Bivariate Distribution Function when a Component is Randomly Truncated

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Abstract

In random truncation models one observes the i.i.d. pairs $(T_i \le Y_i), i = 1, .., n$. If Y is the variable of interest, then, T is another independent variable which prevents the complete observation of Y and random left truncation occurs. Such type of incomplete data is frequently encountered in medical studies as well as in economy, astronomy or insurance applications. Let (Y, X) be a bivariate vector of random variables, with joint distribution function $F(y, x)$. Suppose the variable Y is randomly truncated from the left. In this study nonparametric estimators for $F(y, x)$ are proposed and a.s. representations as i.i.d. mean processes are obtained, from which the a.s. consistency and weak convergence followed.

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1 Introduction and Preliminaries

In survival or reliability studies, incomplete data is frequently encountered. Random truncation and censoring are two common forms of such data. In random left truncation model, one observes the i.i.d. pairs $(Y_i \geq T_i), i = 1, .., n$, where Y is the variable of interest and T is another independent variable which prevents the complete observation of Y. Random right truncation is similarly defined by interchanging the roles of Y and T. One of the earliest applications of the left truncation model was given by Lynden-Bell(1971), where Y denotes the brightness of celestial objects, which is only partially observable due to a preventing variable T. Random truncation models gained more interest in recent years because it is conveniently used to model several aspects of AIDS data. See e.g. Kalbfleisch and Lawless(1989) and Wang(1989) for AIDS applications.

Now suppose (Y, X) is a bivariate vector of random variables with joint distribution function(d.f) $F(y, x) = P(Y \leq y, X \leq x)$. In this study, nonparametric estimation of $F(y, x)$ is considered when Y is subject to left truncation, ie., only those triplets $(Y_i, X_i, T_i), i = 1, .., n$ are observed for which $T_i \leq Y_i$, where T is a r.v., independent of both Y and X. It is assumed that F is continuous in both components and without loss of generality, the pair (Y, X) is taken to be nonnegative. For the deviations of the estimates from $F(y, x)$, a.s. representations as i.i.d. mean processes are obtained. Strong consistency and weak convergence of the estimators are then easily established. Such representations are available in the literature for the univariate case when observations are censored and/or truncated(see discussion below). For the bivariate case however, no such study seems to have appeared, to the best of our knowledge.

The paper is organized as follows: In the rest of this section, main results for the univariate case are presented. In Section 2, proposed estimators for $F(y, x)$ are introduced. Section 3 contains the main theoretical results of this study. The proofs of this section are deferred to the appendix. Finally in Section 4, results of a simulation study is presented, together with some remarks concerning the possible extensions of the model considered.

Now some preliminary results existing in the literature for univariate truncation models will be presented. Let Y be the variable of interest and T be the truncating variable with d.f.'s F and G

respectively. The pairs (Y,T) are observed only if $(T_i \leq Y_i),\ i = 1,\ldots,n$. Under this sampling scheme, Woodroofe (1985) points out that F and G can be estimated completely, only if $(F,G) \in R_o$, where

$$R_o = \{(F,G) :\ a_G \leq a_F;\ b_G \leq b_F\} \tag{1}$$

and, for any distribution function W,

$$a_W = \inf\{x :\ W(x) > 0\},\ \ b_W = \sup\{x :\ W(x) < 1\}.$$

Then $(F,G) \in R_o$ implies

$$\alpha \equiv P(T \leq Y) = \int_o^\infty G(x)\, dF(x)\ >\ 0.$$

Note that α is the proportion of the bivariate population (Y,T), which can be observed under the truncation scheme. If (1) is not valid, only a conditional distribution function can be estimated, ie., one can only estimate $F(y)/\bar{F}(T_o)$ where $T_o \geq \min(T_1,\ldots,T_n)$ and $\bar{F} = 1 - F$.

Assuming that (1) holds, the nonparametric estimators F_n, G_n of F, G are given by

$$\begin{aligned}\bar{F}_n(y) &= \prod_{i:Y_i \leq y}\left[1 - \frac{s(Y_i)}{nC_n(Y_i)}\right]\\ G_n(t) &= \prod_{i:T_i > t}\left[1 - \frac{r(T_i)}{nC_n(T_i)}\right]\end{aligned}$$

where for $u > 0$

$$r(u) = \#\{i :\ T_i = u\}, \quad s(u) = \#\{i : Y_i = u\} \tag{2}$$

and

$$nC_n(u) = \#\{i :\ T_i \leq u \leq Y_i\}.$$

Several authors have studied the statistical properties of the estimators given above. $F_n(y)$ was first suggested by Lynden-Bell(1971) and is the nonparametric maximum likelihood estimator of F. Woodroofe (1985) proved the uniform weak consistency of F_n and showed the weak convergence of the process $\sqrt{n}(F_n - F)$ to a Gaussian process. Wang, Jewell and Tsai (1986) proved the uniform strong consistency of G_n. Chao and Lo (1988) derived a representation of $(F_n - F)$ as i.i.d. mean processes with a remainder term of $o(n^{-1/2})$ a.s. if $a_G = a_F$ and $O(logn/n)^{\frac{3}{4}}$ if $a_G < a_F$. The order of the remainder term for this representation is improved by Stute(1993) under a slightly more restrictive assumption and it is extended to the left truncated and right censored model by Gijbels and Wang(1993), again with an improved order for the remainder term when $a_G < a_F$ (ref. Theorem 1 below). As an application of their results, Gijbels and Wang illustrated the kernel density estimation. Kernel estimators of the hazard function for truncated/censored data are studied by Uzunoğulları and Wang (1992). Gürler, Stute and Wang(1993), Gu and Lai (1990), Lai and Ying (1991), and Gross and Caroll(1992) extended the results for truncated/censored data in various directions. Keiding and Gill(1990) provided a Markov -Process approach to the model and derived similar results with Martingale methods.

The results of Chao and Lo(1988), Gijbels and Wang(1993), and Stute(1993) will frequently be to in the rest of this paper and therefore they are presented below:

Let (Y_i, T_i), $i = 1, \ldots n$ be the observations. Define

$$L_i(z) \;=\; \alpha \left\{ \frac{I(Y_i \leq z)}{G(Y_i)\bar{F}(Y_i)} \;-\; \int_0^z \frac{I(T_i \leq u \leq Y_i)}{G(u)\bar{F}^2(u)} dF(u) \right\}$$

and

$$\bar{L}_n(z) = \sum_{i=1}^{n} n^{-1} L_i(z)$$

Theorem 1: Suppose F is continuous and $b < b_F$. Then the following representation holds:

$$\begin{aligned} F_n(z) - F(z) \quad &= \quad n^{-1}\bar{F}(z)\sum_{i=1}^{n} L_i(z) \; + \varepsilon_n(z) \\ &= \quad \bar{F}(z)\bar{L}_n(z) + \varepsilon_n(z) \end{aligned}$$

(i)(Chao and Lo(1988)).(a) Let $a_G = a_F$. If $lim_{x \to 0+} F(x)/G(x) \;\; = 0$ and $\int dF/G \; < \; \infty$, then

$$\sup_{0 \leq z \leq b} |\varepsilon_n(z)| = o(n^{-1/2}) \quad a.s.$$

(b) Let $a_G < a_F$. Then

$$\sup_{0 \leq z \leq b} |\varepsilon_n(z)| = O((logn/n)^{3/4}) \;\; a.s.$$

(ii) (Gijbels and Wang(1993)). Let $a_G < a_F$. Then

$$\sup_{0 \leq z \leq b} |\varepsilon_n(z)| = O(logn/n) \;\; a.s.$$

(iii)(Stute(1993a)). Let $a_G \leq a_F$. If $\int dF/G^2 < \infty$, then

$$\sup_{a_{F_Y} \leq z \leq b} |\varepsilon_n(z)| = O(log^3 n/n) \quad a.s.$$

Note here that the result of Gijbels and Wang(1993) is an improvement over that of Chao and Lo(1988) for the case $a_G < a_F$, and for $a_G = a_F$, Stute(1993a) provides a better bound than Chao and Lo(1988), but at the expense of assuming a more restrictive integrability condition. See however Stute(1993a) for a discussion of this condition.

Corollary 1: Under the conditions of Theorem 1, the process $W_n(z) = \sqrt{n}(F_n(z) - F(z))$ converges weakly to a zero mean Gaussian process on D[0,b], with covariance structure :

$$Cov(\ W_n(z_1), W_n(z_2)\ ) = \alpha \bar{F}(z_1)\bar{F}(z_2) \int_0^{z_1 \wedge z_2} \frac{F(du)}{G(u)\bar{F}^2(u)} \ .$$

2 Bivariate Model and the Estimators

We now consider the bivariate truncation model, in which one observes the triplets (Y_i, X_i, T_i), $i = 1, \ldots, n$ only if $(T_i \leq Y_i)$. The purpose is to estimate the bivariate d.f. $F(y, x)$ of (Y, X). T here is a nuisance random variable, which is assumed to be independent of (Y, X), with d.f. G. All the variables are assumed to be continuous, nonnegative. The marginal d.f.'s of Y and X are denoted by F_Y and F_X respectively. To avoid the identifiability problems, it is assumed that $a_{F_Y} \geq a_G$ and $b_{F_Y} \geq b_G$, as in the univariate model. Given this model, the observed triplets can be considered to arise from the following trivariate conditional distribution H:

$$\begin{aligned} H_{Y,X,T}(y, x, t) &= P\,(\,Y \leq y, X \leq x, T \leq t \mid Y \geq T\,) \\ &= \alpha^{-1} \int_0^x \int_0^y G(t \wedge u)\, dF(u, v) \end{aligned}$$

where α is as defined before, and $t \wedge u = min(y, u)$. The observed pairs then have the following distributions:

$$F^*_{Y,X}(y, x) \equiv H_{Y,X,T}(y, x, \infty) = \alpha^{-1} \int_0^x \int_0^y G(u)\, dF(u, v)\ .$$

$$H_{X,T}(x,t) = \alpha^{-1}\int_0^x \int_0^\infty G(t\wedge u)\, dF(u,v) \tag{3}$$
$$H_{Y,T}(y,t) = \alpha^{-1}\int_0^\infty \int_0^y G(t\wedge u)\, dF(u,v).$$

Univariate marginals are also easily obtained as:

$$F_Y^*(y) = \alpha^{-1}\int_0^\infty \int_0^y G(u)\, dF(u,v)$$
$$F_X^*(x) = \alpha^{-1}\int_0^x \int_0^\infty G(u)\, dF(u,v)$$
$$G_T^*(t) = H_{Y,T}(\infty,t) = \alpha^{-1}\int_0^\infty \int_0^\infty G(t\wedge u)\, dF(u,v).$$

Assuming the existence of the densities(denoted in lowercase letters), we also have

$$f_{Y,X}^*(y,x) = \alpha^{-1}G(y)\, f_{Y,X}(y,x)$$
$$f_Y^*(y) = \alpha^{-1}G(y)\, f_Y(y)$$
$$f_X^*(x) = \alpha^{-1}\int_0^\infty G(u)f(u,x)\, du$$

The following function is of importance in truncation model, and the empirical countarpart of it defined in (2) is the so called 'risk set at time z':

$$\begin{aligned} C(z) &= \alpha^{-1}G(z)\bar{F}_Y(z-) \\ &= G_T^*(z) - F_Y^*(z) \end{aligned} \tag{4}$$

It is interesting to note here that, the conditional density of X given Y in the truncation model is the same as that of untruncated model. This implies that inference for the conditional distribution of X given Y can be based on the observed truncated sample. However the reverse is not true for Y given X and the procedures proposed in this study can be used to handle this case.

The first estimator considered for $F(y,x)$ is the following:

$$F_n^1(y,x) = \frac{1}{n}\sum_i \frac{\bar{F}_{Y,n}(Y_i-)}{C_n(Y_i)}.I(Y_i \le y, X_i \le x),$$

where $\bar{F}_{Y,n}$ and C_n are as given in (2). This estimator is motivated from the observation that

$$f(y,x) = [\alpha^{-1}G(y)]^{-1}f^*(y,x)$$

where

$$[\alpha^{-1}G(y)]^{-1} \;=\; \frac{\bar{F}_Y(y)}{C(y)}$$

which follows from (3) and (4) .

It can easily be verified that $F_n^1(y,x)$ is a bivariate distribution function. This is an important nice feature of this estimator, since many of the existing estimators for the bivariate d.f. in the censored case, lack this property (see eg. the discussion in Dabrowska(1988)). Stute (1993,b) however proposes an estimator for the censored case which is similar in structure to F_n^1 and which is a bivariate d.f.. He proves the strong consistency of his estimator and consider some applications. Recently van der Laan(1992) proposes a new class of estimates which are asymptotically efficient and are bivariate survival functions. He also presents an extensive discussion of the existing estimators for censored observations. The other two estimators suggested below also lack this property of being a proper d.f.. $F_n^1(y,x)$ reduces to the product limit estimator (2) of $F_Y(y)$ when $x \to \infty$, and as $y \to \infty$, we have the following estimator for the marginal d.f of X:

$$F_{X,n}(x) = \frac{1}{n}\sum_i \frac{F_{Y,n}(Y_i-)}{C_n(Y_i)}.I(X_i \leq x) \tag{5}$$

Theoretical results relating to these estimators will be presented in the next section.

In the rest of this section two other estimators for $F(y,x)$ will be presented and discussed. These estimators are based on a bivariate "diverse hazard" function defined below. This development is motivated by the Dabrowska(1988) estimator for the censored observations. Recall that for i.i.d. univariate observations, the d.f. is expressed in terms of the cumulative d.f. in q unique way(see eg. Woodroofe 1985). For the bivariate case however, there is not a single definition of a cumulative hazard function(see Marshall(1975), Cox , Johnson and Kotz(1975) etc. for several definitions). Dabrowska(1988) provides a nice representation of a bivariate survival function in terms of her hazard function which is a vector of three components that correspond to double and single failures. In what follows a similar approach will be adapted which utilizes an analogous relation between the bivariate d.f. and a slightly different hazard function, which we call "diverse hazard". For the description of the estimators we consider the following function. Somewhat abusing the notation, define

$$\bar{F}(y,x) = P(Y \geq y, X \leq x)$$

Letting $R(y,x) = log\bar{F}(y,x)$, we have

$$\begin{aligned}\bar{F}(y,x) &= \bar{F}_Y(y)F_X(x)exp\{-\int_x^\infty \int_0^y R(du,dv)\} \\ &\equiv \bar{F}_Y(y)F_X(x)exp\{-\Lambda(y,x)\}\end{aligned}$$

and

$$F(y,x) = F_X(x)[1-\bar{F}_Y(y)exp\{-\Lambda(y+,x)\}]$$

In the above decomposition, the marginal distributions of Y and X can be estimated by (2) and (5). For $exp\{-\Lambda(y,x)\}$, two possible estimators will be proposed and they will lead to two alternative estimators for $F(y,x)$. Define

$$\begin{aligned}C_2(y,x) &= H_{T,X}(y,x) - F^*(y-,x) &(6)\\ &= \alpha^{-1}G(y)\bar{F}(y,x) &(7)\end{aligned}$$

It can be seen from the relations presented (3) and (7) that

$$\begin{aligned}\Lambda_{12}(du,dv) &\equiv -\frac{\bar{F}(du,dv)}{\bar{F}(u,v)} = \frac{F^*(du,dv)}{C_2(u,v)}\\ \Lambda_1(du,v) &\equiv -\frac{\bar{F}(du,v)}{\bar{F}(u,v)} = \frac{F^*(du,v)}{C_2(u,v)}\end{aligned}$$

and

$$\Lambda_2(u,dv) \equiv \frac{\bar{F}(u,dv)}{\bar{F}(u,v)} = -\frac{C_2(u,dv)}{C_2(u,v)}.$$

We define the bivariate "diverse hazard" vector as $\tilde{\Lambda}(u,v)$, where

$$\tilde{\Lambda}(u,v) = (\Lambda_{12}(du,dv), \Lambda_1(du,v), \Lambda_2(u,dv))$$

This is analogous to the bivariate hazard vector given in Dabrowska(1988), and a similar equivalence relation between this vector and a general bivariate survival function can be established. We therefore omit the details of such a representaion and refer the interested reader to the above mentioned paper. Now, since

$$\Lambda(y,x) = \int_x^\infty \int_0^y \Lambda_1(du,v)\Lambda_2(u,dv) - \Lambda_{12}(du,dv),$$

an estimator, $\Lambda_n(y,x)$ can be obtained by replacing the integrands with their empirical equivalents in terms of the observable distributions. In particular, for a bivariate function $\phi(u,v)$, which is left-continuous in the u component, and right-continuous in the v component, letting

$$\phi(\delta u, x) = \phi(u+,v) - \phi(u,v)$$

$$\phi(u,\delta v) = \phi(u,v) - \phi(u,v-)$$

$$\phi(\delta u,\delta v) = \phi(u+,v) - \phi(u,v) - \phi(u+,v-) + \phi(u,v-)$$

we have the following natural estimates

$$\Lambda_{12,n}(\delta u,\delta v) = \frac{F_n^*(\delta u,\delta v)}{C_{2,n}(u,v)}$$

$$\Lambda_{1,n}(\delta u,v) = \frac{F_n^*(\delta u,v)}{C_{2,n}(u,v)}$$

$$\Lambda_{2,n}(u,\delta v) = \frac{C_n(u,\delta v)}{C_{2,n}(u,v)}$$

where,

$$nC_{2,n}(u,v) \;=\; \#\{i : T_i \le u \le Y_i\;,\; X_i \le v\}.$$

Then, an estimator for $\Lambda(y,x)$ is obtained as

$$\Lambda_n(y,x) = \sum_{u\le y}\sum_{v>x}[\Lambda_{1,n}(\delta u,v)\Lambda_{2,n}(u,\delta v) - \Lambda_{12,n}(\delta u,\delta v)] \tag{8}$$

Hence we have the following alternative estimator:

$$F_n^2(y,x) = F_{X,n}(x)[1 - \bar{F}_{Y,n}(y)exp\{-\Lambda_n(y,x)\}],$$

where $F_{X,n}(x)$ and $\bar{F}_{Y,n}(y)$ are as given in (2) and (5). The other alternative is based on the Jordan decomposition of $\Theta(y,x) \equiv exp\{-\Lambda(y,x)\}$ presented in Dabrowska(1988). Let

$$\Gamma_n(\delta u,\delta v) = \frac{\Lambda_{1,n}(\delta u,v)\Lambda_{2,n}(u,\delta v) - \Lambda_{12,n}(\delta u,\delta v)}{[1-\Lambda_{1,n}(\delta u,v)][1-\Lambda_{2,n}(u,\delta v)]}$$

The decomposition provided in Dabrowska(1988) then suggests the following estimator for $\Theta(y,x)$:

$$\Theta_n(y,x) = \prod_{u\le y; v>x}[1 - \Gamma_n(\delta u,\delta v)].$$

Our last estimators is then obtained as follows:

$$F_n^3(y,x) = F_{X,n}(x)[1 - \bar{F}_{Y,n}(y)\Theta_n(y,x)],$$

where the marginal distributions of Y and X are again obtained from (2) and (5).

3 Almost Sure Representations

Observe that $F_n^1(y,x)$ is a weighted sum of i.i.d. variables, where the weights are the jump sizes of the truncation product limit estimator $F_{Y,n}^1(y)$ at the data points. Therefore theoretical properties of this estimator are strongly related to those of $F_{Y,n}(y)$. The following theorem provides an i.i.d. representation of $F_n^1(y,x)$, the proof of which is given in the appendix.

Theorem 2: Assume F is continuous in both components, $b < b_{F_Y}$ and let $T_b = \{(y,x) : 0 < y < b; 0 < x < \infty\}$. Then $F_n^1(y,x)$ admits the following representation:

$$\begin{aligned} F_n^1(y,x) - F(y,x) &= \int_0^y \frac{\bar{F}(u)}{C(u)}[F_n^*(du,x) - F^*(du,x)] \\ &+ \int_0^y \frac{\bar{F}(u)}{C^2(u)}[C(u) - C_n(u) - \bar{L}_n(u)C(u)]F^*(du,x) + R_n(y,x) \\ &\equiv \bar{\xi}_n(y,x) + R_n(y,x) \end{aligned} \tag{9}$$

and

(i) If $a_G < a_{F_Y}$, then

$$\sup_{(y,x)\,\epsilon\, T_b} |R_n(y,x)| = O(log^2 n/n)$$

(ii) If $a_G = a_{F_Y}$, and $\int G^{-2}(u)F_Y(du) < \infty$, then

$$\sup_{(y,x)\,\epsilon\, T_b} |R_n(y,x)| = O(log^3 n/n) = o(n^{-1/2}).$$

Notice here that the order of the remainder term for part (i) is better than that of Chao and Lo(1988), but not as good as the result of Gijbels and Wang(1993). It may be another task to further improve this result to achieve a similar bound. The magnitute of the remainder term for part (ii) derives from the result of Stute(1993a) and therefore the integrability condition here is more restrictive than that of Chao and Lo(1988). Note however that, starting with the result of Chao and Lo(1988), one can obtain the same order of magnitude $o(n^{-1/2})$ as theirs for the remainder term, by merely the assuming the conditions of part(ii) of Theorem 1. The following Corollary is immediate from Theorem 2, SLLN and functional LIL.

Corollary 2: Under the assumptions of part (ii) of Theorem 2,

(a) For $(y, x) \epsilon T_b$

$$F_n^1(y, x) \to F(y, x) \quad a.s.$$

(b) $\sup_{(y,x)\, \epsilon\, T_b} |F_n(y, x) - F(y, x)| = O((logn/n)^{1/2})$

(c) An i.i.d representation for the estimator of the marginal distribution of X immediately follows:

$$F_{X,n}(x) - F_X(x) \;=\; \bar{\xi}_{X,n}(x) \equiv \bar{\xi}_n(\infty, x) + R_{X,n}(x), \tag{10}$$

where the order of the remainder term is as given in the theorem.

Theorem 2 can also be utilized to establish the weak convergence of the process $Z_n(y, x) = \sqrt{n}[F_n^1(y, x) - F(y, x)]$. The weak convergence of empirical processes in multidimensional time is considered in Neuhaus(1971) and Straf(1972). Campbell(1981) used the results of Neuhaus(1971) to establish the weak convergence of the bivariate process in the censoring case. For the present work also, the construction of Neuhaus(1971) will be utilized and only the result will be stated. The interested reader however is referred to this article for a detailed discussion. Define the processes:

$$W_n(z_1, z_2) = \sqrt{n}[F_n^*(z_1, z_2) - F^*(z_1, z_2)]$$

$$V_n(z) = \sqrt{n}[C_n(z) - C(z)]$$

$$U_n(z) = \sqrt{n}[F_{Y,n}^*(z) - F_Y^*(z)]$$

We have the following covariance relations:

$$Cov(W_n(z_1, z_2), W_n(t_1, t_2) = F^*(z_1 \wedge t_1, z_2 \wedge t_2) - F^*(z_1, z_2)F^*(t_1, t_2)$$

$$Cov(V_n(z_1), V_n(z_2)) = H_T(z_1 \wedge z_2) - H_{Y,T}(z_1 \vee z_2, z_1 \wedge z_2)$$

$$Cov(U_n(z_1), U_n(z_2) = F_Y^*(z_1 \wedge z_2) - F_Y^*(z_1)F_Y^*(z_2)$$

And for $s < z,\ s < z_1$

$$Cov(V_n(s), U_n(z) = H_{Y,T}(z, s) - H_{Y,T}(s, s) - F_Y^*(z)C(s)$$

$$Cov(W_n(z_1, z_2), U_n(z)) = F^*(z, z_2) - F^*(z_1, z_2)F_Y^*(z)$$

$$Cov(W_n(z_1,z_2),V_n(z)) = H_{Y,X,T}(z_1,z_2,z) - H_{Y,X,T}(s,z_2,z) - F^*(z_1,z_2)C(z)$$

Then it follows from standard results that $V_n(.)$ and $U_n(.)$ converge weakly to mean zero Gaussian processes on D[0,b) with covariance structures given above. The weak convergence of $W_n(z_1,z_2)$ to a mean zero two-time parameter Gaussian process on the complete seperable metric space (D_2,d) defined on $[0,1]\times[0,1]$ described in Neuhaus(1971) follows from the arguments in that article.

The representation below, which is obtained from Theorem 2, by integration by parts will be more convenient for the presentation of the following corollary:

$$\begin{aligned} F_n^1(y,x) - F(y,x) &= [F_n^*(y,x) - F^*(y,x)]\frac{\bar{F}_Y(y)}{C(y)} \\ &- \int_0^y [F_n^*(u,x) - F^*(u,x)]d(\frac{\bar{F}_Y(u)}{C(u)}) \\ &+ \int_0^y \frac{\bar{F}_Y(u)}{C^2(u)}[C(u) - C_n(u) + \bar{L}_n(u)C(u)]F^*(du,x) + R_n(y,x) \end{aligned}$$

Corollary 3: Suppose the conditions of part (ii) of Theorem 3 hold. Then for $(y,x)\,\epsilon\,T_b$, $Z_n(y,x)$ converges weakly to a mean zero, two dimentional time Gaussian process on (D_2,d). The covariance structures can be obtained from the expressions given above and Corollary 1.

The results for the other estimators will be obtained via the following lemmas, which are of interest on their own also. In Lemma 2 below, it will be shown that the estimators F_n^i, (i=1,2) are asymptotically equivalent, which enables us to carry out the inference together. Let

$$H_n(y,x) = \bar{F}_{Y,n}(y)\,exp\{-\Lambda_n(y,x)\}$$

The following lemma provides an i.i.d. representation for $\Lambda_n(y,x)$ and $H_n(y,x)$:

Lemma 1: Let $a > a_{F_X}$ $b < b_{F_Y}$ be such that $\bar{F}(b,a) > 0$ and $T_{a,b} = \{(y,x) : 0 < y \le b, x \ge a\}$. If $\int F(dy)/G^2(y) < \infty$, then

(i)

$$\begin{aligned} \Lambda_n(y,x) - \Lambda(y,x) &= \int_0^y \int_x^\infty \left\{ \frac{C^2(u,v) - C_n^2(u,v)}{C^4(u,v)} F^*(du,v)C(u,dv) \right. \\ &+ \left. \frac{C(u,v) - C_n(u,v)}{C^2(u,v)}\, F^*(u,v) \right\} + R_{1,n}(y,x) \\ &\equiv \xi_{\Lambda,n}(y,x) + R_{1,n}(y,x) \end{aligned}$$

(ii)

$$\begin{aligned} H_n(y,x) - H(y,x) &= exp\{-\Lambda(y,x)\}.[\bar{L}_n(y) - \bar{F}(y)\,\bar{\xi}_{\Lambda,n}(y,x)] + R_{2,n}(y,x) \\ &\equiv \bar{\xi}_{H,n}(y,x) + R_{2,n}(y,x) \end{aligned}$$

where for $i = 1, 2$

$$\sup_{(y,x)\,\epsilon\,T_{a,b}} |R_{i,n}(y,x)| = O(log^3 n/n).$$

Lemma 2: Let $D_n(y,x) = \Theta_n(y,x) - exp(-\Lambda_n(y,x))$. Then, under the conditions of Lemma 1,

$$\sup_{(y,x)\,\epsilon\,T_{a,b}} |D_n(y,x)| = O(log^3 n/n).$$

We now present the main result for the last two estimators:

Theorem 3: If the conditions of Lemma 1 hold, then for $i = 1, 2$,

$$\begin{aligned} F_n^i(y,x) - F(y,x) &= \bar{\xi}_{X,n}(x)[1 - H(y,x)] \\ &+ F_X(x) exp\{-\Lambda(y,x)\}[\bar{L}_n(y) - \bar{F}(y)\,\bar{\xi}_{\Lambda,n}(y,x)] + R_{3,n}(y,x) \end{aligned}$$

where

$$\sup_{(y,x)\,\epsilon\,T_{a,b}} |R_{3,n}(y,x)| = O(log^3 n/n).$$

Theorem 3 implies the strong consistency of the estimators $F_n^i, i = 1, 2$. The weak convergence of the corresponding processes could also be established similar to the previous discussion relating to $F_n^1(y,x)$, therefore it will not be further elaborated. However, it is seen from the i.i.d. representations that the covariance structure of the two limiting processes will be different.

4 Simulations and Concluding Remarks

It is seen from the results of the last section that all three estimators are asymptotically unbiased and have a normal distribution, but with different limiting variances. It is however hard to compare and judge the quality of the estimators on the basis of these large sample results. Therefore their performance are investigated by a moderate size simulation study. For samples of size $n = 50$, 75, and 1000 replications, two different models for (Y, X) and T are considered as follows:

1) (Y, X) : Bivariate Exponential such that $Y = min(U_1, U_{12})$ and $X = min(U_2, U_{12})$ where U_1, U_2 and U_{12} are Exponential r.v.'s with mean $\lambda = 1$ and T is Exponential(μ)

2) (Y, X) : Independent Exponential ($\lambda = 1.5$) and T is Exponential(μ).

The parameter μ of the T variable is adjusted to obtain $\alpha = 0.25, 0.50, 0.75$ approximately.

The results are reported in Tables 1 and 2, in which the α values are the average proportion of observable pairs in 1000 replications. As a performance measure of the estimators, estimated Integrated Mean Squared error is used. The integral is evaluated for equally spaced grids of 40 points in each axis. However the region of integration is restricted to a subset of the support of (Y, X), where about % 95 of the probability lies. The terms bsq, var, and mse refer to the integrated squared bias, variance and mean square respectively. In the tables, the first column corresponds to the bivariate empirical d.f., applied to an untruncated sample (Y, X), for reference purposes.

The results of this simulation indicate that, for practical purposes, the estimators are behaving equally well. It could also seen however that in most of the cases either F_n^2 or F_n^3 have slightly larger bias and F_n^1 has larger variance, which yield an insignificant dominance of F_n^1 by the others. In particular, for the cases considered, F_n^2 seems to behave best in terms of the integrated mean square. These results indicate that it is worthwhile to further study the relative properties of these estimators, since the last last two estimators, besides having the disadvantage of not being a proper bivariate distribution function, have some promising performance. We conclude this section by pointing out some extensions of the proposed methods.

Remarks:

1-) (Extension to k Covariates)

If we observe $(Y_i, X_{1,i}, \ldots, X_{k,i}, T_i)$, $i = 1, \ldots, n$ only if $(T_i \leq Y_i)$, then $F_n^1(.)$ is modified in a simple obvious fashion:

$$F_n^1(y, x) = \sum_{i: Y_i \leq y,\, X_{j,i} \leq x_{ji}\; j=1,2,..,k} \frac{\bar{F}_{Y,n}(Y_i-)}{nC_n(Y_i)}$$

For the other estimators, the extension is possible, however they become quite messy. Interested

Model	α		F_n^o	F_n^1	F_n^2	F_n^3
1	0.24	bsq	0	13	12	12
		var	577	1502	1493	1496
		mse	577	1515	1505	1508
	0.51	bsq	1	1	1	2
		var	553	1234	1229	1230
		mse	554	1235	1230	1232
	0.81	bsq	1	2	2	2
		var	583	761	758	759
		mse	584	763	760	761
2	0.28	bsq	0	3	3	3
		var	547	1577	1569	1570
		mse	547	1580	1572	1573
	0.48	bsq	1	1	1	1
		var	491	1388	1381	1383
		mse	492	1389	1382	1384
	0.76	bsq	0	7	8	8
		var	542	845	842	843
		mse	542	853	850	851

Table 1: IMSE(F_n^i), 1000 replications, n=30, x10^{-5}

Model	α		F_n^o	F_n^1	F_n^2	F_n^3
1	0.23	bsq	1	4	4	4
		var	223	685	682	683
		mse	0224	689	686	687
	0.50	bsq	0	0	0	0
		var	241	553	550	551
		mse	241	553	550	551
	0.80	bsq	0	0	0	0
		var	0228	350	349	349
		mse	0228	350	349	349
2	0.28	bsq	0	2	2	2
		var	204	667	664	664
		mse	204	669	666	666
	0.47	bsq	0	2	3	3
		var	210	543	541	542
		mse	0210	545	544	545
	0.75	bsq	0	1	1	1
		var	202	371	370	370
		mse	203	372	371	371

Table 2: IMSE(F_n^i), Left truncation, 1000 replications, n=75, x10^{-5}

readers could refer to Dabrowska(1988) for such extensions.

2-) (Extension to double truncation)

Let $\dot{Y} = (Y_1, Y_2)$ $\dot{T} = (T_1, T_2)$ be random bivariate vectors and suppose we observe $(\dot{Y}_i, \dot{T}_i)$ only if $\dot{Y}_i \geq \dot{T}_i$ componentwise, for $i = 1, \ldots, n$. The results of the present paper are not directly applicable for this double truncation scheme. To see this note that, as before we have:

$$H(\dot{y}, \dot{t}) = \alpha^{-1} \int^{\dot{y}} G(\dot{t} \wedge \dot{u}) F(d\dot{u})$$

and we have

$$\frac{F(d\dot{y})}{\bar{F}(\dot{y})} = \frac{F^*(d\dot{y})}{C(\dot{y})}$$

where $\bar{F}(\dot{y}) = P(Y_1 > y_1, Y_2 > y_2)$, C is componentwise similarly defined as before and $\alpha = P(\dot{Y} \geq \dot{T})$. For this model, it is not possible to use the decomposition of $F(y, x)$ that gave rise to the estimators F_n^1 and F_n^2, since this time, estimators for the marginals of F are not directly available. Therefore other decompositions based on the conditional densities could be used. Such an approach for censored data was adapted by Tsai, Leurgans and Crowley(1988). Furthermore the identifiability issue becomes more delicate in this model. This problem therefore, will further be addressed in another study.

3-) (Right truncation)

The estimators suggested in this paper could be extended for right truncation model in a natural way, however with some modifications. Since right truncation models receive attention for their applicability to AIDS data, it seems appropriate to indicate the necessary modifications. Suppose we observe $(Y_i, X_i, T_i), i = 1, \ldots n$ only if $Y_i \leq T_i$. Then the observed (Y, X, T) have the distribution $H_{Y,X,T}(y, x, t)$, where

$$\begin{aligned} H_{Y,X,T}(y, x, t) &= P\,(\,Y \leq y, X \leq x, T \leq t \mid Y \geq T) \\ &= \alpha^{-1} \int_t^F (y \wedge u, x)\, dG(u) \end{aligned}$$

Then, $F_n^1(y, x)$ defined similarly except that $\bar{F}_n(y)$ is replaced with $F_n(y)$. For the other estimators we need the following modifications:

$$C_2(y, x) = F^*_{Y,X}(y, x) - H_{T,X}(y-, x) \qquad R(y, x) = log F(y, x).$$

Then,

$$\frac{F(du, dv)}{F(u, v)} = \frac{F^*(du, dv)}{C_2(u, v)}$$

$$\frac{F(du,v)}{F(u,v)} = \frac{F^*(du,v)}{C_2(u,v)}$$
$$\frac{F(u,dv)}{F(u,v)} = \frac{C_2(u,dv)}{C_2(u,v)}.$$

Now, we can estimate the decomposition

$$F(y,x) \;=\; F_Y(y)F_X(x)exp\{\int_x^\infty \int_y^\infty R(du,dv)\}$$

APPENDIX

For the proof of the lemmas and theorems of Section 3, the following lemmas will be useful:

Lemma A1:

(i)

$$\sup_{y\le b} \frac{(C_n(y)-C(y))^2}{C(y)} = O(logn/n)$$

(ii)

$$\sup_{(y,x)\,\epsilon\, T_{a,b}} \frac{(C_{2,n}(y,x)-C_2(y,x))^2}{C_2(y,x)} = O(logn/n)$$

Proof: Part (i) is Lemma A2 of Chao and Lo(1988) and part(ii) could be obtained similarly.

Lemma A2:

(i)

$$\sup_i \frac{C(Y_i)}{C_n(Y_i)} \;=\; O(logn)$$

(ii)

$$\sup_i \frac{C_2(Y_i,X_i)}{C_{2,n}(Y_i,X_i)} \;=\; O(logn)$$

Proof: Part (i) is Corollary 1.3. of Stute(1991) and for part(ii) a similar approach can be used, by defining the process:

$$\bar{H}_N(y,x,t) = N^{-1}\sum I(Y_i \le y, X_i \le x, T_i \le t)$$

and showing that $\bar{H}_N(y,x,t)/F(y,x)G(t)$ is a reverse submartingale.

Proof of Theorem 2:

For the brevity of notation, the arguments of $\bar{F}_Y(u)$, $\bar{F}_{Y,n}(u)$, $C(u)$, and $C_n(u)$ will be supressed and the notation $\int_{y]x]}$ will be used to denote the double integral $\int_0^y \int_0^x$.

$$\begin{aligned}
F_n^1(y,x) - F(y,x) &= \int_{y]x]} \{ \frac{\bar{F}_{Y,n}}{C_n} F_n^*(du,dv) \; - \; \frac{\bar{F}_Y}{C} F^*(du,dv) \} \\
&= \int_{y]x]} \{ \frac{\bar{F}_Y}{C} (F_n^*(du,dv) \; - \; F^*(du,dv)) \; + \; \frac{\bar{F}_Y(C - C_n)}{C^2} F^*(du,dv) \\
&+ \; \frac{\bar{F}_{Y,n} - \bar{F}_Y}{C} F_n^*(du,dv) \; + \; \frac{\bar{F}_Y(C - C_n)}{C^2} (F_n^*(du,dv) - F^*(du,dv)) \\
&+ \; \frac{\bar{F}_Y(C - C_n)^2}{C^2 C_n} F_n^*(du,dv) \; + \; \frac{(\bar{F}_{Y,n} - \bar{F}_Y)(C - C_n)}{C C_n} F_n^*(du,dv) \} \\
&\equiv I + II + III + R_{2,n} + R_{3,n} + R_{4,n}
\end{aligned}$$

From the representation in Theorem 1, the term III above is written as :

$$\int_{y]x]} \{ \frac{\bar{F}_Y \bar{L}_n}{C} F^*(du,dv) \; + \; \frac{\bar{F}_{Y,n} - \bar{F}}{C} (F_n^*(du,dv) - F^*(du,dv)) \} + \varepsilon_n\prime(y)$$

$$\equiv \int_{y]x]} \frac{\bar{F}_Y \bar{L}_n}{C} F^*(du,dv) \; + \; R_{1,n} + \varepsilon_n\prime(y)$$

where

$$\sup_{(0<y<b)} |\varepsilon_n\prime(y)| = O(|\varepsilon_n(y)|)$$

Hence after evaluating the above integrals w.r.t. v, we obtain the representation of Theorem 2, with

$$R_n(y,x) = \sum_{i=1}^{4} R_{i,n}(y,x) + O(|\varepsilon_n(y)|)$$

The orders of $R_{i,n}, (i = 1, \ldots, 4)$ can now be obtained from Theorem 1, Lemma A1 (i), Lemma A2 (i) and the following facts:

(i) For $a_G \leq a_{F_Y}, b_G \leq b_{F_Y}$

$$\sup_{0 \leq y < \infty} |C_n(y) - C(y)| = O((logn/n)^{1/2})$$

since C_n is a difference of empirical d.f.'s.

(ii)For $(y,x) \,\epsilon\, [0,\infty) \;\times\; [0,\infty)$

$$\sup |F_n^*(y,x) - F^*(y,x)| = O((logn/n)^{1/2})$$

(iii)For $(y,x)\,\epsilon\,[0,\infty)\;\times\;[0,\infty)$

$$\int_{y]x]}\frac{\bar{F}}{C^2}F^*(du,dv)\leq\alpha\int_0^y\frac{F_Y(du)}{Gu}.$$

As an illustration, consider

$$R_{3,n}(y,x)=\int^y\frac{\bar{F}_Y(C-C_n)^2}{C^2C_n}F_n^*(du,x).$$

We have,

$$\begin{aligned}\sup|R_{3,n}(y,x)| &\leq \sup\frac{(C-C_n)^2}{|C|}\int^y\frac{\bar{F}_Y}{C.C_n}F_n^*(du,x)\\ &\leq O(logn/n)\,\sup_i\frac{C(Y_i)}{C_n(Y_i)}\int^y\frac{\bar{F}_Y}{C^2}F_n^*(du,x)\\ &= O(logn/n)\,O(logn)\,O(1)\;=\;O(log^2n/n),\end{aligned}$$

by Lemma A1(i) and A2(i).

Proof of Lemma 1: In the following proof, the double integral $\int_0^y\int_x^\infty$ will be denoted by $\int_{y][x}$ and the arguments of $C_2(y,x)$ and its empirical counterpart will be dropped to avoid a messy presentation (i) The representation is easily obtained with the following remainder term:

$$\begin{aligned}R_{1,n}(y,x) &= \int_{y][x}\{\frac{C_2-C_{2,n}}{C_2^2}[F_n^*(du,dv)\;-F^*(du,dv)]\\ &+ \frac{(C_2-C_{2,n})^2}{C_2^2C_{2,n}}F_n^*(du,dv)\\ &+ \frac{C_2^2-C_{2,n}^2}{C_2^4}[F_n^*(du,v)C_{2,n}(u,dv)\;-F^*(du,v)C_2(u,dv)]\\ &+ \frac{(C_2^2-C_{2,n}^2)^2}{C_2^4C_{2,n}^n}F_n^*(du,v)C_{2,n}(u,dv)\}\\ &\equiv \sum_{i=1}^4R_{1,i}(y,x) \qquad (11)\end{aligned}$$

where, for $(y,x)\,\epsilon\,T_{a,b}$, it follows from L.I.L. and S.L.L.N. that

$$\sup|R_{1,1}(y,x)|=O(logn/n).$$

From Lemma A1 and A2 parts (ii), we have

$$\sup|R_{1,2}(y,x)|\;=\;O(log^2n/n)$$

and

$$\begin{aligned}\sup|R_{1,3}(y,x)| &\leq cons.\sup|C_2 - C_{2,n}|\left|\int_{y][x}\frac{[F_n^*(du,v)C_{2,n}(u,dv)\ - F^*(du,v)C_2(u,dv)]}{C_2^4}\right|\\ &\leq O((logn/n)^{1/2})\,O((logn/n)^{1/2})\end{aligned}$$

The last inequality above is obtained by applying the LIL to the sum of i.i.d. variables represented by the integral there and the following relations:

$$F^*(du,v) = \alpha^{-1}G(u)F(du,v)$$

and

$$C_2(u,dv) = \alpha^{-1}[G(u)F_X(dv) - G(u)F(u,dv)]$$

hence,

$$\begin{aligned}\int_{y][x} C_2^{-4}F^*(du,v)C_2(u,dv) &= \int_{y][x}\alpha\frac{F(du,v)F_X(dv)}{G(u)\bar{F}^3(u,v)}\\ &- \int_{y][x}\alpha\frac{F(du,v)F(u,dv)}{G(u)\bar{F}^3(u,v)}\end{aligned}$$

and it is in absolute value less than

$$\alpha\bar{F}^{-4}(a,b)\int F_Y(du)/G^2(u) = O(1).$$

For $R_{1,4}(y,x)$, note that by Lemma A2 (ii),and A1(ii)

$$\begin{aligned}\frac{(C_2^2 - C_{2,n}^2)^2}{C_2^4C_{2,n}^2} &\leq \sup\left|\frac{(C_2 - C_{2,n})^2}{C_2}.\frac{(C_2 + C_{2,n})^2}{C_2^3C_{2,n}^2}\right|\\ &\leq O(logn/n).2\ \max\left(\frac{1}{C_2^3},\frac{1}{C_{2,n}^2C_2}\right)\\ &= O(logn/n)O(log^2n)\frac{1}{C_2^3}\\ &= O(log^3n/n)\frac{1}{C_2^3}\end{aligned}$$

The bound for $R_{1,4}(y,x)$ is now obtained from the SLLN, since the remaining integral represents a sum of i.i.d. random variables with finite mean , bounded similar to term $R_{1,3}(y,x)$ (ii) Follows from part(i)

Proof of Lemma 2: After some algebra we can write:

$$\begin{aligned}
|D_n(y,x)| &= |\Theta_n(y,x) - exp(-\Lambda_n(y,x))| \\
&\leq |log\Theta_n(y,x) \ + \ \Lambda_n(y,x)| \\
&= \Bigg|\sum_{u\leq y}\sum_{v>x} log[1-\Gamma_n(\delta u,\delta v)] \\
&+ \ \Lambda_{1,n}(\delta u, v)\Lambda_{2,n}(u,\delta v) - \Lambda_{12,n}(\delta,\delta v)| \\
&= \Bigg|\sum_{u\leq y}\sum_{v>x} log\left(\frac{C_{2,n}(u,v-)C_{2,n}(u,v) + C_{2,n}(u,v-)F_n^*(\delta u,v)}{C_{2,n}(u,v-)C_{2,n}(u,v) + C_{2,n}(u,v)F_n^*(\delta u,v-)}\right) \\
&- \left(\frac{C_{2,n}(u,v-)F_n^*(\delta u,v) - C_{2,n}(u,v)F_n^*(\delta u,v-)}{C_{2,n}^2(u,v)}\right)\Bigg|
\end{aligned}$$

Now, applying a two term Taylor expansion to the logarithm term above, and rearranging the terms, the above expression can be written(up to a term of a smaller order) as

$$\Bigg|\sum_{u\leq y}\sum_{v>x} C_{2,n}^{-2}(u,v)F_n^*(\delta u,\delta v)[F_n^*(\delta u,v) + \ F_n^*(\delta u,v-) + C_{2,n}(u,\delta v)]$$

$$- \frac{F_n^*(\delta u,v-)C_{2,n}(u,\delta v)}{C_{2,n}^2(u,v)C_{2,n}(u,v-)}[C_{2,n}(u,\delta v) + F_n^*(\delta u,v-)]\Bigg|$$

Now observing that

$$\frac{F_n^*(\delta u,v-)}{C_{2,n}(u,v-)} \leq \frac{F_{Y,n}(\delta u)}{C_n(u)},$$

and using Lemma A1, we obtain

$$\begin{aligned}
\sup_{(y,x)\,\epsilon\,T_{a,b}} |D_n(y,x)| &\leq O(log^2 n/n)\int \frac{F_n^*(\delta u,\delta v)}{C_2(u,v)} \\
&+ \ O(log^3 n/n)\int \frac{F_{Y,n}(\delta u)C_{2,n}(u,\delta v)}{C_2^3(u,v)} \\
&= O(log^3 n/n).
\end{aligned}$$

Proof of Theorem 3 now follows from a little algebra, Lemma 1 and Lemma 2.

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