which

$$\limsup_{n \to \infty} P\{\sup_{[\delta]} |V_{n1}(U_n, X_n t, a) - V_{n1}(U_n, X_n, s, b)| > 2\eta | \, ||X_n||\} < \epsilon \tag{7.8}$$

where $[\delta] = \{((t,a),(s,b)) : ||ta - sb|| \le \delta\}$. Since the limiting properties is being investigated as $n \to \infty$, n is always considered to be large enough below which simplifies some arguments of proof.

It is easy to show that if a d-dimensional variable u is uniformly distributed on S^d, then u can be expressed as $e \cdot u^*$ where $e =$ with probability one half, u^* has the same distribution as u and $e = \pm$ and u^* are independent. From this, the LHS of (7.8) can be written as

$$\begin{aligned} & P\{\sup_{[\delta]} \sqrt{n} |P_n(\sin(ta'e \cdot u^* ||x|| - \sin(sb'e \cdot u^* ||x||)| > \eta | X_n\} \\ & = P\{\sup_{[\delta]} \sqrt{n} |P_n^\circ(\sin(ta'u^* ||x|| - \sin(sb'u^* ||x||)| > \eta | X_n\} \end{aligned} \tag{7.9}$$

where P_n° is the signed measure that places mass e_i/n at $u_i ||x_i||$ which is similar to that in Pollard (1984, p.14).

We now consider conditional probability given $U_n^* = (u_1^*, \cdots, u_n^*)$ and $||X_n||$. Combining (7.9) with that $|\sin(ta'u^*||x|| - \sin(sb'u^*||x||| < ||ta - sb||\,||x||$, the Hoeffding inequality implies that

$$P\{\sup_{[\delta]} \sqrt{n}|(P_n^\circ(\sin(ta'u^*||x||) - \sin(sb'u^*||x||) > \eta c||ta - sb|||\,||X_n||, U_n^*\}$$
$$\leq 2\,\exp(-\eta^2/32). \tag{7.10}$$

In order to apply the chaining lemma (e.g. Pollard 1984, p.144), we need to check the covering integral

$$J_2(\delta, ||\cdot||, I \times S^d) = \int_0^\delta \{2\log\{(N_2(u, ||\cdot||, I \times S^d))^2/u\}\}^{1/2} du \tag{7.11}$$

is finite for small $\delta > 0$, where $||\cdot||$ is the Euclidean norm in R^d and the covering number $N_2(u, ||\cdot||, I \times S^d)$ is the smallest m for which there exist m points $t_1, \cdots, t_m$ with $\min_{1\leq i\leq m} ||ta - t_i a_i|| \leq u$ for every $(t, a) \in I \times S^d$. Let

$$< c\delta >= \{(\sin(t\cdot), \sin(s\cdot)) : \sqrt{P_n(\sin(t'e\cdot(x-\mu)) - P_n\sin(s'e\cdot(x-\mu)))^2} \leq \delta\}.$$

It is clear that

$$N_2(u/c, ||\cdot||, I \times S^d) \leq cu^{-d} \tag{7.12}$$

Consequently, for small $\delta > 0$,

$$J_2(\delta, ||\cdot||, I \times S^d) \leq c\int_0^\delta (\log u)^{1/2} du \leq c\delta\log\delta \leq c\delta^{1/2}. \tag{7.13}$$

Applying now the chaining lemma, there exists a countable dense subset $[\delta]^*$ of $[\delta]$ such that

$$P\{\sup_{[\delta]^*} \sqrt{n}|(P_n^\circ(\sin(ta'u^*||x||) - \sin(sb'u^*||x||))| > 26cJ_2(\delta, ||\cdot||, I \times S^d)|\,|X_n|\}$$
$$\leq 2c\delta. \tag{7.14}$$

The countable dense subset $[\delta]^*$ can be replaced by $[\delta]$ itself because $\sqrt{n}P_n^\circ\{\sin(ta'u||x||) - \sin(sb'u||x||)\}$ is a continuous function with respect to ta and sb for each fixed $||X_n||$. Hence, choosing properly small δ (7.8) is proved. The proof of the lemma is completed.

Proof of Theorem 3.4. The convergence of the processes $V_{n2}(U_n)$ and $V_{n3}(U_n)$ can be concluded following the proof of Theorem 3.2 and noticing $\hat{A}-A = O_p(1/\sqrt{n})$ and $\bar{x}-\mu = O_p(1/\sqrt{n})$. Limits of $V_{n2}(U_n)$ and $V_{n3}(U_n)$ are V_1 and V_2 the ones of their unconditional counterparts. The asymptotic validity of E_{n2} and E_{n3} then follows. Completing the proof.

7.3 Proof of Theorems in Section 4

Proof of Theorem 4.1. Note that condition (4.1) is satisfied for any nonsigular matrix Σ_n say which converges to the covariance matrix of the variable z, Σ say. Assume no loss of generality that the center $\mu = 0.$ and the covariance matrix of x is $\Sigma_n = (A_n)^{-2}$. Applying the Taylor expansion to *sine* function, for any $(t, a) \in I \times S^d$,

$$\begin{aligned}
& \sqrt{n} P_n\{\sin(ta' A_n(z + y/n^{1/(2l)}))\} \\
= \; & \sqrt{n} P_n\{\sin(ta' A_n z)\} + \sum_{i=1}^{l-1} (1/i!) n^{-i/(2l)} \sqrt{n} P_n\{(ta' A_n y)^i \sin^{(i)}(ta' z)\} \\
& + (1/l!) n^{-1} \sum_{j=1}^{n} \{(ta' A_n y_j)^l \sin^{(l)}(ta' A_n(z_j + (t' y_j)^*/n^{1/(2l)})) - \sin^l(ta' A_n z_j))\} \\
& + (1/l!) P_n\{(t' y)^l \sin^{(l)}(ta' A_n z)\}, \qquad (7.15)
\end{aligned}$$

where $(ta' A_n y_j)^*$ is a value between 0 and $ta' A_n y_j$. We need to show that the second and third summands in RHS of the equality tend to zero in probability as $n \to \infty$, and the fourth summand converges in probability to $E\{(ta' A_n y)^l) \sin^{(l)}(ta' z)\}$. The convergence of the fourth term is obvious. Noticing that $E\{(ta' A_n y)^i) \sin^{(i)}(ta' z)\} = 0$ for $1 \leq i \leq l-1$, and following the argument used in proof of Theorem II. 37 of Pollard (1984, pp.36, or see Theorem 3.1 of Zhu, 1993), it is easily showed that the second term converges in probability to zero, and similarly for the third term. Completing the proof.

Proof of Theorem 4.2. Note that $\sqrt{n}(\hat{A}_n - A_n) = O_p(1)$, $A_n - A = o(1)$, and $\max_{1 \leq j \leq n} ||y_j||/n^{1/(2l)} \to 0$, $a.s.$. Similar to the proof of Theorem 3.4, the proof can be concluded.

Proof of Theorem 4.3. Note that $\sqrt{n}(\hat{A}_n - A_n) = O_p(1)$, $A_n - A = o(1)$,

$$\begin{aligned} & \sup_{(t,a)\in I\times S^d} |P_n(\sin(ta'\hat{A}_n(z-Ez)+(y-Ey/n^{1/(2l)}))-\sin(ta'\hat{A}_n(z-Ez)))| \\ < \; & cP_n||\hat{A}_n(y-Ey)||/n^{1/(2l)} = O(n^{-1/(2l)}) \quad \text{a.s.}, \end{aligned}$$

$$\begin{aligned} & \sup_{(t,a)\in I\times S^d} |1-\cos(ta'P_n(\hat{A}_n(z-Ez)+\hat{A}_n(y-Ey/n^{1/(2l)})))| \\ < \; & c(||P_n\hat{A}_n(z-Ez)||^2+||P_n\hat{A}_n(y-Ey)||^2/n^{1/l} = O_p(n^{-(l+1)/l}), \end{aligned}$$

and

$$\begin{aligned} & \sup_{(t,a)\in I\times S^d} |\sqrt{n}(\sin(ta'P_n\hat{A}_n((z-Ez)+(y-Ey/n^{1/(2l)})))-\sin(ta'P_n\hat{A}_n(z-Ez)))| \\ & < c\sqrt{n}||\hat{A}_n(P_ny-Ey)||/n^{1/(2l)} = O_p(n^{-1/(2l)}). \end{aligned}$$

Based on these inequalities, a triangle identity and the argument in the proof of Theorem 4.2, it is easy to see that

$$\begin{aligned} & \sqrt{n}P_n(\sin(ta'\hat{A}_n(z+y/n^{1/(2l)}-(\bar{z}+\bar{y}/n^{1/(2l)})))) \\ = \; & \sqrt{n}P_n(\sin(ta'A_n(z+y/n^{1/(2l)}-(Ez+Ey/n^{1/(2l)})))) \\ & -\sqrt{n}P_n(\cos(ta'A_n(z-Ez))\sin(ta'P_nA_n(z-Ez))+O_p(n^{-1/(2l)}) \\ = \; & \sqrt{n}P_n(\sin(ta'A(z-Ez)))+(1/l!)E\{(ta'A(y-Ey))^l\sin^{(l)}(ta'A(z-Ez))\} \\ & -\sqrt{n}\sin(ta'P_nA(z-Ez))E(\cos(ta'A(z-Ez))+o_p(1). \end{aligned}$$

This is just the formula in (4.4). The proof is then concluded from applying Theorem 2.1.

Proof of Theorem 4.4. Along with the argument in Theorem 4.3 and 4.2, the conclusion can be derived.

Acknowledgment. The author thanks Nobert Henze for some valuable comments when a draft of the paper was written.

References

[1] Aki, S. (1987), On non-parametric tests for symmetry, *Ann. Inst. Statist. Math.* **39**, 457-472.

[2] Antille, L.,Kersting, G., and Zucchini,W. (1982), Testing symmetry, *I. Amer. Statist. Assoc.*, **77**, 639-646.

[3] Baringhaus, L. (1991). Testing for spherical symmetry of a multivariate distribution, *Ann. Statist.*, **19**, 899-917.

[4] Baringhaus, L. and Henze, N. (1991). Limit distributions for measures of skewness and kurtosis based on projections, *J. Multivariate Anal.* **38**, 51-69

[5] Beran, R.(1979). Testing for elliptical symmetry of a multivariate density. *Ann. Statist.*, **7**, 150-162.

[6] Blough, D.K. (1989), Multivariate symmetry via projection pursuit, *Ann. Inst. Statist. Math.*, **41**, 461-475.

[7] Csorgo, S., and Heathcote, C.R. (1987), Testing for symmetry, *Boimetrika*, **74**, 177-184

[8] Doksum, K. A., Fenstad, G., and Aaberge, R. (1977), Plots and tests for symmetry, *Biometrika*, **64**, 473-487.

[9] Fang, K. T., Zhu, L. X., and Bentler, P. M. (1993), A necessary test of goodness of fit for sphericity, *J. Multivariate Anal.*, **44** ,34-55.

[10] Feuerverger, A., and Mureika, R. A. (1977), The empirical characteristic function and is applications, *Ann. Statist.*, **5**, 88-97.

[11] Ghosh, S., and Ruymgaart, F.H. (1992), Applications of empirical characteristic functions in some multivariate problems, *Canad. J. Statist.* , **20** ,429-440.

[12] Heathcote, C. R. , Rachev, S.T., and Cheng, B. (1995), Testing multivariate symmetry, *J. Multivariate Anal.*, **54**, 91-112

[13] KARIYA, T., EATON, M. L. (1982), Robust tests for spherical symmetry, *Ann. Statist.*, **1**, 206-215.

[14] POLLARD, D. (1984), *Convergence of Stochastic Processes*, Springer-Verlag, New York.

[15] ROTHMAN, E. D., and WOODROOFE, M. (1972) A Cramer-von Mses type statistic for testing symmetry, *Ann. Math. Statist.*, **43**, 2035-2038.

[16] SCHUSTER, E. F., and BARKER, R. C. (1987), Using the bootstrap in testing symmetry and asymmetry, *Camm. Statist. Simul. Comp.*, **16**, 69-84.

[17] SHORACK, G., and Wellner, J. A. (1986), *The Empirical Processes with Applications to Statistics*, Wiley, New York.

[18] ZHU, L.X., FANG, K.T., and ZHANG, J. T. (1995). A projection NT-type test for spherical symmetry of a multivariate distribution, *Multivariate Statistics and Matrices in Statistics eds, E. -M. Tiit, Kollo, T. and Niemi, H.* TEV & VSP, Holland, 109-122.

Institute of Statistics
Catholique University of Louvain
Louvain-la-Neuve, BELGIUM
and
Institute of Applied Mathematics
Chinese Academy of Sciences
Bejing, 100080, CHINA